An energy-enstrophy method for global stability in two-dimensional hydrodynamics

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Kolmogorov Flows

$$\zeta_t + u\zeta_x + v\zeta_y + \beta\psi_x = -\mu\zeta + \cos x \; + \; \nu\nabla^2\zeta$$

velocity: $(u, v) = (-\psi_y, \psi_x)$ *(2-D periodic domain)*

vorticity: $\zeta(x, y) = v_x - u_y = \nabla^2\psi$

- *single-scaled* sinusoidal body force (at $k_f = 1$)

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- *2D flows*: dual conservation of energy E and enstrophy Z

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  - E is transferred up ($k < 1$) and down ($k > 1$) scale
  - large-scale dissipation to halt the inverse E cascade

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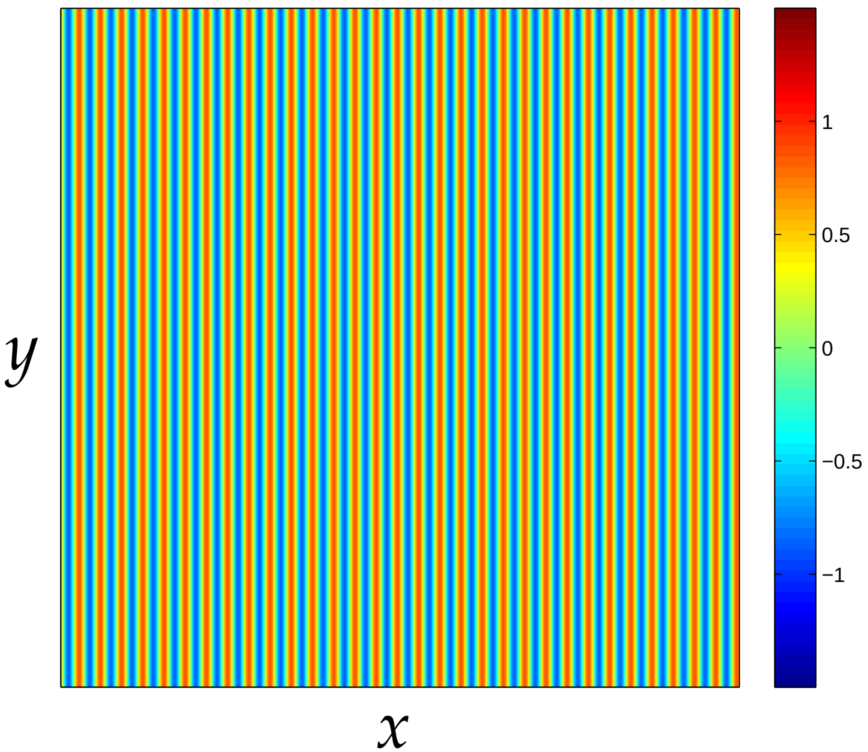
We shall first consider the inviscid case: $\nu = 0$

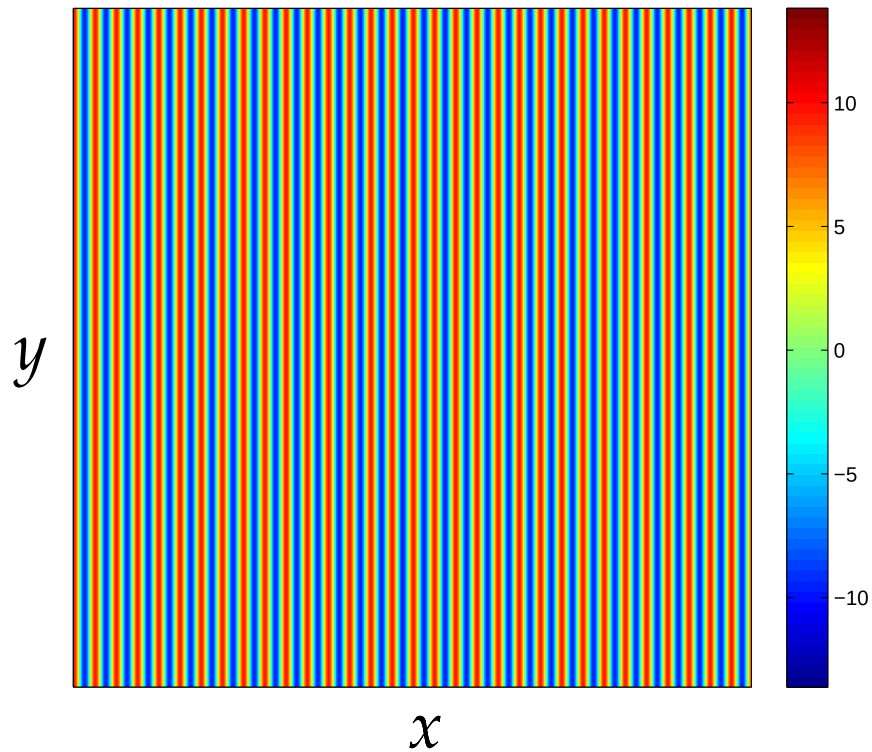
An Inviscid Laminar Solution

$$\zeta_{\mathrm{L}}(x) = a \cos(x - x_\beta)$$

$$a = \frac{1}{\sqrt{\beta^2 + \mu^2}} \quad , \quad x_\beta = \tan^{-1} \frac{\beta}{\mu}$$



$\mu = 0.5 \quad \beta = 1.0$



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Stability of the Laminar Solution

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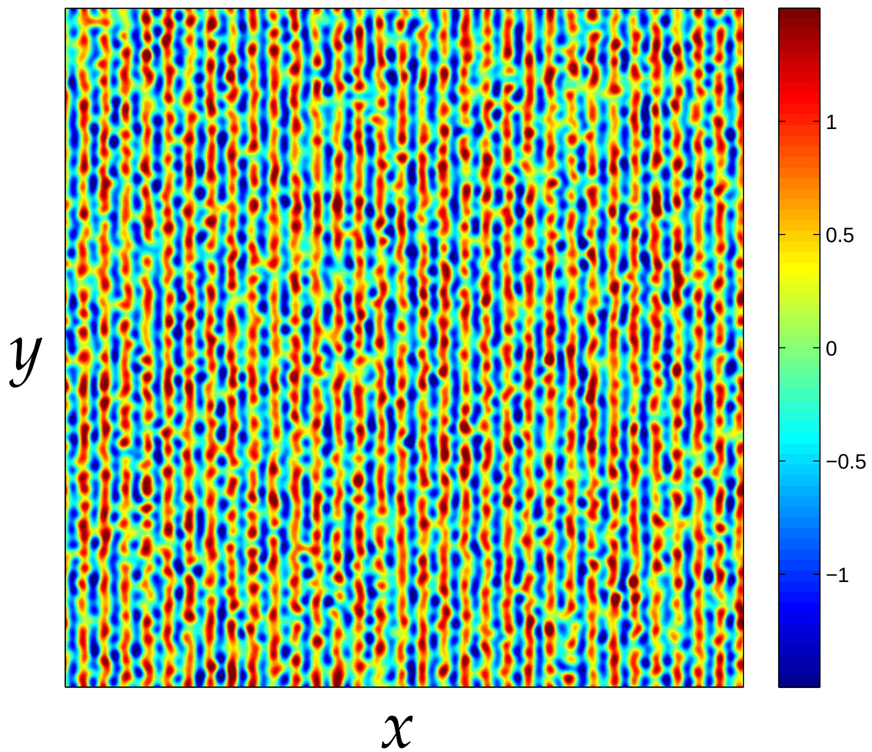

