Scaling of the geomagnetic secular variation time scales

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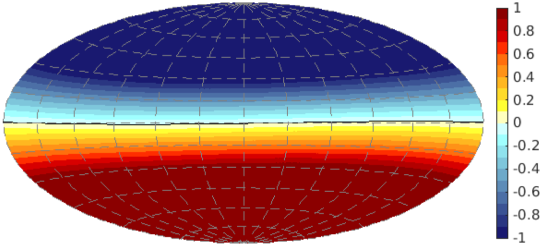
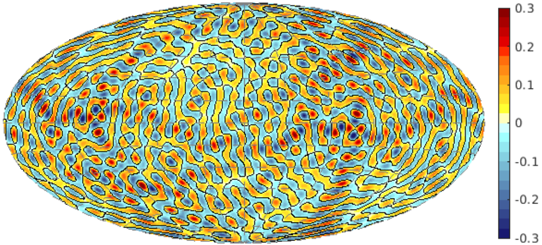
Time variation of the geomagnetic field at different spatial scales

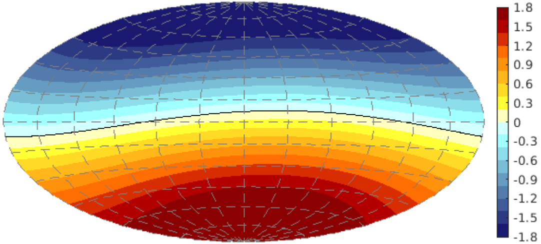
B_r for $l = 1$ at $r = 1.00\, r_{\text{cmb}}$ and $t = 2.01799$



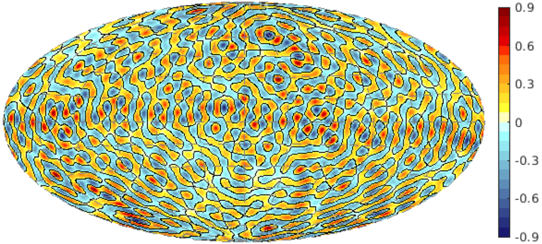
B_r for $l = 40$ at $r = 1.00\, r_{\text{cmb}}$ and $t = 2.01799$



B_r for $l = 1$ at $r = 0.50\, r_{\text{cmb}}$ and $t = 2.01799$



B_r for $l = 40$ at $r = 0.50\, r_{\text{cmb}}$ and $t = 2.01799$



Spectra: to study properties at different spatial scales

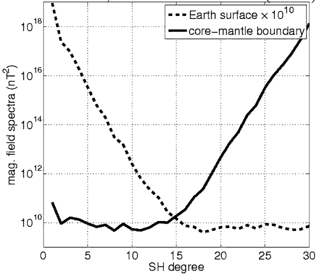
(1) Lowes spectrum $(r \geqslant r_{\text{cmb}})$

$$R(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[\, g_{lm}^2(t) + h_{lm}^2(t) \,\right],$$

$$\sum_{l=1}^{\infty} R(l,r,t) = \frac{1}{4\pi} \oint |\boldsymbol{B}(r,\theta,\phi,t)|^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi$$

(a = Earth's radius)

Gillet, Lesur and Olsen (2010)

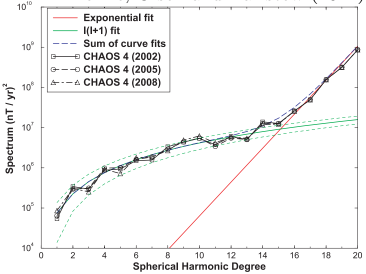


(2) Secular variation spectrum $(r \geqslant r_{\text{cmb}})$

$$R_{\text{sv}}(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[\, \dot{g}_{lm}^2(t) + \dot{h}_{lm}^2(t) \right]$$

$$\sum_{l=1}^{\infty} R_{\text{sv}}(l,r,t) = \frac{1}{4\pi} \oint |\dot{\boldsymbol{B}}(r,\theta,\phi,t)|^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi\,, \quad \dot{\boldsymbol{B}} = \frac{\partial \boldsymbol{B}}{\partial t}$$

