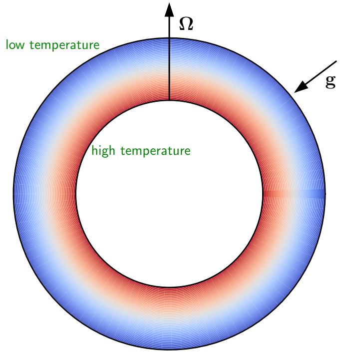
Oscillatory double-diffusive convection in a rotating spherical shell

Yue-Kin Tsang

School of Mathematics, Statistics and Physics
Newcastle University

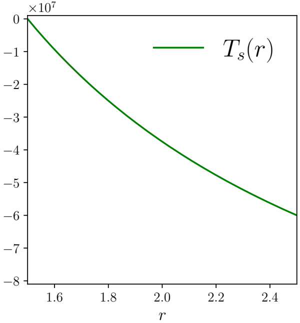
Céline Guervilly, Graeme R. Sarson

Pure thermal convection



$$\left.\frac{\partial T}{\partial r}\right|_{r_i} = -q_i < 0$$

$$\left.\frac{\partial T}{\partial r}\right|_{r_o} = -q_o < 0$$



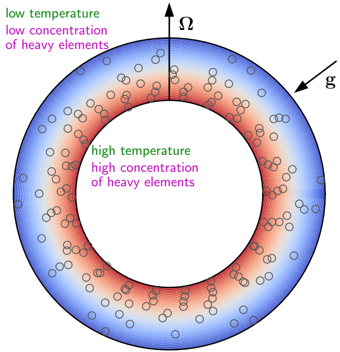
$$T_s(r) = q_i r_i^2 \left(\frac{1}{r} - \frac{1}{r_i}\right)$$

$$\frac{\mathrm{d}T_s}{\mathrm{d}r} < 0 \quad \textit{destabilising}$$

Consider a Boussinesq fluid in a rotating spherical shell of inner radius r_i and outer radius r_o

- thermal diffusivity: κ_T
- density: $\rho(T) = \rho_m[1 - \alpha_T(T - T_m)]\,, \quad \boldsymbol{F}_{\text{buoy}} = \rho \boldsymbol{g}/\rho_m$
- buoyancy frequency: $N^2 = -\dfrac{g}{\rho_m}\dfrac{\mathrm{d}\rho(T_s)}{\mathrm{d}r} \implies N^2 = g\alpha_T \dfrac{\mathrm{d}T_s}{\mathrm{d}r} < 0 \quad \text{(top-heavy)}$

Oscillatory double-diffusive convection (ODDC)

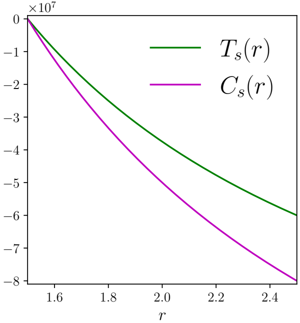


$$\left.\frac{\partial T}{\partial r}\right|_{r_i} = -q_i < 0$$

$$\left.\frac{\partial T}{\partial r}\right|_{r_o} = -q_o < 0$$

$$\left.\frac{\partial C}{\partial r}\right|_{r_i} = -f_i < 0$$

$$\left.\frac{\partial C}{\partial r}\right|_{r_o} = -f_o < 0$$



$$T_s(r) = q_i r_i^2 \left(\frac{1}{r} - \frac{1}{r_i}\right)$$

$$\frac{\mathrm{d}T_s}{\mathrm{d}r} < 0 \quad \textit{destabilising}$$

$$C_s(r) = f_i r_i^2 \left(\frac{1}{r} - \frac{1}{r_i}\right)$$

$$\frac{\mathrm{d}C_s}{\mathrm{d}r} < 0 \quad \textit{stabilising}$$

Consider a Boussinesq fluid in a rotating spherical shell of inner radius r_i and outer radius r_o