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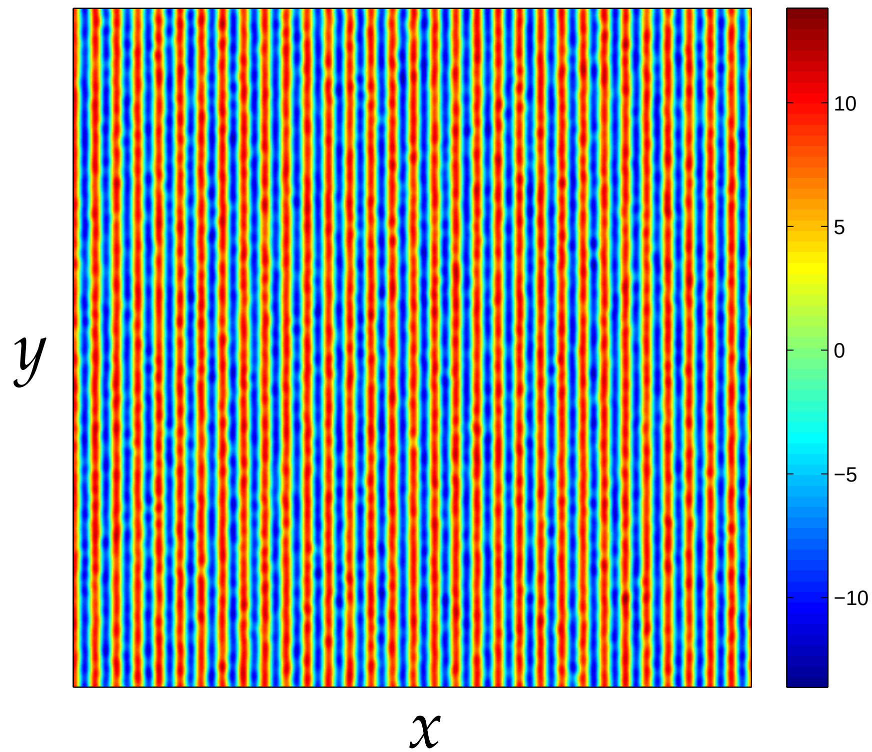

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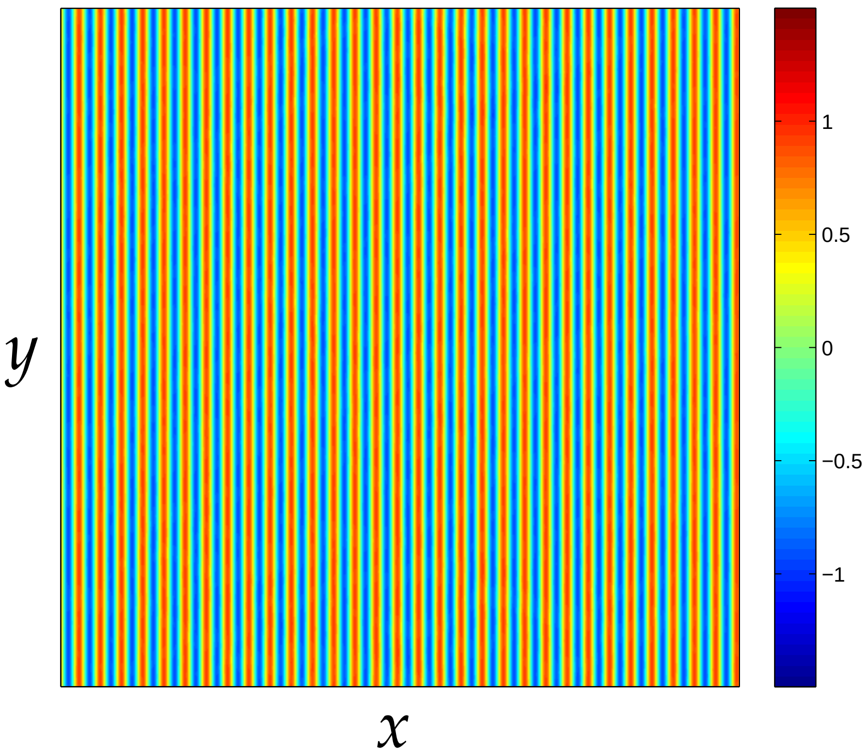

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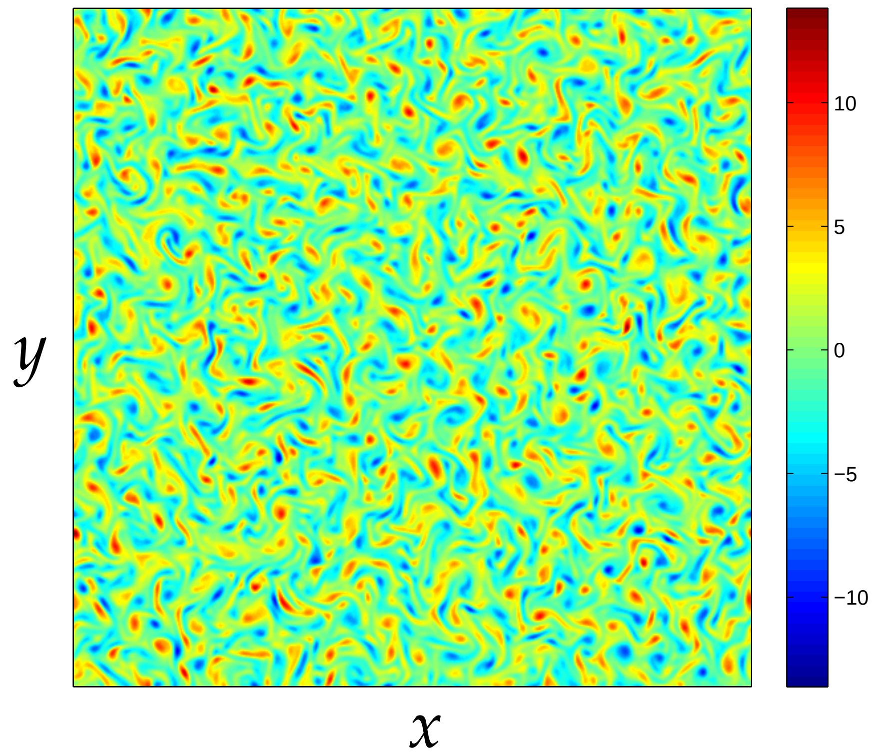
stable



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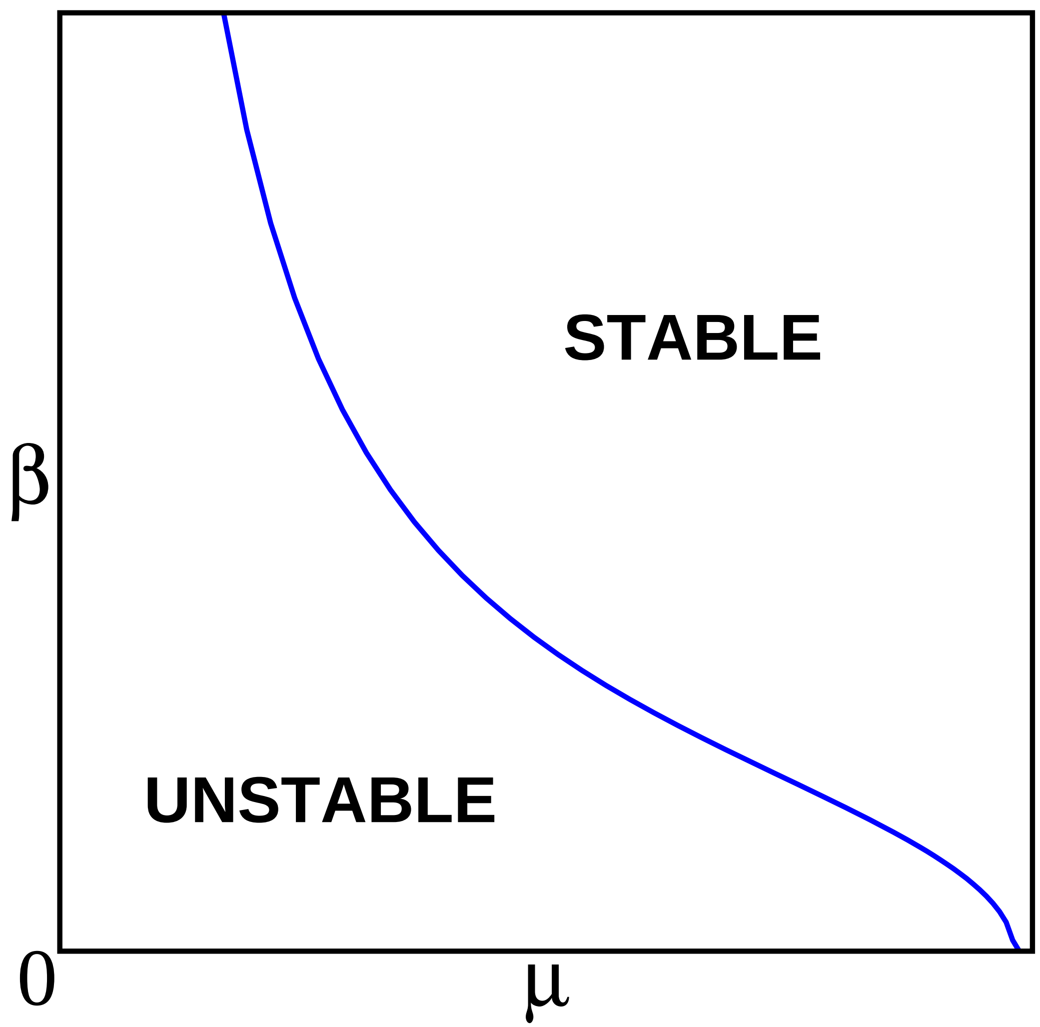
unstable