Holme, Olsen and Bairstow (2011)



Secular variation time-scale spectrum

$$R(l) \sim \text{“amount” of } B^2 \text{ in spatial scale } l$$

$$R_{\rm sv}(l) \sim \text{“amount” of } \dot{B}^2 \text{ in spatial scale } l$$

$$\tau_{\rm sv}(l,t) = \sqrt{\frac{R}{R_{\rm sv}}} = \sqrt{\frac{\sum_{m=0}^{l}\left(g_{lm}^2 + h_{lm}^2\right)}{\sum_{m=0}^{l}\left(\dot{g}_{lm}^2 + \dot{h}_{lm}^2\right)}} \quad (r \geqslant r_{\rm cmb})$$

- characteristic time scale of magnetic field structures with spatial scale characterised by l
- numerical simulations and *some* satellite data support the simple power-law: $\tau_{\rm sv}(l) \sim l^{-1}$ (but there are debates about this)
- theoretically, there is an argument based on the frozen-flux hypothesis that involves the radial magnetic field and the horizontal derivative:

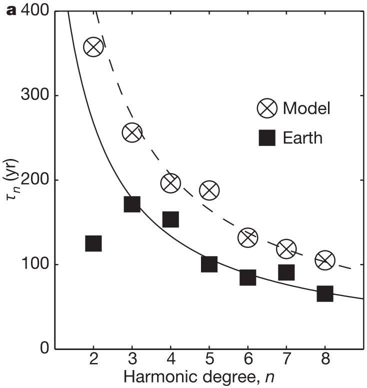
$$\dot{B}_r = -\nabla_{\rm h} \cdot (\boldsymbol{u}_{\rm h} B_r)$$

$$\nabla_{\rm h} \sim \sqrt{l(l+1)} \sim l \quad \text{and} \quad \boldsymbol{u}_{\rm h} \sim U$$

$$\tau_{\rm sv} \sim B_r/\dot{B}_r \sim l^{-1}$$

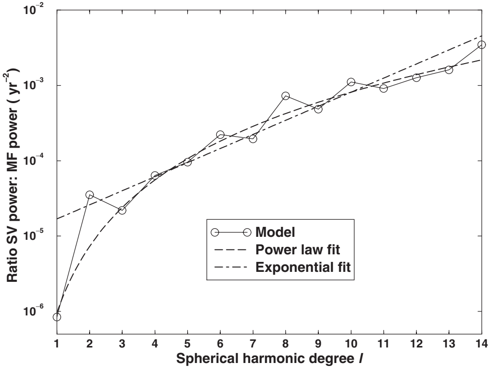
Scaling of $\tau_{\rm sv}(l)$: observations and numerical models

Christensen and Tilgner (2004)
observation data 1840 – 1990
and numerical dynamo models



$$\tau_{\rm sv} \sim l^{-1}$$

Holme and Olsen (2006)
satellite data 1999 – 2003



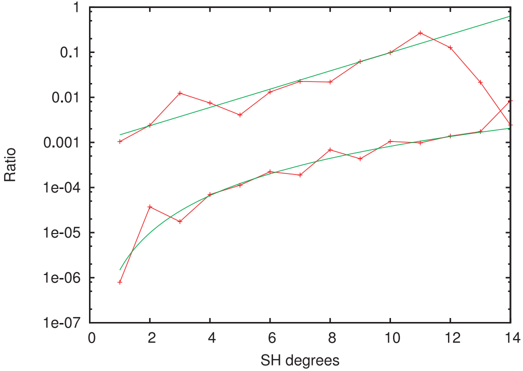
$$\frac{R_{\rm sv}}{R} \sim l^{2.9}$$

$$\Longrightarrow \quad \tau_{\rm sv} \sim l^{-1.45}$$

Scaling of $\tau_{\text{sv}}(l)$: observations and numerical models

Lesur et al. (2008)