- composition diffusivity: $\kappa_C \ll \kappa_T$
- density: $\rho(T, C) = \rho_m[1 - \alpha_T(T - T_m) + \alpha_C(C - C_m)]$, $\quad \boldsymbol{F}_{\mathrm{buoy}} = \rho\boldsymbol{g}/\rho_m$
- buoyancy frequency: $N^2 = -\dfrac{g}{\rho_m}\dfrac{\mathrm{d}}{\mathrm{d}r}\left[\rho(T_s, C_s)\right] \implies N^2 = g\alpha_T\dfrac{\mathrm{d}T_s}{\mathrm{d}r} - g\alpha_C\dfrac{\mathrm{d}C_s}{\mathrm{d}r}$

Governing equations of ODDC

Perturbations temperature and perturbation composition:

$$T(\boldsymbol{x},t) = T_s(r) + \Theta(\boldsymbol{x},t), \quad C(\boldsymbol{x},t) = C_s(r) + \xi(\boldsymbol{x},t)$$

Non-dimensional equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{2}{Ek}\hat{\boldsymbol{z}}\times\boldsymbol{u} = -\nabla\Pi + (\Theta-\xi)r\,\hat{\boldsymbol{r}} + \nabla^2\boldsymbol{u},$$

$$\nabla\cdot\boldsymbol{u} = 0,$$

$$\frac{\partial\Theta}{\partial t} + \boldsymbol{u}\cdot\nabla\Theta = \frac{Ra_T}{Pr}\left(\frac{\gamma}{1-\gamma}\right)^2\frac{u_r}{r^2} + \frac{1}{Pr}\nabla^2\Theta,$$

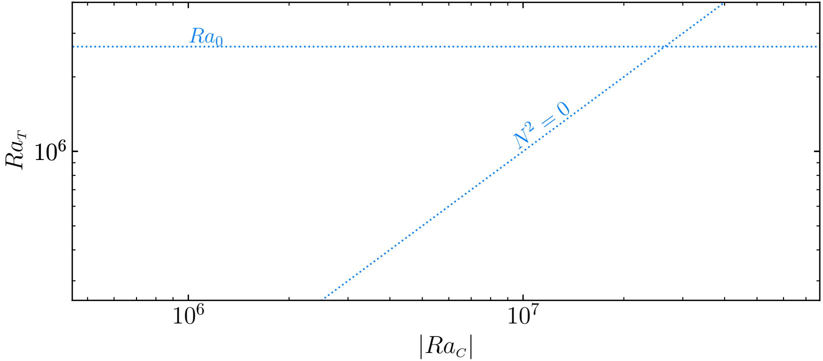
$$\frac{\partial\xi}{\partial t} + \boldsymbol{u}\cdot\nabla\xi = \frac{|Ra_C|}{Sc}\left(\frac{\gamma}{1-\gamma}\right)^2\frac{u_r}{r^2} + \frac{1}{Sc}\nabla^2\xi,$$

Dimensionless numbers:

$$\gamma = \frac{r_i}{r_o} = 0.6, \qquad Ek = \frac{\nu}{\Omega D^2} = 10^{-5}, \qquad Pr = \frac{\nu}{\kappa_T} = 0.3, \qquad Sc = \frac{\nu}{\kappa_C} = 3$$

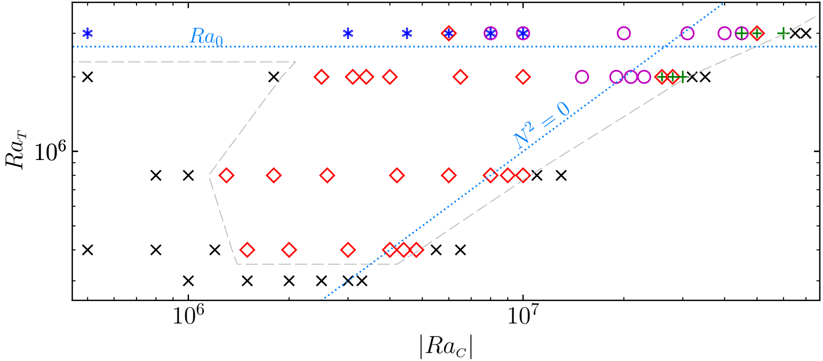
$$Ra_T = \frac{g_o\alpha_T q_i D^5}{r_o\nu\kappa_T} \quad \text{and} \quad Ra_C = -\frac{g_o\alpha_C f_i D^5}{r_o\nu\kappa_C}$$