Goal: Neutral Curve

$$\zeta_t + u\zeta_x + v\zeta_y + \beta\psi_x = -\mu\zeta + \cos x$$



Stability Analysis

$$\psi(x,y,t) = \psi_{\mathrm{L}}(x) + \varphi(x,y,t)$$

- Linear Instability
  - assume infinitesimal disturbance $\varphi \sim e^{-i\omega t}$
  - $\Im\{\omega\} > 0 \;\Rightarrow\; \psi_{\mathrm{L}}$ is unstable
  - gives sufficient condition for instability
- Global Stability (Asymptotic Stability)
  - φ is *not* assumed to be small
  - **disturbance energy**

$$E_{\varphi}(t) = \frac{1}{2}\left\langle |\nabla\varphi|^2 \right\rangle \to 0 \quad as \quad t \to \infty$$

  - gives sufficient condition for stability

Energy Method

$$\nabla^2\psi_t + J(\psi, \nabla^2\psi) + \beta\psi_x = -\mu\nabla^2\psi + \cos x$$

$$\psi_{\mathsf{L}}(x) = -a\cos(x - x_\beta)$$

Time evolution equation for φ $(\psi = \psi_{\mathsf{L}} + \varphi)$:

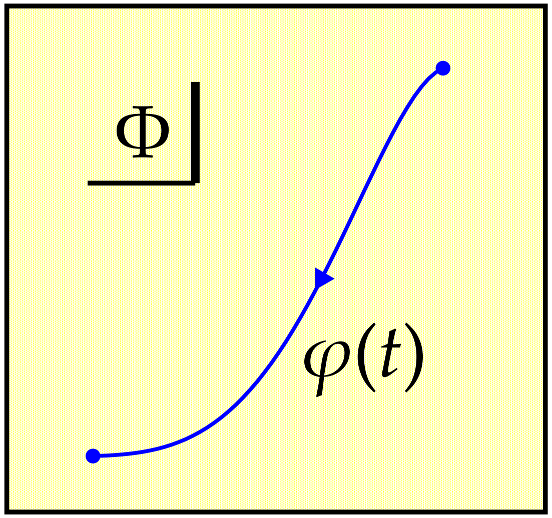
$$\nabla^2\varphi_t + J(\psi_{\mathsf{L}}, \nabla^2\varphi) + J(\varphi, \nabla^2\psi_{\mathsf{L}}) + J(\varphi, \nabla^2\varphi) + \beta\varphi_x = -\mu\nabla^2\varphi$$

$$\therefore \qquad \frac{dE_\varphi}{dt} = \left\langle \varphi J(\psi_{\mathsf{L}}, \nabla^2\varphi) \right\rangle - \mu \left\langle |\nabla\varphi|^2 \right\rangle$$

$$= a\left\langle \varphi_x\varphi_y \cos x \right\rangle - 2\mu E_\varphi$$

Energy Method

$$\frac{dE_\varphi}{dt} = 2\left(a\mathcal{R}[\varphi] - \mu\right)E_\varphi$$



where $$\mathcal{R}[\varphi] \equiv \frac{\langle \varphi_x \varphi_y \cos x \rangle}{\langle |\nabla\varphi|^2 \rangle}$$

Now define $$\mathcal{R}_* \equiv \max_{\varphi \in \Phi} \mathcal{R}[\varphi]$$

Φ : *set of all functions satisfying periodic boundary conditions*

Then,

$$\frac{dE_\varphi}{dt} < 2\left(a\mathcal{R}_* - \mu\right)E_\varphi$$

$$\Rightarrow \quad E_\varphi(t) < E_\varphi(0)\, e^{2(a\mathcal{R}_* - \mu)t}$$ (Gronwall's inequality)

Energy Method

$$E_\varphi(t) < E_\varphi(0)\, e^{2(a\mathcal{R}_* - \mu)t}$$

$$\therefore \quad E_\varphi(t \to \infty) \to 0 \quad \textit{if} \quad a\mathcal{R}_* - \mu < 0$$

Neutral condition

$$a = \frac{1}{\mathcal{R}_*}\mu \quad \Rightarrow \quad \boxed{\beta = \sqrt{\frac{\mathcal{R}_*^2}{\mu^2} - \mu^2}}$$

Optimal solution

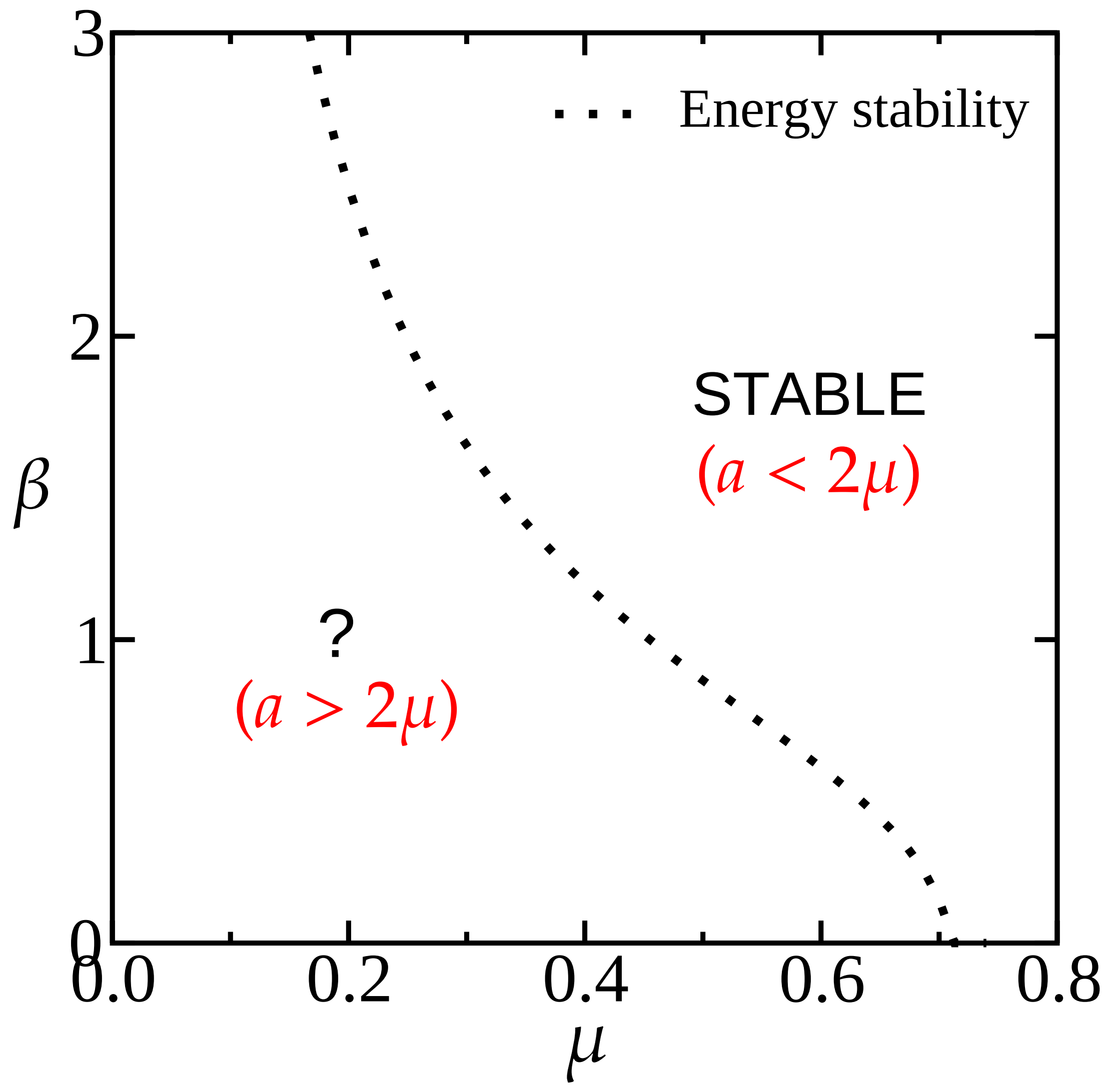
Maximize: $\mathcal{R}[\varphi] \equiv \dfrac{\left\langle \varphi_x \varphi_y \cos x \right\rangle}{\langle |\nabla\varphi|^2 \rangle}$ over the set Φ.