6yr CHAMP + 5yr observatory data

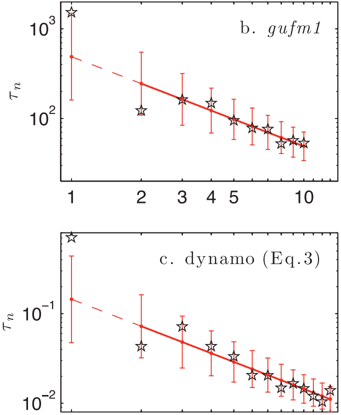


$$\frac{R_{\text{sv}}}{R} \sim l^{2.75}$$

$$\Longrightarrow \ \tau_{\text{sv}} \sim l^{-1.38}$$

Lhuillier et al. (2011)

"historical data" 1840 − 1990, satellite data (2005) and numerical dynamo models



$$\tau_{\text{sv}} \sim l^{-1}$$

The scaling exponent γ

$$\tau_{\rm sv}(l) \sim l^{-\gamma} \qquad \text{(excluding } l = 1\text{)}$$

Questions:

1. $\tau_{\rm sv}$ is defined using the Gauss coefficients obtained from $\boldsymbol{B}$ outside the outer core.
   Do $\tau_{\rm sv}$ and the scaling law $\tau_{\rm sv} \sim l^{-1}$ describe the time variation of $\boldsymbol{B}$ inside the outer core?
   [*No. Inside the outer core, $\boldsymbol{B}$ is not potential. $\dot{B}_r$ $\dot{B}_\theta$ and $\dot{B}_\phi$ may all be important.*]

2. Does the frozen-flux argument explain the scaling $\gamma = 1$ observed at the surface?
   [*No. Magnetic diffusion is important near the CMB.*]

3. What are the mechanisms controlling the scaling $\tau_{\rm sv} \sim l^{-\gamma}$?
   [*It varies, depending on locations and the boundary conditions.*]

Generalisation to inside the dynamo region (outer core)

Recall the definition of the Lowes spectrum $R(l,r,t)$ for $r \geqslant r_{\text{cmb}}$,

$$\boldsymbol{B} = -\nabla\Psi\,, \quad \Psi(r,\theta,\phi,t) = a\sum_{l=1}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^{l+1}\hat{P}_{lm}(\cos\theta)\left[g_{lm}(t)\cos m\phi + h_{lm}(t)\sin m\phi\right]$$

$$\frac{1}{4\pi}\oint|\boldsymbol{B}(r,\theta,\phi,t)|^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \sum_{l=1}^{\infty}R(l,r,t)$$

$$R(l,r,t) = \left(\frac{a}{r}\right)^{2l+4}(l+1)\sum_{m=0}^{l}\left[\,g_{lm}^2(t) + h_{lm}^2(t)\,\right]$$

For any r, expand in vector spherical harmonics,

$$\boldsymbol{B}(r,\theta,\phi,t) = \sum_{lm}\left[q_{lm}(r,t)\hat{\boldsymbol{Y}}_{lm}(\theta,\phi) + s_{lm}(r,t)\hat{\boldsymbol{\Psi}}_{lm}(\theta,\phi) + t_{lm}(r,t)\hat{\boldsymbol{\Phi}}_{lm}(\theta,\phi)\right]$$

We define the magnetic energy spectrum $F(l,r,t)$ for all r:

$$\sum_{l=1}^{\infty}F(l,r,t) \equiv \frac{1}{4\pi}\oint|\boldsymbol{B}(r,\theta,\phi,t)|^2\,\mathrm{d}\Omega = \sum_{l=1}^{\infty}\left[\frac{1}{(2l+1)}\sum_{m=0}^{l}\left(|q_{lm}|^2 + |s_{lm}|^2 + |t_{lm}|^2\right)(4-3\delta_{m,0})\right]$$

Generalisation to inside the dynamo region (outer core)

$$F(l,r,t) = \frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|q_{lm}|^2 + |s_{lm}|^2 + |t_{lm}|^2\right)(4 - 3\delta_{m,0})$$