ODDC at low Rayleigh numbers



- $N^2 = -\dfrac{Ra_T}{Pr}\dfrac{r_i^2}{r} + \dfrac{|Ra_C|}{Sc}\dfrac{r_i^2}{r}$
- $N^2 < 0$: top-heavy, $\quad N^2 > 0$: bottom-heavy
- Ra_0: critical Rayleigh number for pure thermal convection
- pure thermal convection: unstable when $N^2 < 0$ and $Ra_T > Ra_0$

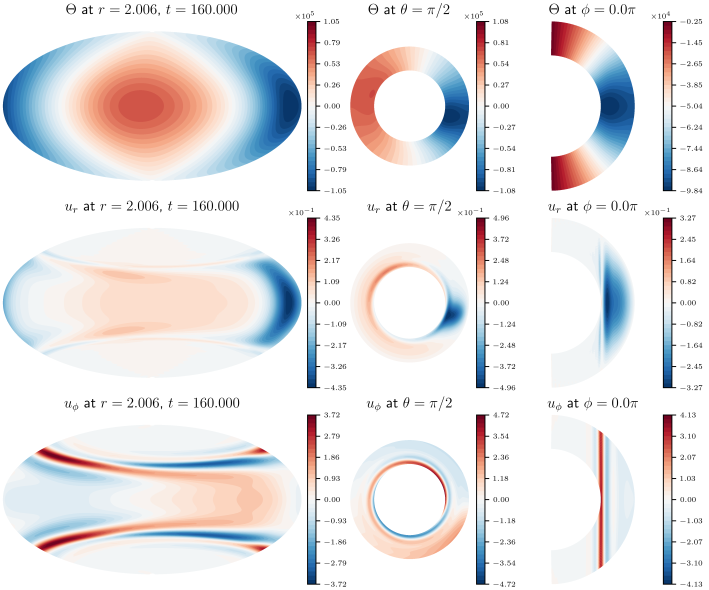
Phase diagram: $Ek = 10^{-5}$



- Expected features:
  - at small $|Ra_C|$, similar to pure thermal convection
  - at large $|Ra_C|$, compositional effects stabilise the system
- Counter-intuitive features (intermediate $|Ra_C|$):
  - the system can become unstable even when $N^2 > 0$ (bottom-heavy)
  - sustained motion is possible at some $Ra_T < Ra_0$

$Ra_T = 8 \times 10^5 \,(< Ra_0)\,,\ Ra_C = 1.8 \times 10^6$

- large-scale structures
- retrograde
- only exists at small Ek (rapid rotation)