$$\mathcal{R}_* = \mathcal{R}[\varphi_*] = \frac{1}{2}$$

$$\varphi_*(x,y) = \mathfrak{R}\left\{ e^{i l y}\, \tilde{\varphi}(x) \right\} \quad \textit{with} \quad l \to \infty$$

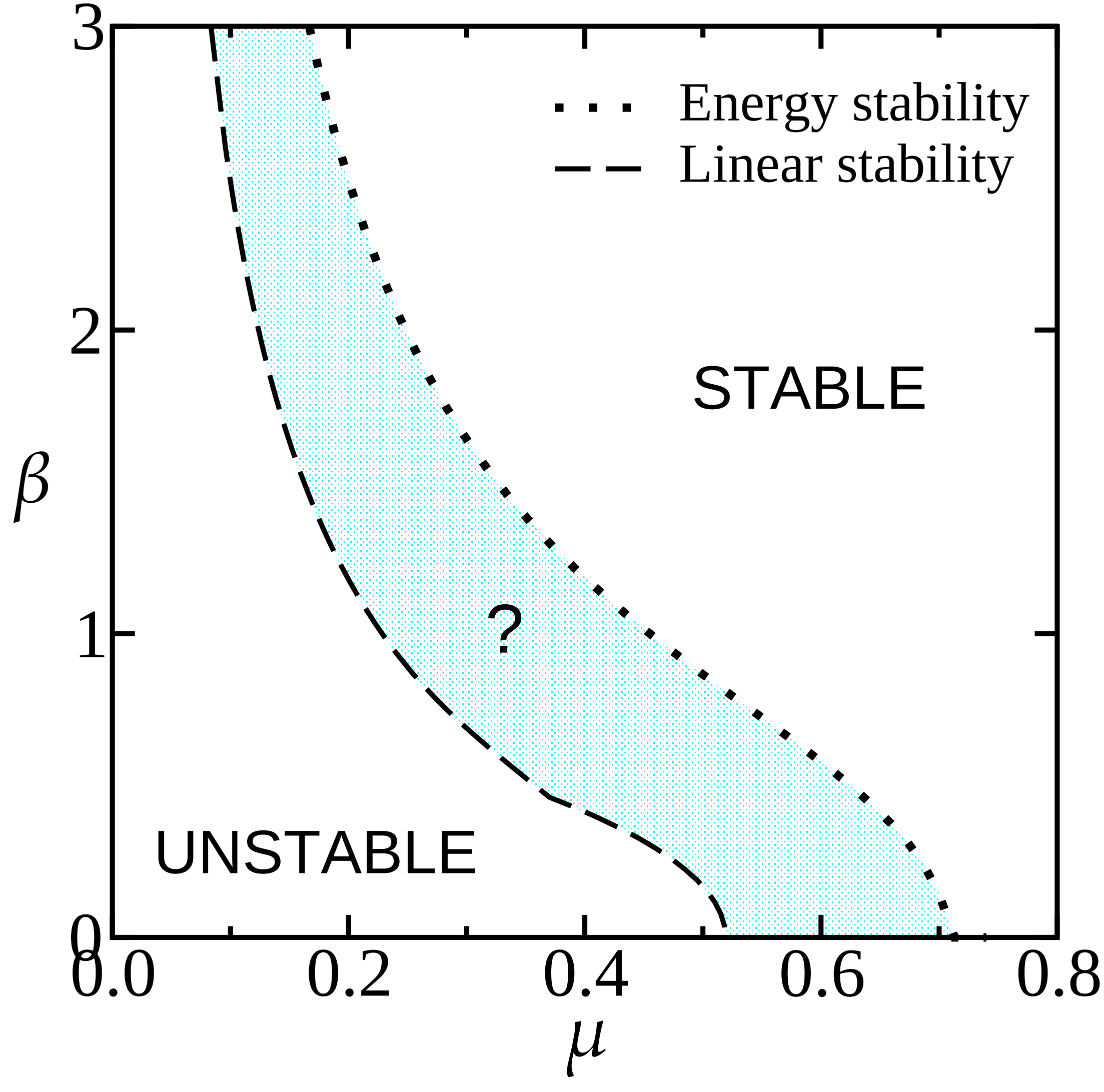
Energy Stability Curve

$$\beta = \sqrt{\frac{1}{4\mu^2} - \mu^2} \qquad (a = 2\mu)$$



Energy Stability and Linear Stability Curve

$$\beta = \sqrt{\frac{1}{4\mu^2} - \mu^2} \qquad (a = 2\mu)$$



Energy-Enstrophy Balance

Limitations of the energy method: requires $E_\varphi(t)$ to decrease monotonically for all φ, thus excludes transient growth of $E_\varphi(t)$

Disturbance enstrophy: $Z_\varphi = \frac{1}{2}\langle(\nabla^2\varphi)^2\rangle$

$$\frac{dZ_\varphi}{dt} = a\left\langle \varphi_x \varphi_y \cos x \right\rangle - 2\mu Z_\varphi$$

Recall,

$$\frac{dE_\varphi}{dt} = a\left\langle \varphi_x \varphi_y \cos x \right\rangle - 2\mu E_\varphi$$

Then,

$$\frac{d}{dt}(E_\varphi - Z_\varphi) = -2\mu(E_\varphi - Z_\varphi)$$

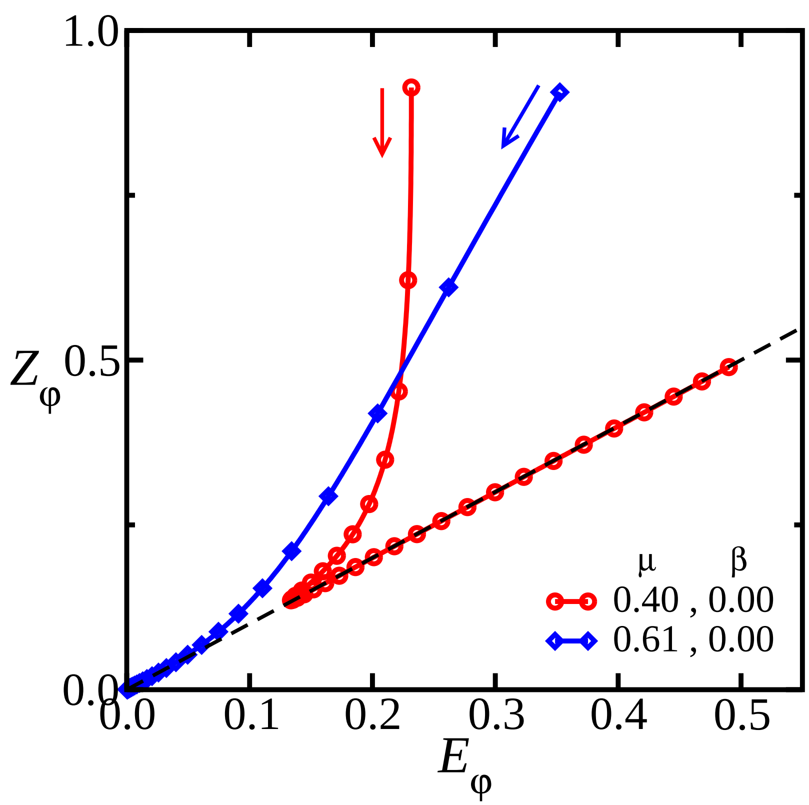
$$\boxed{E_\varphi(t) - Z_\varphi(t) = e^{-2\mu t}\left[E_\varphi(0) - Z_\varphi(0)\right]}$$

