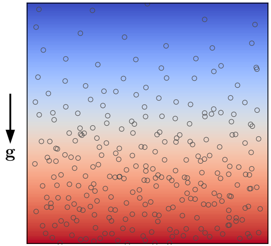
Double-diffusive Convection in a Model of Saturn's Stably Stratified Layer

Yue-Kin Tsang

School of Mathematics, Statistics and Physics
Newcastle University

Céline Guervilly, Graeme R. Sarson

Oscillatory double-diffusive convection (semi-convection)

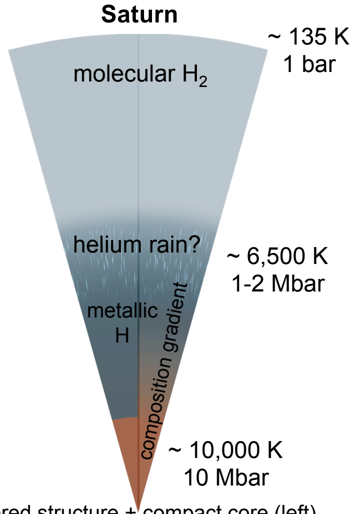


- Density variation comes from two different components of the fluid
- Consider a heavy element in the fluid, e.g. salt in seawater, He in H–He mixture
- Let C be the concentration of the heavy element (composition):

$$\rho(T,C) = \rho_m \left[1 - \alpha_T (T - T_m) + \alpha_C (C - C_m)\right], \quad \alpha_T, \alpha_C > 0$$

- What distinguishes T and C is their diffusivities: $\kappa_T \gg \kappa_C$
- Rayleigh numbers: thermal $Ra_T > 0$ (destabilizing), compositional $Ra_C < 0$ (stabilizing)

Possible scenarios of ODDC in Saturn



layered structure + compact core (left),
inhomogeneous structure + fuzzy core (right)

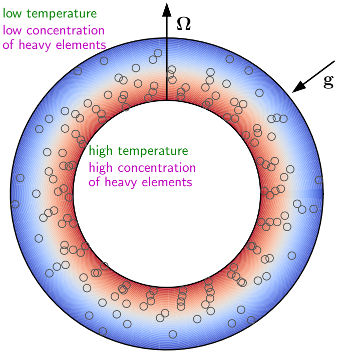
Stabilizing compositional gradient may arise inside Saturn in different ways:

- **Helium rain**: near the bottom of the molecular H_2 envelop, temperature and pressure are such that H and He become immiscible, the heavier He falls inward and accumulates above the He-enriched deep interior *(Salpeter 1973; Stevenson and Salpeter 1977)*
- **Dilute/fuzzy core**: recent observation of Saturn's rings by the Cassini mission reveals the trapping of internal gravity waves in the deep interior, suggesting an extended region with stabilizing compositional gradient created by heavy elements dissolving from an inner core at the center *(Mankovich and Fuller 2021)*

There are discussions about whether ODDC can exist in these layers.

Fortney et al. in "Saturn: The Grand Finale", K. H. Baines et al., eds., Cambridge University Press

Mathematical model: ODDC in a rotating spherical shell



Boundary condition

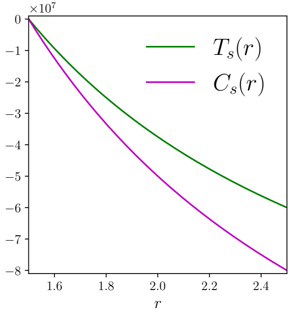
fixed T gradient:

$$\left.\frac{\partial T}{\partial r}\right|_{r_i}, \left.\frac{\partial T}{\partial r}\right|_{r_o} < 0$$

fixed C gradient:

$$\left.\frac{\partial C}{\partial r}\right|_{r_i}, \left.\frac{\partial C}{\partial r}\right|_{r_o} < 0$$

Equilibrium profiles (at $\boldsymbol{u} = 0$)



$\frac{\mathrm{d}T_s}{\mathrm{d}r} < 0$ *destabilising*

$\frac{\mathrm{d}C_s}{\mathrm{d}r} < 0$ *stabilising*

$$\rho(T_s, C_s) = \rho_m[1 - \alpha_T(T_s - T_m) + \alpha_C(C_s - C_m)]$$

Consider a Boussinesq fluid in a rotating spherical shell of inner radius r_i and outer radius r_o