$Ra_T = 3 \times 10^6 > Ra_0$

$\ast$ $Ra_C = 5 \times 10^5$

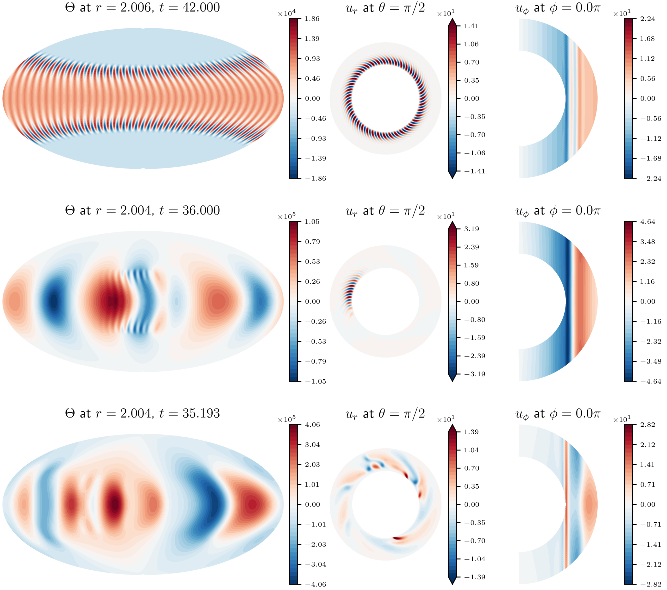
prograde

$\circledast$ $Ra_C = 8 \times 10^6$

prograde/retrograde

$\bigcirc$ $Ra_C = 2 \times 10^7$

retrograde



$Ra_T = 3 \times 10^6 > Ra_0$

pure thermal convection

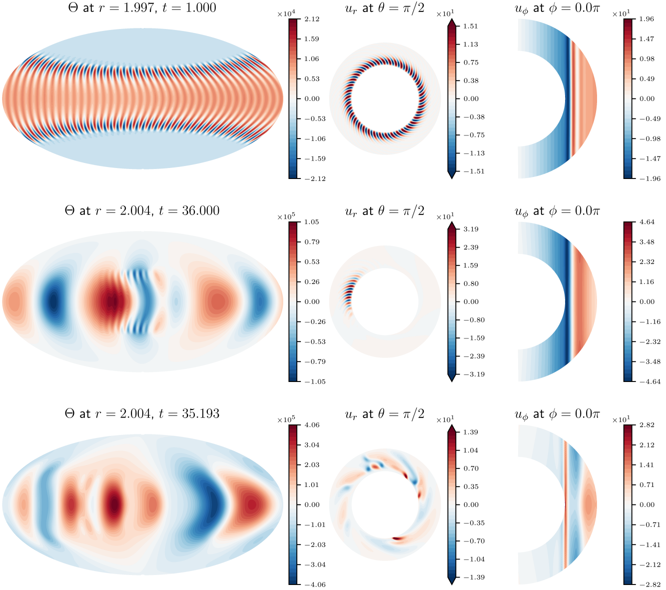
prograde

$\circledast$ $Ra_C = 8 \times 10^6$

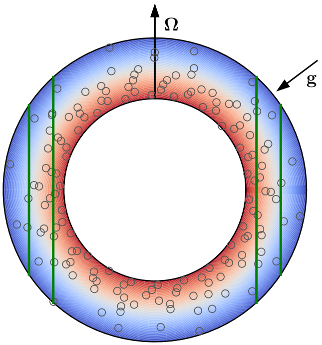
prograde/retrograde

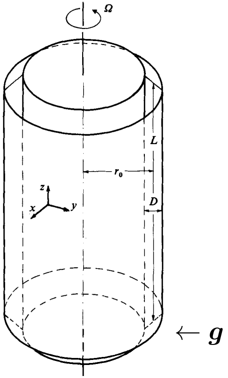
$\circ$ $Ra_C = 2 \times 10^7$

retrograde



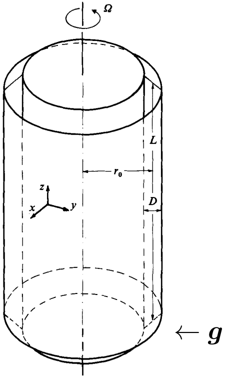
Thin cylindrical annulus model (Busse, 1986)

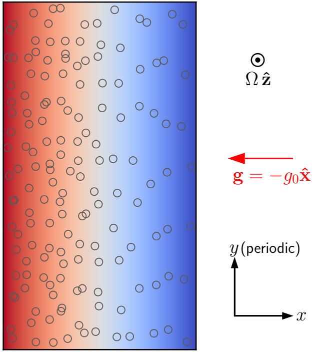




- cylinder annulus with top and bottom tilted at a constant angle χ
- captures two physics of the spherical shell
  1. rotation
  2. curvature of the spherical geometry

Thin cylindrical annulus model (Busse, 1986)





- rapid rotation: geostrophic balance at leading order (columnar structures)
- integration (average) over height $\Rightarrow$ two-dimensional system
- thin annulus $\Rightarrow$ Cartesian coordinate (x, y)

ODDC on a two-dimensional β-plane

$$(u, v) = \big(-\partial_y \psi, \partial_x \psi \big)$$

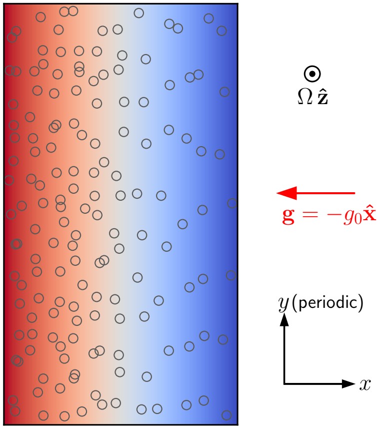
$$J(A, B) = \partial_x A \, \partial_y B - \partial_y A \, \partial_x B$$