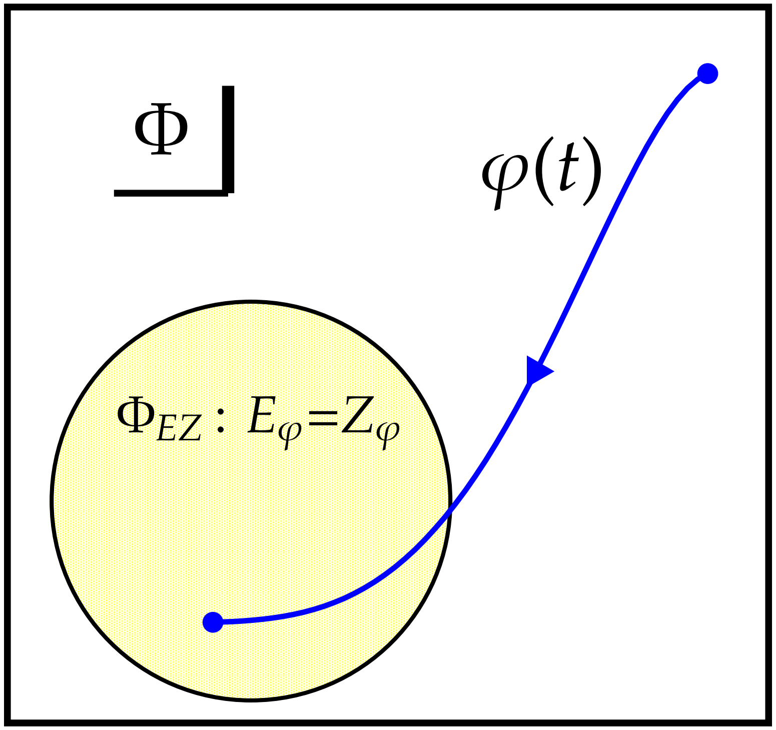
Energy-Enstrophy Balance

$$E_\varphi(t) - Z_\varphi(t) \to 0 \quad as \quad t \to \infty$$

$$\Phi_{EZ} = \{\varphi \in \Phi \text{ such that } E_\varphi = Z_\varphi\}$$

$$\Rightarrow \Phi_{EZ} \text{ attracts all initial conditions}$$





Optimization with Constraints

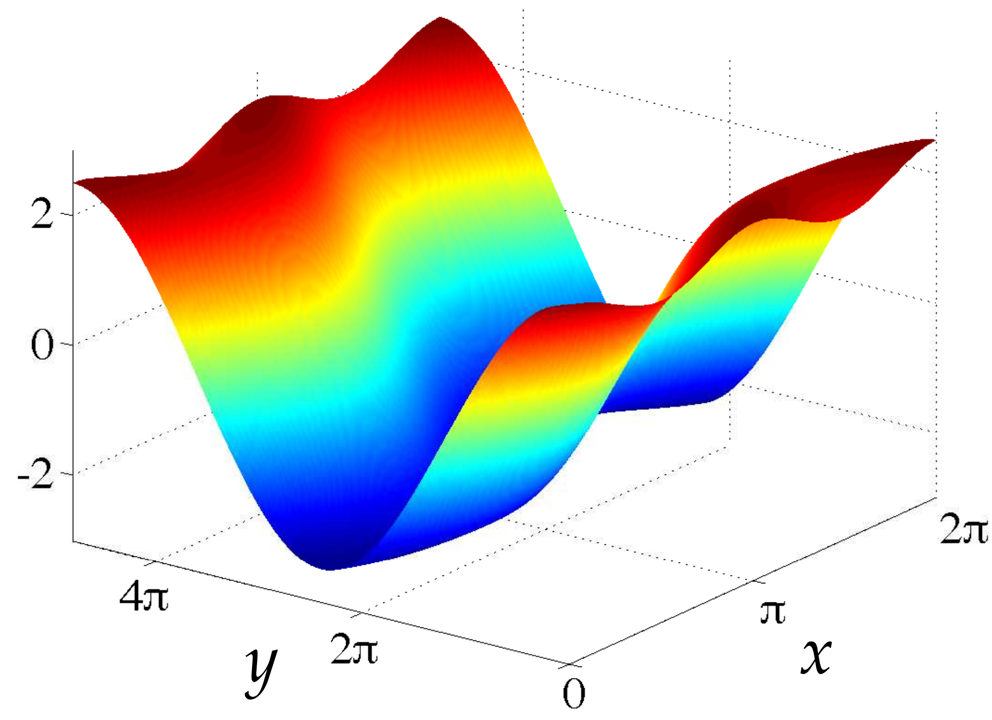
Maximize: $$\mathcal{R}[\varphi] \equiv \frac{\langle \varphi_x \varphi_y \cos x \rangle}{\langle |\nabla\varphi|^2 \rangle}$$

with constraint $$\langle |\nabla\varphi|^2 \rangle = \langle (\nabla^2\varphi)^2 \rangle$$

Optimal solution

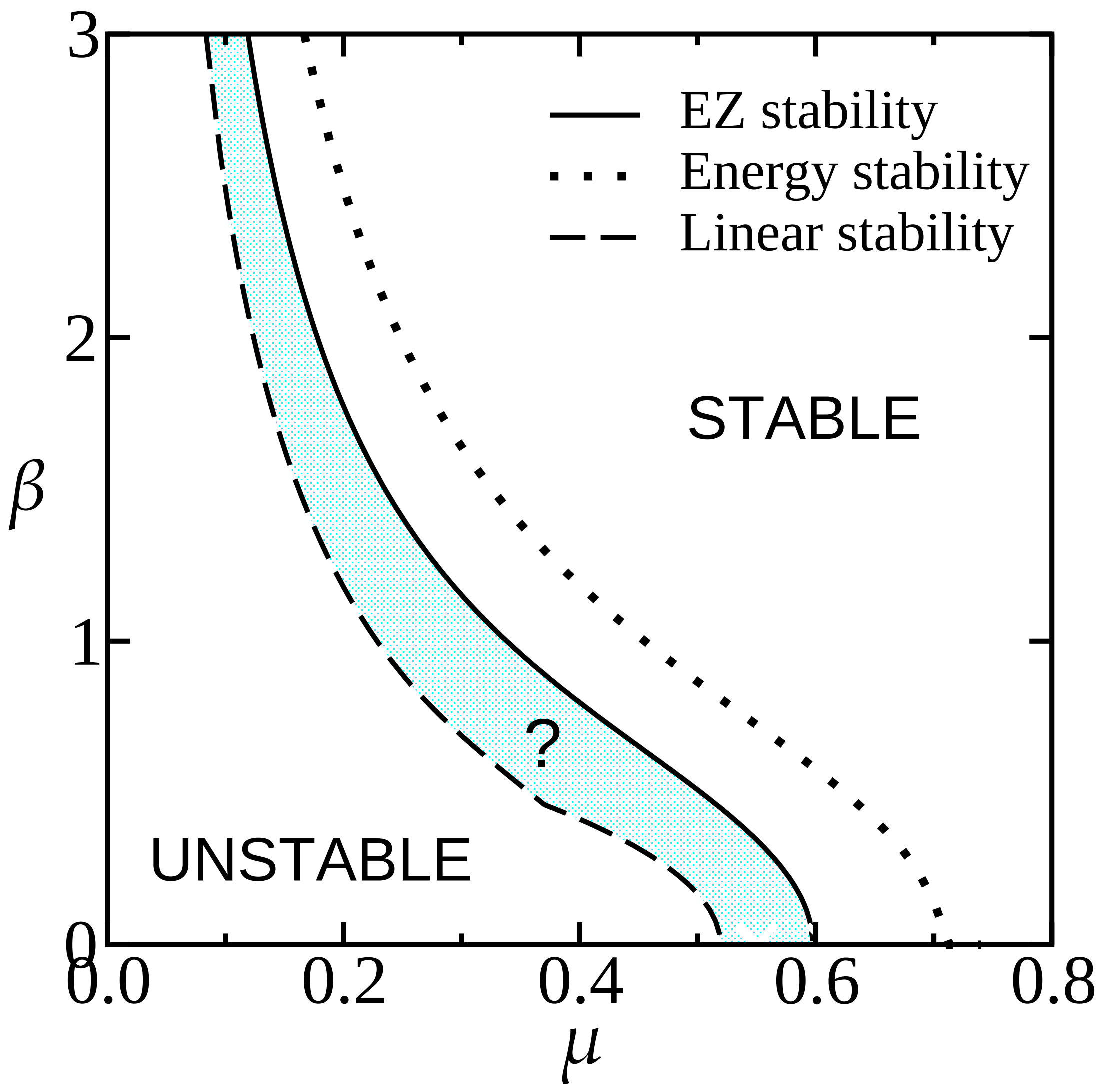
$$\mathcal{R}_* = \mathcal{R}[\varphi_*] = 0.3571$$

$$\varphi_*(x,y) = \mathfrak{R}\left\{e^{\mathrm{i}\,l\,y}\,\tilde{\varphi}(x)\right\} \quad \textit{with} \quad l \approx 0.4166$$