- (equilibrium) buoyancy frequency: $N_0^2 = -\frac{g}{\rho_m}\frac{\mathrm{d}}{\mathrm{d}r}\rho(T_s, C_s) \implies N_0^2 = g\alpha_T\frac{\mathrm{d}T_s}{\mathrm{d}r} - g\alpha_C\frac{\mathrm{d}C_s}{\mathrm{d}r}$
- inverse density ratio: $R_\rho^{-1} = \frac{\alpha_C|\mathrm{d}C_s/\mathrm{d}r|}{\alpha_T|\mathrm{d}T_s/\mathrm{d}r|}$

Governing equations

Non-dimensional equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{2}{Ek}\hat{\boldsymbol{z}}\times\boldsymbol{u} = -\nabla\Pi + (\Theta - \xi) r\,\hat{\boldsymbol{r}} + \nabla^2\boldsymbol{u},$$

$$\nabla\cdot\boldsymbol{u} = 0,$$

$$\frac{\partial \Theta}{\partial t} + \boldsymbol{u}\cdot\nabla\Theta = \frac{Ra_T}{Pr}\left(\frac{\Gamma}{1-\Gamma}\right)^2\frac{u_r}{r^2} + \frac{1}{Pr}\nabla^2\Theta, \quad \Theta(\boldsymbol{x},t) = T(\boldsymbol{x},t) - T_s(r)$$

$$\frac{\partial \xi}{\partial t} + \boldsymbol{u}\cdot\nabla\xi = \frac{|Ra_C|}{Sc}\left(\frac{\Gamma}{1-\Gamma}\right)^2\frac{u_r}{r^2} + \frac{1}{Sc}\nabla^2\xi, \quad \xi(\boldsymbol{x},t) = C(\boldsymbol{x},t) - C_s(r)$$

Dimensionless numbers:

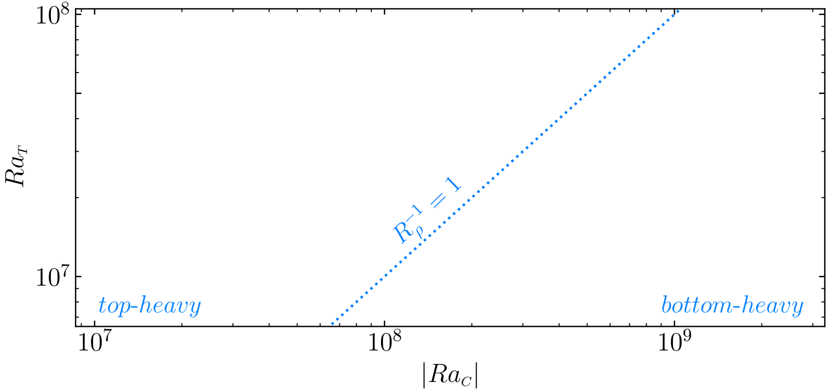
$\Gamma = \frac{r_i}{r_o} = 0.6, \quad Ek = \frac{\nu}{\Omega D^2} = 10^{-4},$ (small) $Pr = \frac{\nu}{\kappa_T} = 0.3, \quad Sc = \frac{\nu}{\kappa_C} = 3 \quad \left(\tau = \frac{\kappa_C}{\kappa_T} = 0.1\right)$

(Yan & Stanley 2021)

$$Ra_T = \frac{g_o \alpha_T D^5}{r_o \nu \kappa_T}\left|T_s'(r_i)\right| \quad \text{and} \quad Ra_C = -\frac{g_o \alpha_C D^5}{r_o \nu \kappa_C}\left|C_s'(r_i)\right|$$

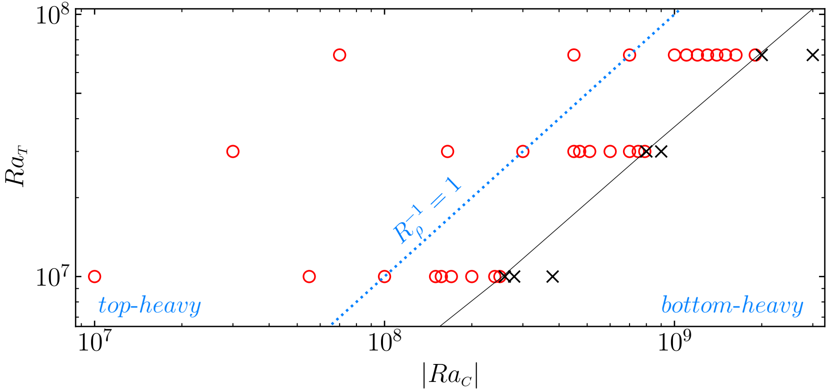
Numerical simulations using XSHELLS by Nathanaël Schaeffer (Université Grenoble Alpes)