Similarly, define the time variation spectrum $F_{\dot{B}}(l,r,t)$:

$$\dot{\boldsymbol{B}}(r,\theta,\phi,t) = \sum_{lm} \left[\dot{q}_{lm}(r,t)\hat{\boldsymbol{Y}}_{lm}(\theta,\phi) + \dot{s}_{lm}(r,t)\hat{\boldsymbol{\Psi}}_{lm}(\theta,\phi) + \dot{t}_{lm}(r,t)\hat{\boldsymbol{\Phi}}_{lm}(\theta,\phi)\right]$$

$$\sum_{l=1}^{\infty} F_{\dot{B}}(l,r,t) \equiv \frac{1}{4\pi} \oint |\dot{\boldsymbol{B}}(r,\theta,\phi,t)|^2 \, \mathrm{d}\Omega = \sum_{l=1}^{\infty} \left[\frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|\dot{q}_{lm}|^2 + |\dot{s}_{lm}|^2 + |\dot{t}_{lm}|^2\right)(4 - 3\delta_{m,0})\right]$$

Then, the magnetic time-scale spectrum is defined as:

$$\tau(l,r) = \left\langle \sqrt{\frac{F(l,r,t)}{F_{\dot{B}}(l,r,t)}} \right\rangle_t$$

Outside the dynamo region: $F = R$, $F_{\dot{B}} = R_{\mathrm{sv}}$, $\tau = \tau_{\mathrm{sv}}$

A numerical model of geodynamo

Boussinesq, compositional driven, rotating convection of a electrically conducting fluid:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\frac{Pm}{Ek}\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\frac{Pm}{Ek}\nabla\Pi' + \left(\frac{RaPm^2}{Pr}\right)C'\boldsymbol{r} + \frac{Pm}{Ek}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + Pm\nabla^2\boldsymbol{u},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla^2\boldsymbol{B}$$