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) - \beta \frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y} (\Theta - \xi) + \nabla^4 \psi$$

$$\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) = -\frac{Ra_T}{Pr} \frac{\partial \psi}{\partial y} + \frac{1}{Pr} \nabla^2 \Theta$$

$$\frac{\partial \xi}{\partial t} + J(\psi, \xi) = -\frac{|Ra_C|}{Sc} \frac{\partial \psi}{\partial y} + \frac{1}{Sc} \nabla^2 \xi$$

$$\beta = \frac{4D \, \tan \chi}{L \cdot Ek}$$



Linear stability analysis

Linearised equations:

$$\frac{\partial}{\partial t}\nabla^2\psi - \beta\frac{\partial\psi}{\partial y} = -\frac{\partial\Theta}{\partial y} + \frac{\partial\xi}{\partial y} + \nabla^4\psi$$

$$\frac{\partial\Theta}{\partial t} = -\frac{Ra_T}{Pr}\frac{\partial\psi}{\partial y} + \frac{1}{Pr}\nabla^2\Theta$$

$$\frac{\partial\xi}{\partial t} = -\frac{|Ra_C|}{Sc}\frac{\partial\psi}{\partial y} + \frac{1}{Sc}\nabla^2\xi$$

Eigenmodes:

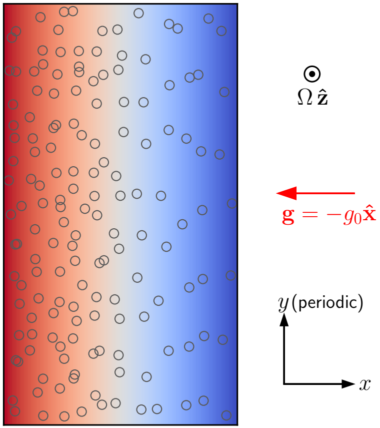
$$\psi(x,y,t) = \hat{\psi}\sin(kx)e^{ily}e^{\lambda t}$$

$$\Theta(x,y,t) = \hat{\Theta}\cos(kx)e^{ily}e^{\lambda t}$$

$$\xi(x,y,t) = \hat{\xi}\cos(kx)e^{ily}e^{\lambda t}$$

Complex growth rate:

$$\lambda = \sigma + i\omega \quad (\sigma,\omega\in\mathbb{R})$$



Maximum growth rate

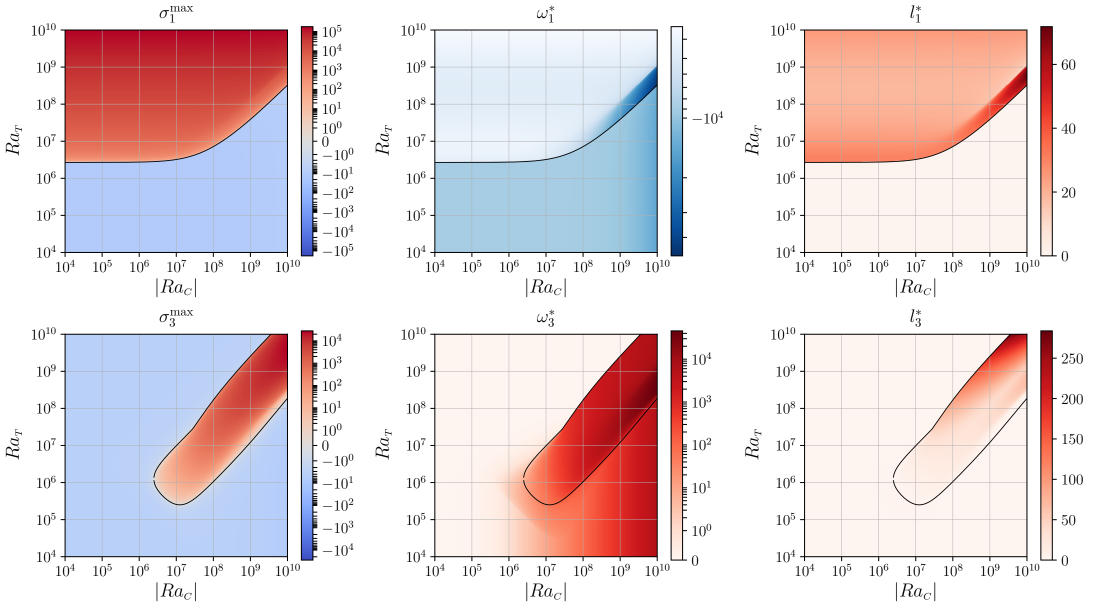
Solvability condition:

$$\lambda^3 + \left(\frac{Pr+1+\tau}{Pr}k_\mathrm{h}^2 + i\frac{\beta l}{k_\mathrm{h}^2}\right)\lambda^2 + \left[\frac{Pr(1+\tau)+\tau}{Pr^2}k_\mathrm{h}^4 + \frac{\tau|Ra_C| - Ra_T}{Pr}\frac{l^2}{k_\mathrm{h}^2} + i\frac{\beta(1+\tau)}{Pr}l\right]\lambda$$
$$+\frac{\tau}{Pr^2}\left[k_\mathrm{h}^6 + (|Ra_C| - Ra_T)l^2 + i\beta k_\mathrm{h}^2 l\right] = 0.$$

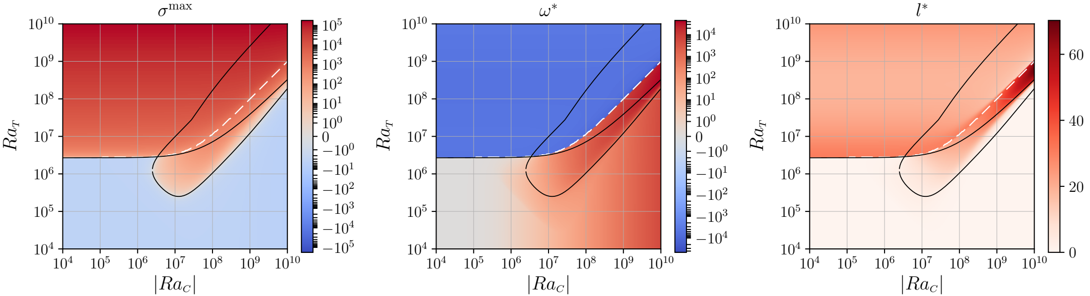
$k = m\pi, \quad k_\mathrm{h}^2 \equiv k^2 + l^2, \quad m, n \in \mathbb{Z}\,; \quad \tau = Pr/Sc = 0.1, \quad \beta = 1.78 \times 10^5$

- three roots: $\lambda_q = \sigma_q + i\omega_q, \quad q = 1, 2, 3$
- for each $(k, l), Ra_T, Ra_C :$ calculate $\lambda_q(k, l; Ra_T, Ra_C)$, $q = 1, 2, 3$
- $\sigma_q^{\max}(Ra_C, Ra_T) = \max\limits_{k,l}\left\{\sigma_q(k, l\,; Ra_C, Ra_T)\right\}$
- $\sigma^{\max}(Ra_C, Ra_T) = \max_q \left\{\sigma_q^{\max}\right\}$
- There is always a decaying solution: $\sigma_2 < 0$

Results: $\sigma_1^{\max}$ and $\sigma_3^{\max}$



Understanding the nonlinear spherical results using $\sigma^{\max}$



- $\lambda_1 = \sigma_1 + i\omega_1$: short-wavelength, prograde mode
  - unstable when $Ra_T > Ra_0$
  - $\sim$ pure thermal convection (modified by compositional effects)
- $\lambda_3 = \sigma_3 + i\omega_3$: long-wavelength, retrograde mode
  - can exists at some $Ra_T < Ra_0$ (strong rotation needed)
  - a “genuine” double-diffusive effect
- there is a range of $|Ra_C|$ in which the two modes can coexists with minimal interaction