Phase diagram: top-heavy vs. bottom heavy



- $N_0^2 = -\frac{Ra_T}{Pr}\frac{r_i^2}{r} + \frac{|Ra_C|}{Sc}\frac{r_i^2}{r}$, $N^2 < 0$: unstable stratification, $N^2 > 0$: stable stratification
- $R_\rho^{-1} = \frac{\tau |Ra_C|}{Ra_T}$, $R_\rho^{-1} < 1$: unstable stratification, $R_\rho^{-1} > 1$: stable stratification
- critical Rayleigh number for pure thermal convection $\approx 1.77 \times 10^5$

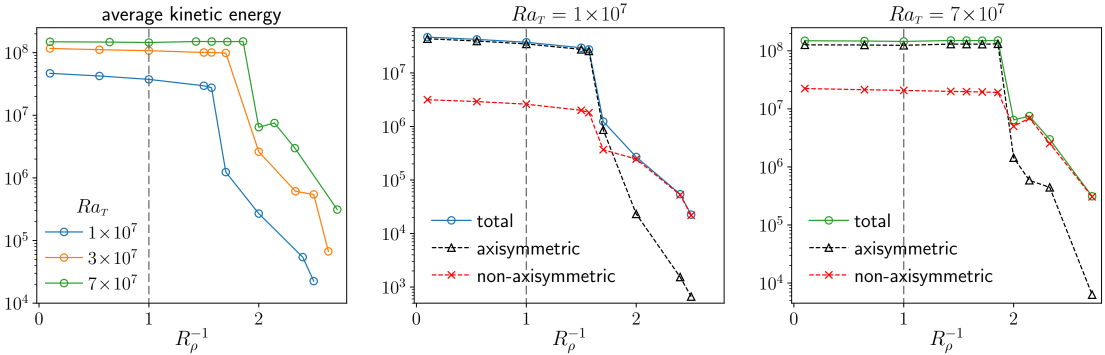
Phase diagram: global stability



black crosses ($\times$): stable   red circles ($\circ$): unstable

- Convection occurs for $R_\rho^{-1} < 1$ *(overturning convection)*
- Convection occurs for $R_\rho^{-1} > 1$ at some moderately large $|Ra_C|$ *(double-diffusive effect)*
- For a given Ra_T, system is stable when R_ρ^{-1} (or $|Ra_C|$) is larger than some critical value

Kinetic energy

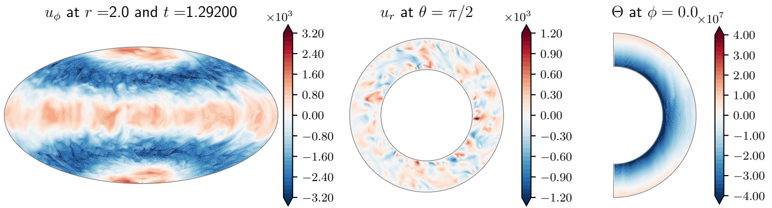


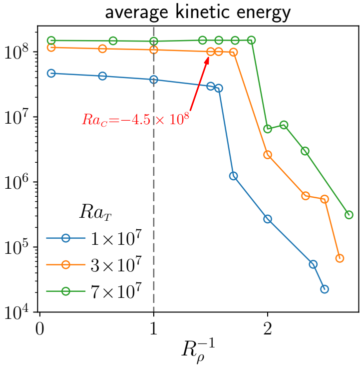
For a given Ra_T:

- KE remains roughly constant for R_ρ^{-1} less than some threshold $R_* > 1$
- KE drops sharply at $R_\rho^{-1} \approx R_*$ and remains small for $R_\rho^{-1} > R_*$
- $R_\rho^{-1} < R_*$: KE is dominated by the axisymmetric $(m = 0)$ velocity
- $R_\rho^{-1} > R_*$: KE is dominated by the non axisymmetric $(m \neq 0)$ velocity

Field morphology: thermal-like behavior

$Ra_T = 3 \times 10^7,\ Ra_C = -4.5 \times 10^8$

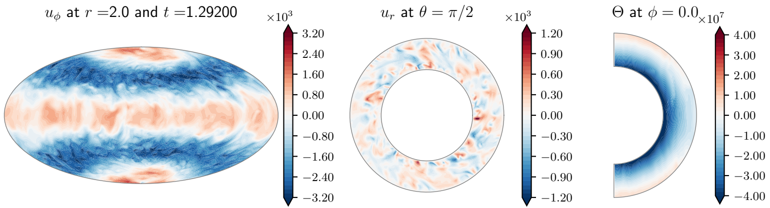




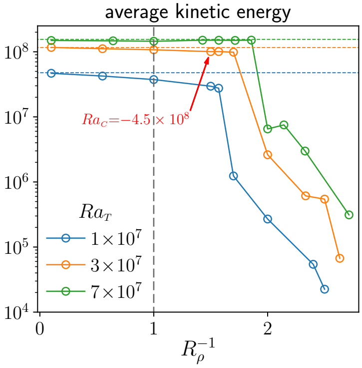
- similar to pure thermal (overturning) convection—even though $R_\rho^{-1} > 1$