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{Pm}{Pr}\nabla^2 C - 1$$

$$\nabla \cdot \boldsymbol{u} = 0$$

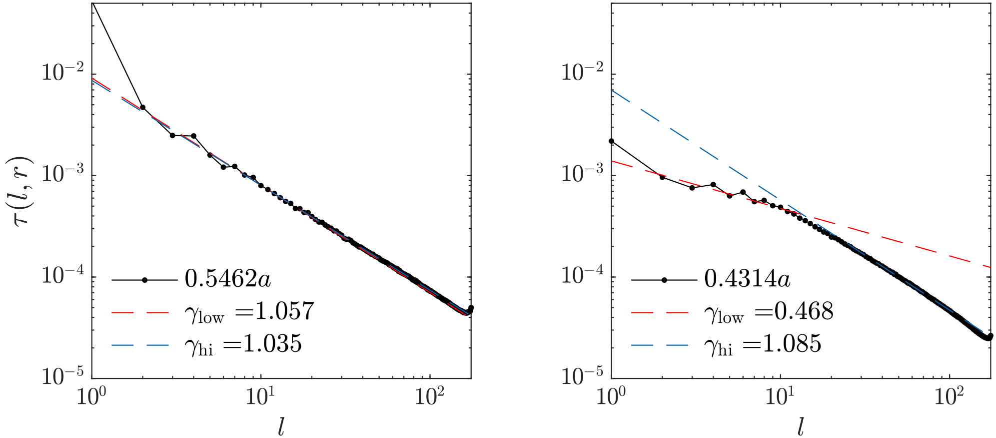
$$\nabla \cdot \boldsymbol{B} = 0$$

Boundary conditions: **no-slip** for $\boldsymbol{u}$, Neumann for C

Domain: a spherical shell $0.1912\,a \leqslant r \leqslant 0.5462\,a$

$Ra = 2.7 \times 10^8\,, Ek = 2.5 \times 10^{-5}\,, Pm = 2.5\,, Pr = 1$

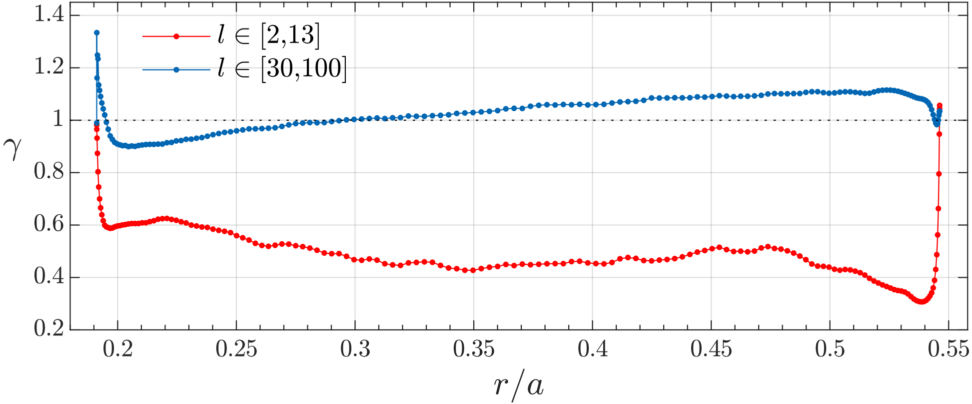
Magnetic time-scale spectrum $\tau(l, r)$ at different depth



For the large-scale modes (small l),

- at the surface: $\tau \sim l^{-1}$
- in the interior: $\tau \sim l^{-0.5}$, *the large-scale modes speeds up in the interior!*

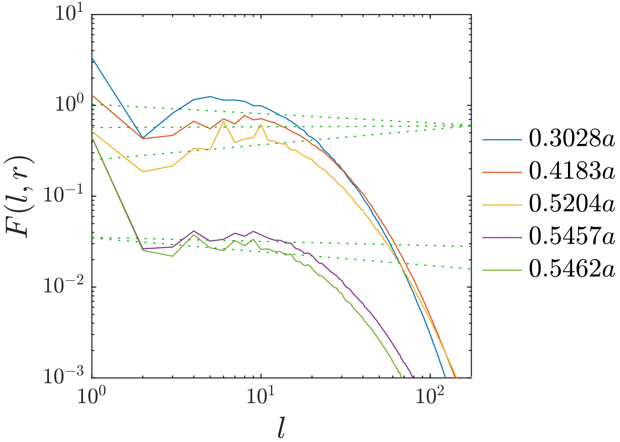
Change in the scaling of τ: where does it occur?



- γ for the large-scale modes increases sharply within a boundary layer under CMB

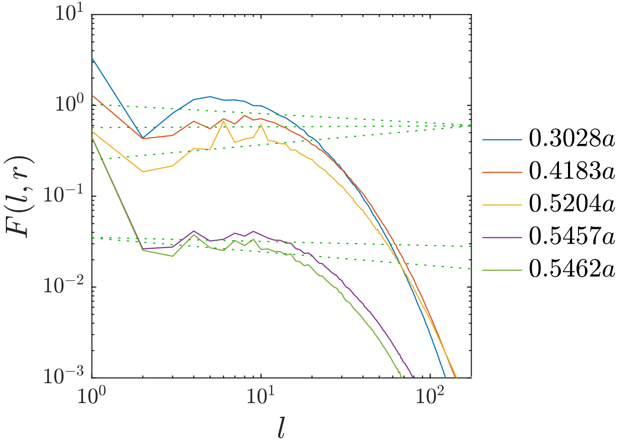
Focus on the large scales in following discussion . . .

Change in the scaling of τ: who causes it?



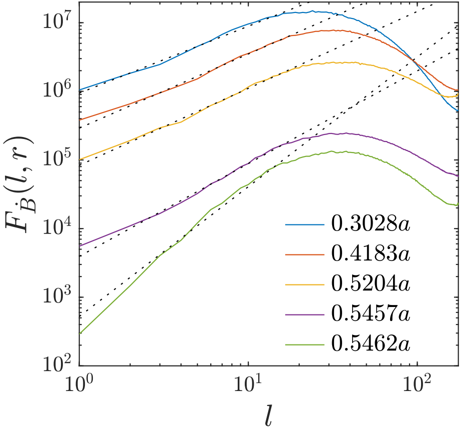
$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}}$$

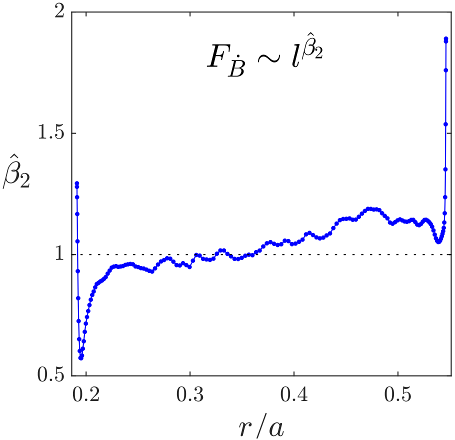
Change in the scaling of τ: who causes it?