Energy-Enstrophy (EZ) Stability $(\nu = 0)$

$$\beta = \sqrt{\frac{0.13}{\mu^2} - \mu^2} \qquad (a = 2.8\mu)$$



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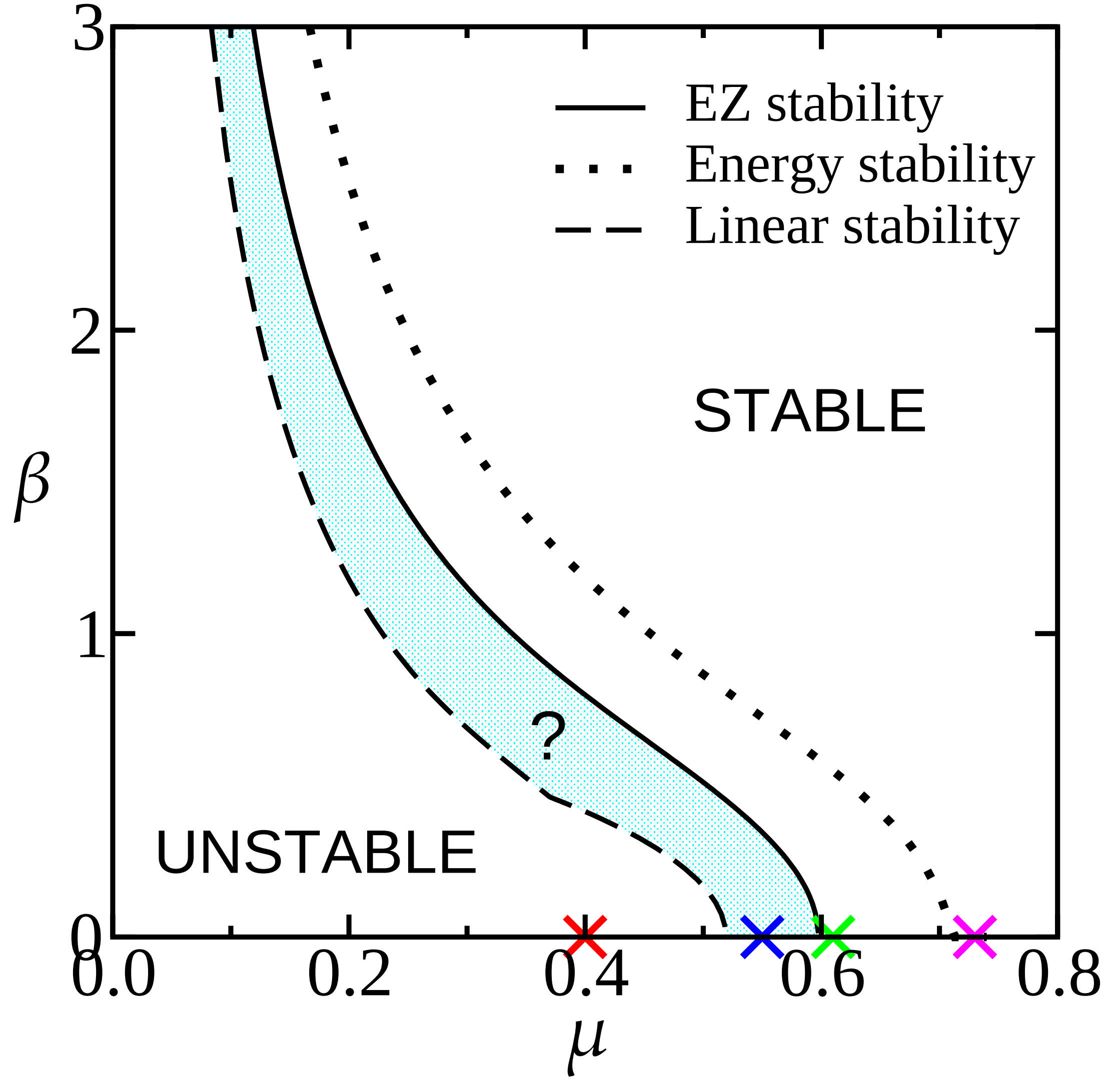

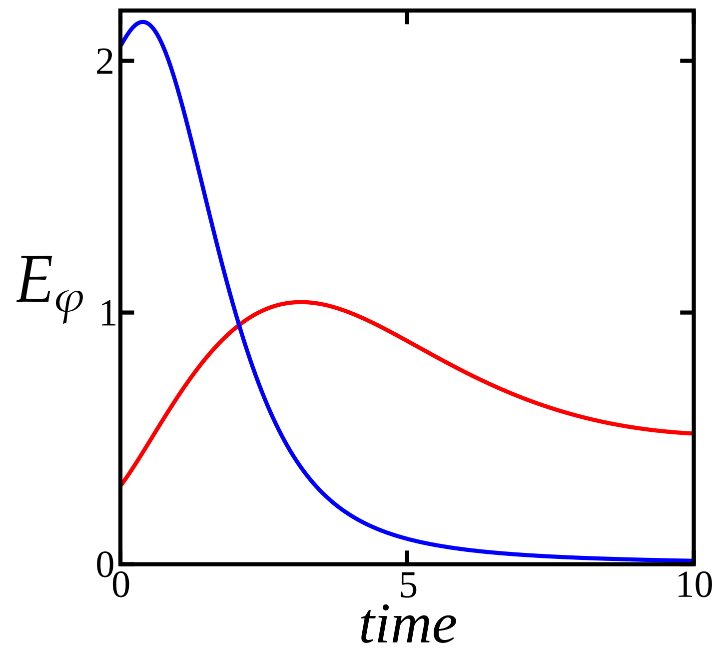

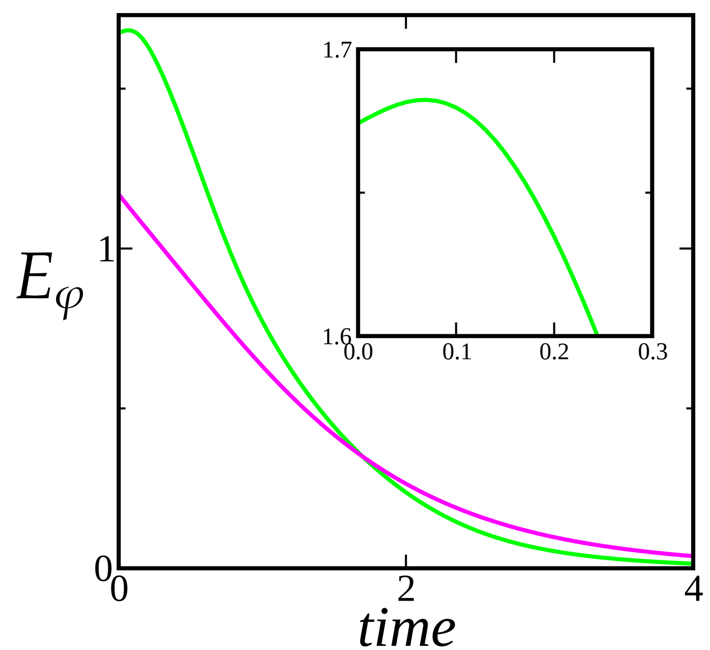

Limitation of the EZ method

The EZ method applies only for

- inviscid flow $\nu = 0$, and
- basic state consists of a single Helmholtz mode:

$$\nabla^2 \psi_L + k_f^2 \psi_L = 0$$

Evolution of perturbation energy and enstrophy:

$$\frac{dE_\varphi}{dt} = -\left\langle \psi_L \, J(\varphi, \nabla^2 \varphi) \right\rangle - 2\mu E_\varphi + 2\nu Z_\varphi$$

$$\frac{dZ_\varphi}{dt} = \left\langle \zeta_L \, J(\varphi, \nabla^2 \varphi) \right\rangle - 2\mu Z_\varphi + 2\nu P_\varphi$$

$$P_\varphi = \frac{1}{2} \left\langle |\nabla(\nabla^2 \varphi)|^2 \right\rangle$$

Extended Energy-Enstrophy (EEZ) Stability

Consider a family of norm with the parameter α:

$$Q(\alpha) = (1 - \alpha)\, E_\varphi + \alpha\, Z_\varphi\ , \quad 0 \leqslant \alpha \leqslant 1$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 2\left\{ \mathcal{R}_Q[\varphi; \alpha, \nu, a] - \mu \right\} Q$$

Global stability : $Q(\alpha) \to 0$ as $t \to \infty$ for some α .