$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{F_{\dot{B}}}} \sim {F_{\dot{B}}}^{-\frac{1}{2}}$$

Change in the scaling of τ: who causes it?



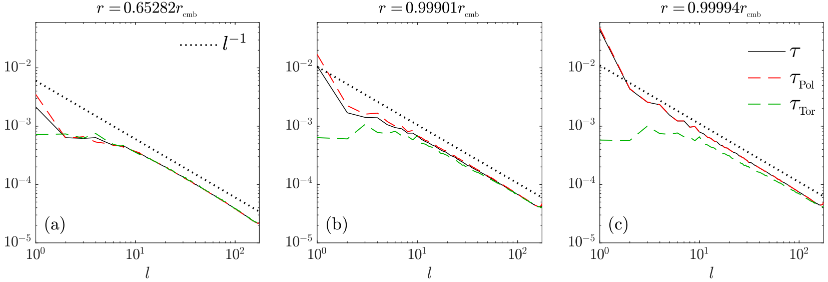


$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{F_{\dot{B}}}} \sim F_{\dot{B}}^{-\frac{1}{2}}$$

$$F_{\dot{B}} \sim l \implies \tau \sim l^{-0.5} \quad \text{(interior)}$$

$$F_{\dot{B}} \sim l^2 \implies \tau \sim l^{-1} \quad \text{(surface)}$$

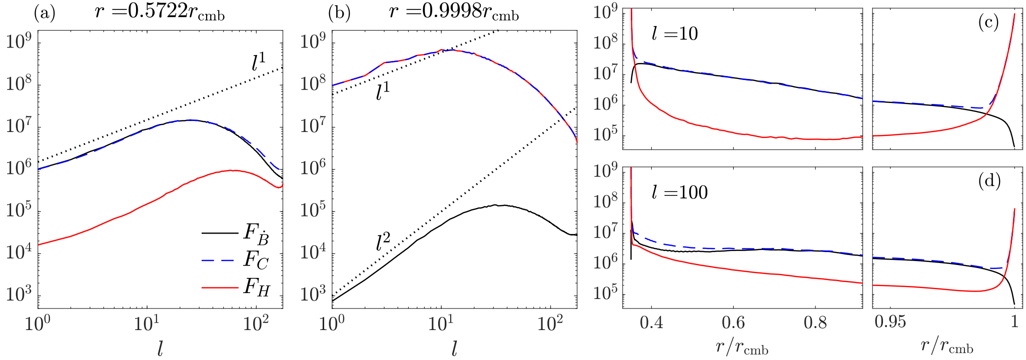
Transition in terms of poloidal and toroidal time scales



$$\boldsymbol{B} = \boldsymbol{B}_{\text{Pol}} + \boldsymbol{B}_{\text{Tor}} \,, \qquad \tau_{\text{Pol}} = \sqrt{\frac{\text{spectrum of } \boldsymbol{B}_{\text{Pol}}}{\text{spectrum of } \dot{\boldsymbol{B}}_{\text{Pol}}}} \,, \qquad \tau_{\text{Tor}} = \sqrt{\frac{\text{spectrum of } \boldsymbol{B}_{\text{Tor}}}{\text{spectrum of } \dot{\boldsymbol{B}}_{\text{Tor}}}}$$