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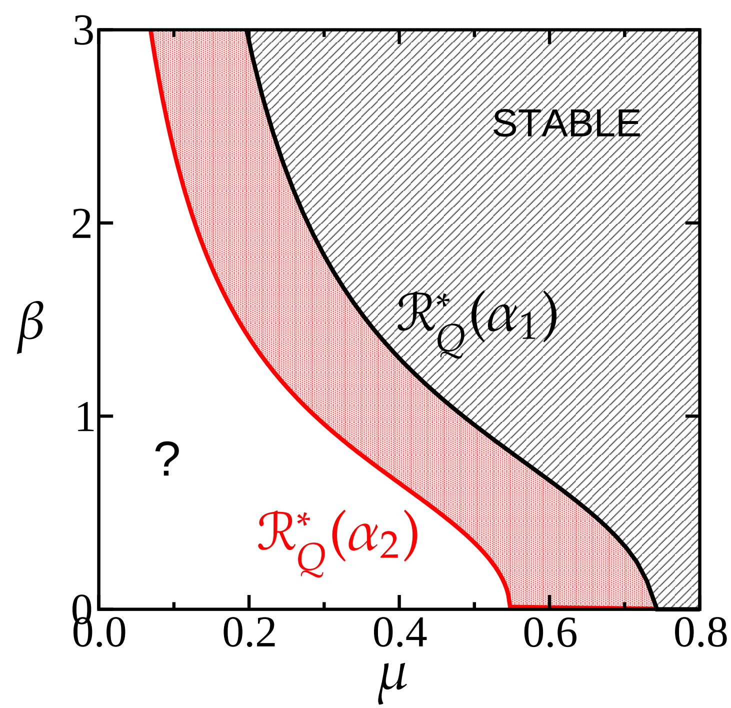
$$\mathcal{R}^*_Q(\alpha,\nu,a) = \max_{\varphi\in\Phi} \mathcal{R}_Q[\varphi;\alpha,\nu,a]$$

For each α, neutral condition : $\mathcal{R}^*_Q(\alpha,\nu,a) = \mu$



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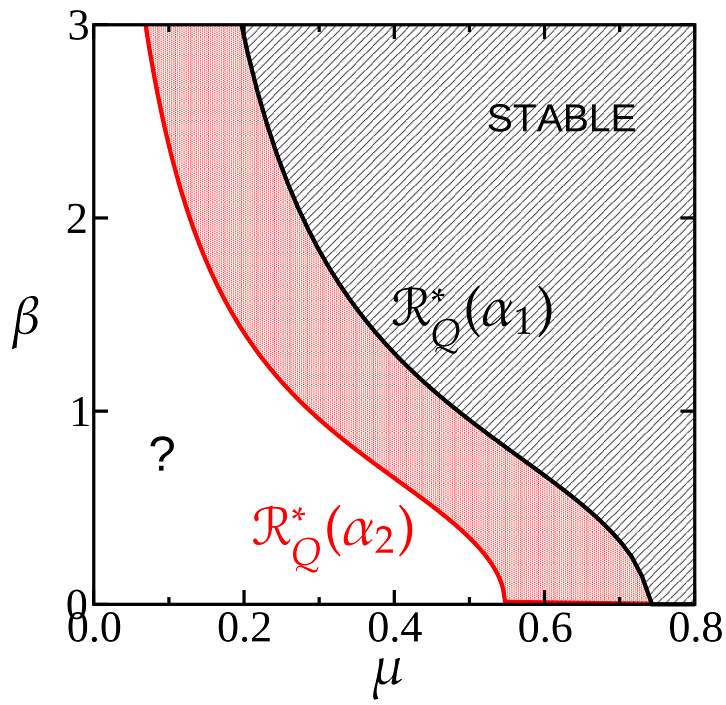
"Optimal" neutral condition : $\min_{\alpha} \mathcal{R}^*_Q(\alpha,\nu,a) = \mu$



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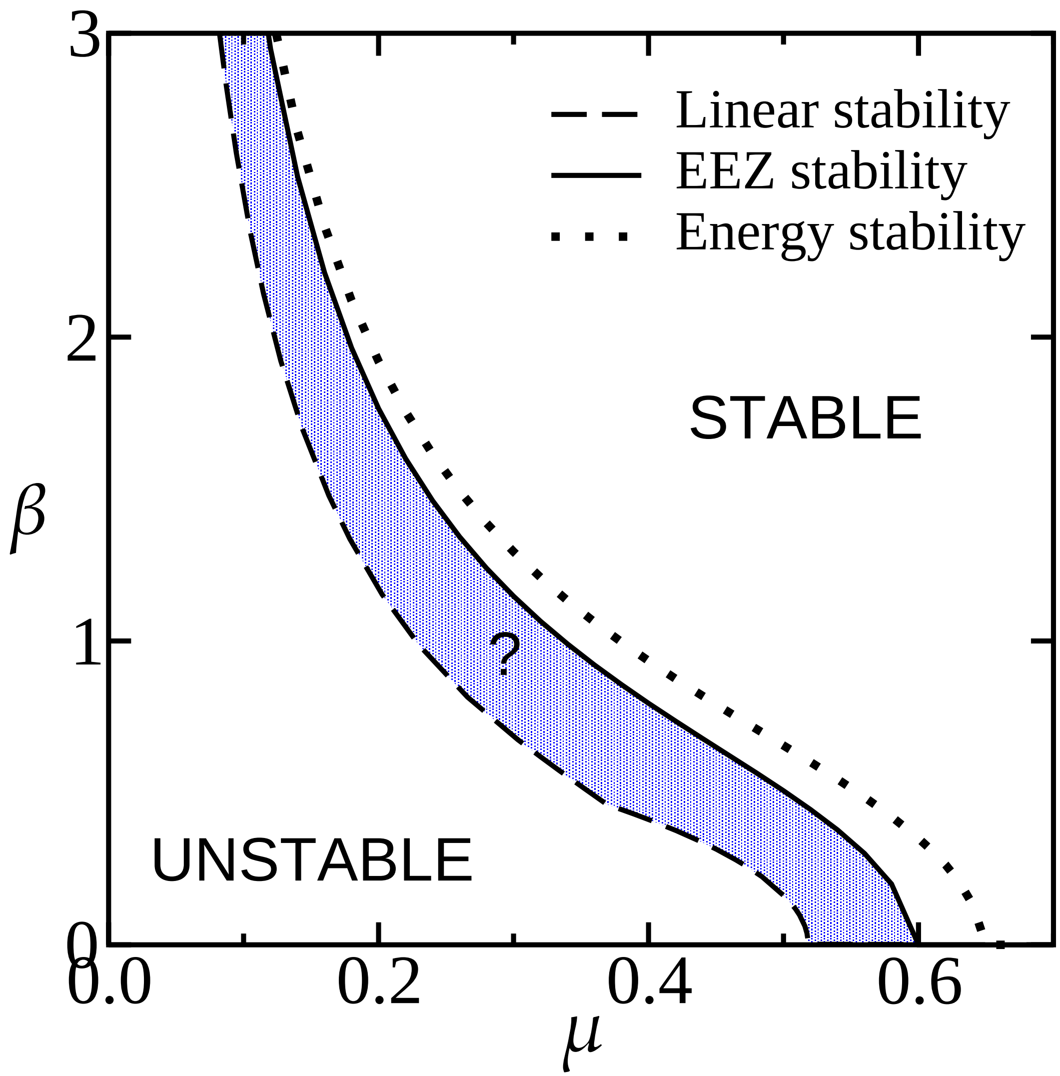
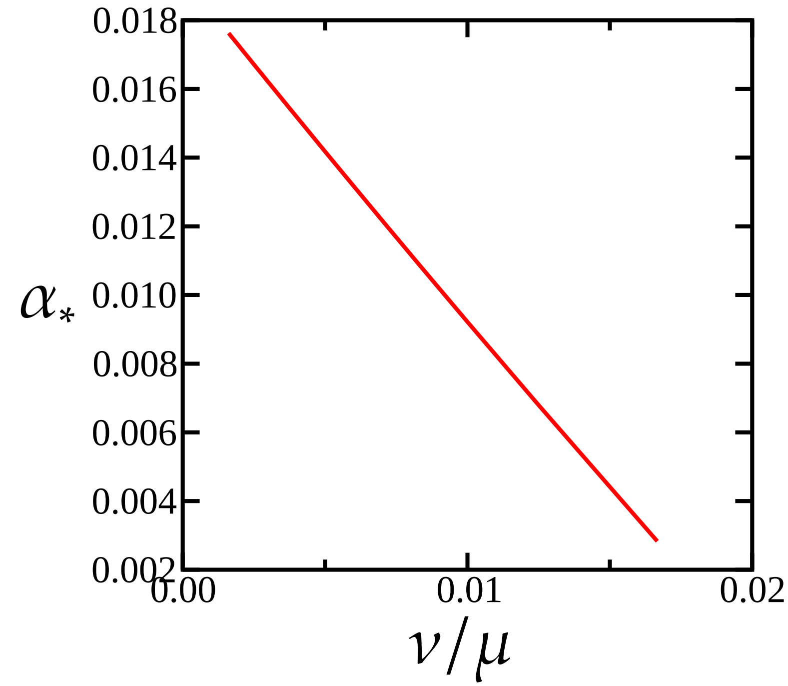
Extended EZ Stability ($\nu = 10^{-3}$)

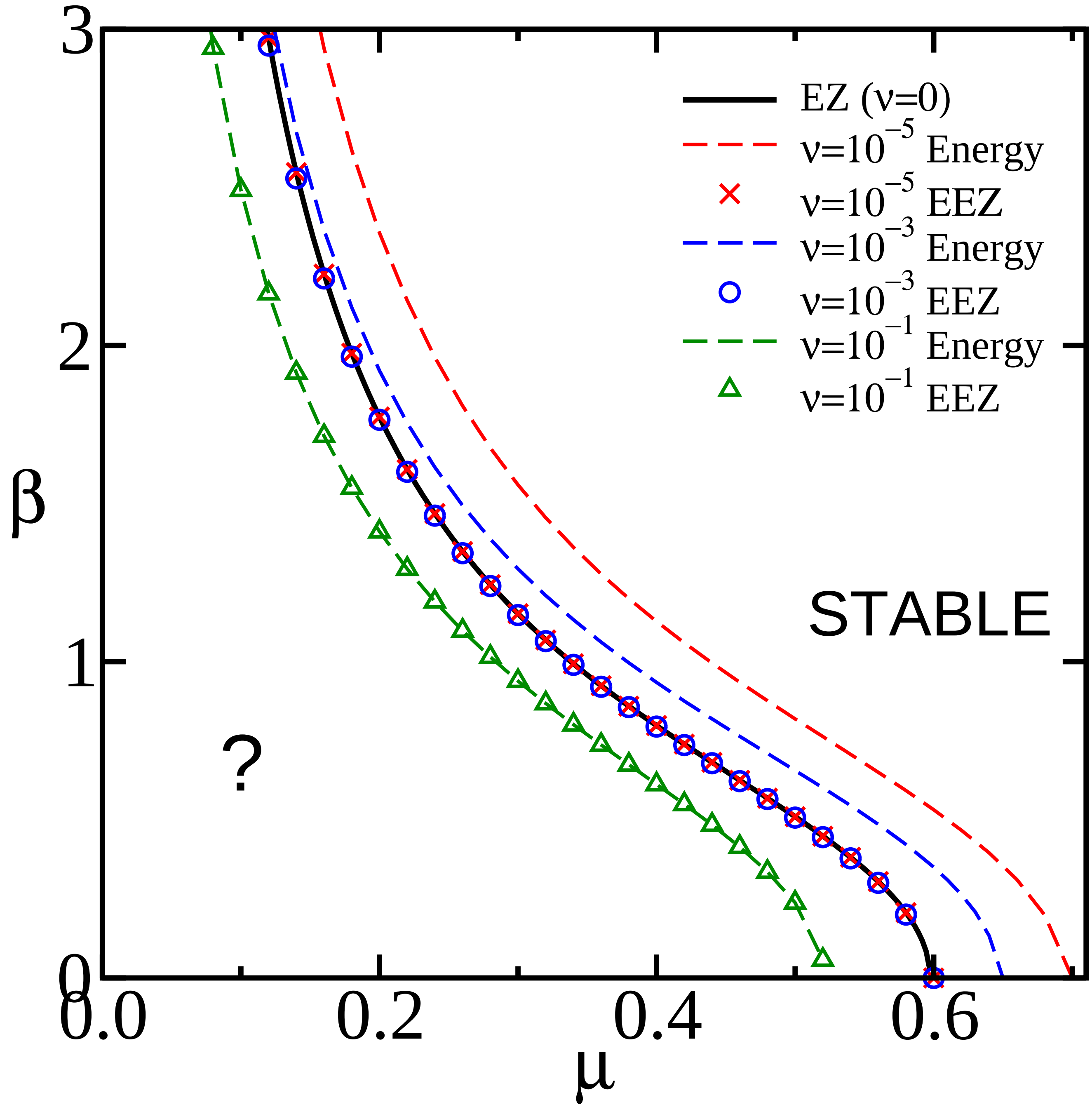
$$\nabla^2\psi_t + J(\psi, \nabla^2\psi) + \beta\psi_x = -\mu\nabla^2\psi + \cos x \; + \; \nu\nabla^2\zeta$$

$$Q(\alpha) = (1-\alpha)\,E_\varphi + \alpha\,Z_\varphi$$





Extended EZ Stability for different ν



Non-Kolmogorov flows

- multiple Helmholtz modes:

$$\psi_{\mathsf{L}} = -a\left(\cos x + \delta \sin mx\right)$$

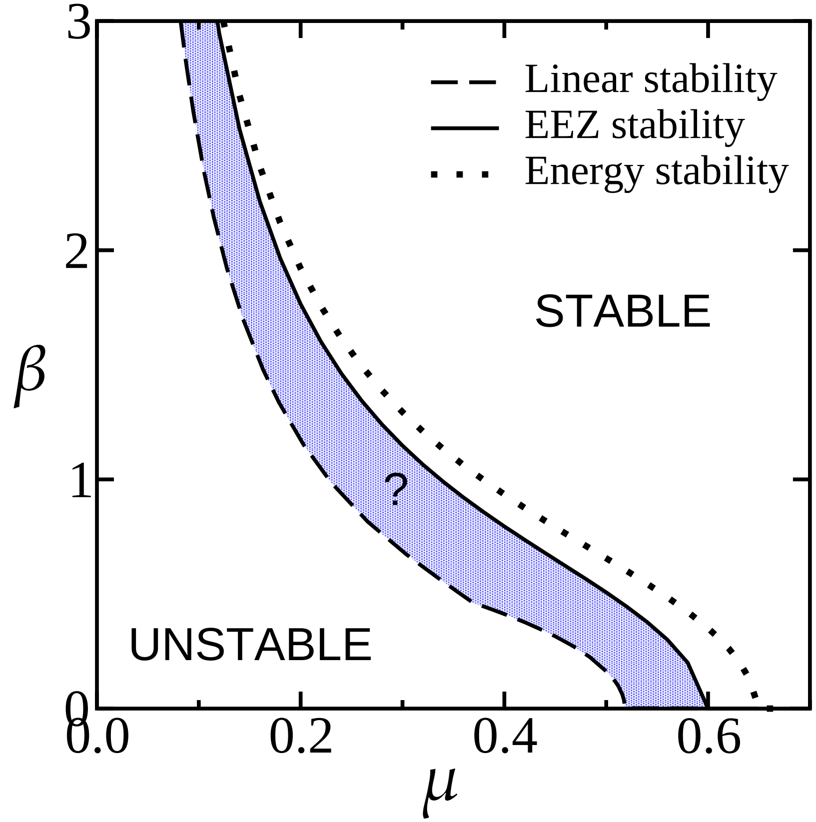
- a street of counter-rotating vortices, Mallier-Maslowe solution:

$$\psi_{\mathsf{L}} = \ln\left(\frac{\cosh \varepsilon y - \varepsilon \cos x}{\cosh \varepsilon y + \varepsilon \cos x}\right)$$

Summary

By incorporating information based on the enstrophy, we develop the EZ and EEZ stability method which

- allows <u>transient</u> growth in $E_\varphi(t)$ $\;(\varphi(t=0) \notin \Phi_{EZ})$
- identifies a physically <u>realistic</u> most-unstable disturbance
- lies <u>closer</u> to the linear stability neutral curve



EZ stability: Tsang & Young, Phys. Fluids **20**, 084102 (2008)