- interior: $\boldsymbol{B}_{\text{Pol}}$ and $\boldsymbol{B}_{\text{Tor}}$ are equally important, $\tau \approx \tau_{\text{Pol}} \approx \tau_{\text{Tor}} \sim l^{-0.5}$
- no change in shape for τ_{Tor}, but the magnetic boundary condition requires $\boldsymbol{B}_{\text{Tor}} \to \boldsymbol{0}$ as $r \to r_{\text{cmb}}$, so τ_{Tor} becomes irrelevant near the CMB
- As $r \to r_{\text{cmb}}$: $\tau_{\text{Pol}} \sim l^{-0.5}$ transitions to $\tau_{\text{Pol}} \sim l^{-1}$ and $\boldsymbol{B} \approx \boldsymbol{B}_{\text{Pol}} \implies \tau \approx \tau_{\text{Pol}} \sim l^{-1}$
- contribution of $\dot{\boldsymbol{B}}_{\text{Tor}}$ ($\dot{B}_\theta$ and $\dot{B}_\phi$) to $\dot{\boldsymbol{B}}$ in the interior masked by the boundary conditions

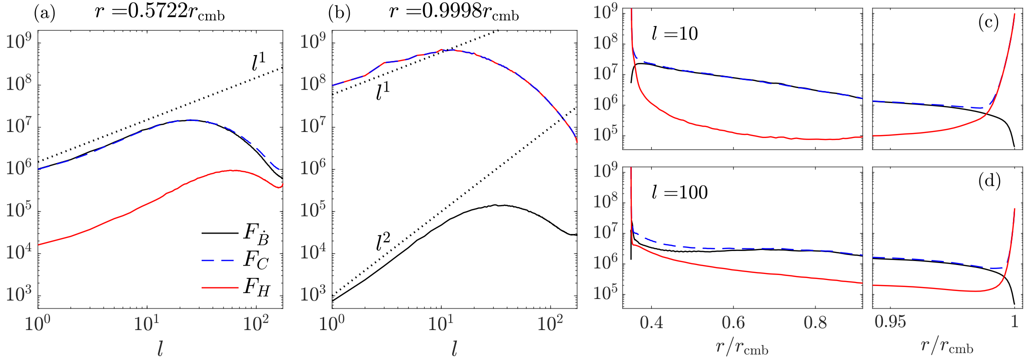
What processes control the scaling of τ in the interior?



$$\dot{\boldsymbol{B}} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} = \boldsymbol{C} + \boldsymbol{H}$$

- $F_{\dot{B}} \approx F_C$ (magnetic diffusion negligible), $F_{\dot{B}} \sim F_C \sim l \implies \tau \sim l^{-0.5}$
- frozen-flux argument not applicable:
  - $\dot{B}_\theta$ and $\dot{B}_\phi$ contribute significantly to $\dot{\boldsymbol{B}}$
  - $\dot{B}_\theta$ and $\dot{B}_\phi$ dominated by radial derivatives in the induction term $\boldsymbol{C}$

What processes control the scaling of τ near the CMB?



$$\dot{\boldsymbol{B}} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} = \boldsymbol{C} + \boldsymbol{H}$$

- $F_{\dot{B}} \sim l^2 \implies \tau \sim l^{-1}$, but ...
- $F_C \approx F_H$, $F_C \sim F_H \sim l$ ($\boldsymbol{C}$ and $\boldsymbol{H}$ cancel to leading order)
- $\boldsymbol{H}$ is important $\Rightarrow$ frozen-flux argument is not applicable in explaining $\tau \sim l^{-1}$ at the CMB

Summary

- scaling of $\tau(l,r)$ with l observed outside the outer core is different from that in the interior
- for the large scales:

$$\begin{aligned} \tau &\sim l^{-0.5}, \quad &\text{in the interior} \\ \tau &\sim l^{-1}, \quad &\text{at the CMB} \end{aligned}$$

  the transition occurs within a boundary layer under the CMB
- time variation of $\boldsymbol{B}_{\mathrm{Tor}}$ in the interior is hidden from surface observation by the magnetic boundary condition at the CMB
- for the large scales, $F_{\dot{B}}$ is responsible for the transition ($\tau = \sqrt{F/F_{\dot{B}}}$)
  - in the interior, induction term $\boldsymbol{C}$ dominates, $\dot{\boldsymbol{B}} \approx \boldsymbol{C}$ and $F_{\dot{B}} \sim l$
  - at the CMB (no-slip), balance between the induction term and magnetic diffusion leads to $F_{\dot{B}} \sim l^2$, meaning frozen-flux argument not applicable