Field morphology: comparing to pure thermal convection

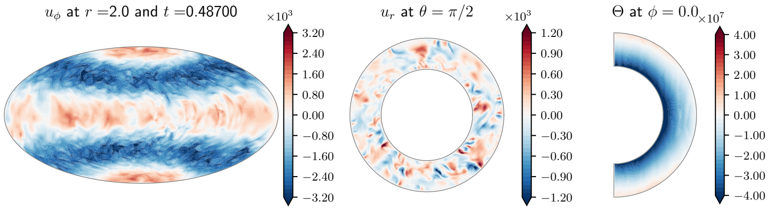
$Ra_T = 3 \times 10^7$, $Ra_C = -4.5 \times 10^8$



- similar to pure thermal (overturning) convection—even though $R_\rho^{-1} > 1$



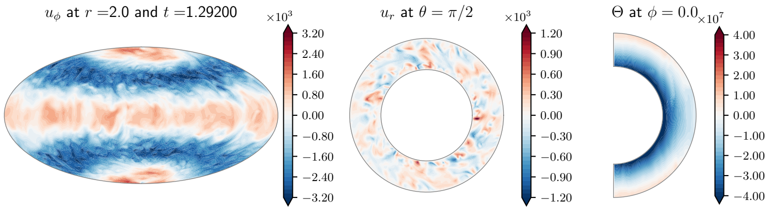
Pure thermal convection at $Ra_T = 3 \times 10^7$



- pure thermal convection

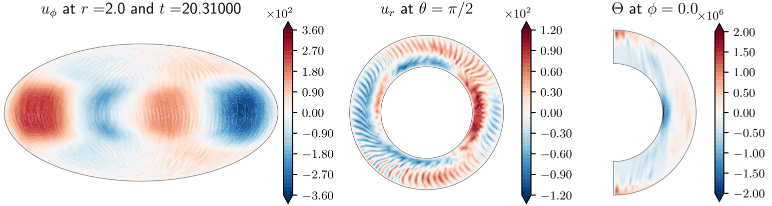
Field morphology: oscillatory double-diffusive behavior

$Ra_T = 3 \times 10^7,\ Ra_C = -4.5 \times 10^8$

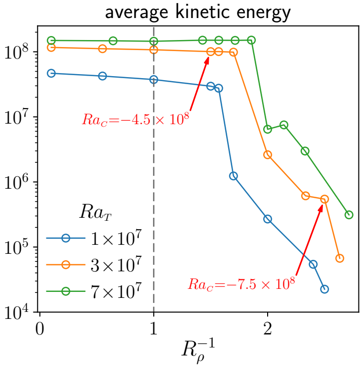


- similar to pure thermal (overturning) convection—even though $R_\rho^{-1} > 1$

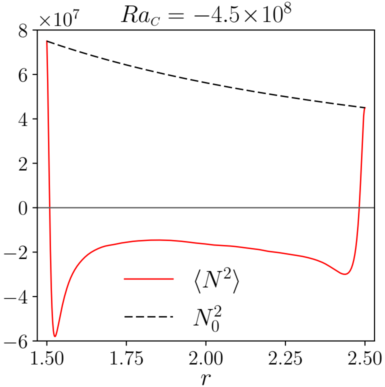
$Ra_T = 3 \times 10^7,\ Ra_C = -7.5 \times 10^8$

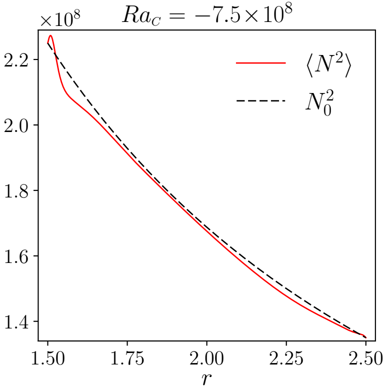


- overlapping of large-scale pattern and small-scale wave motions



Buoyancy frequency $(Ra_T = 3 \times 10^7)$

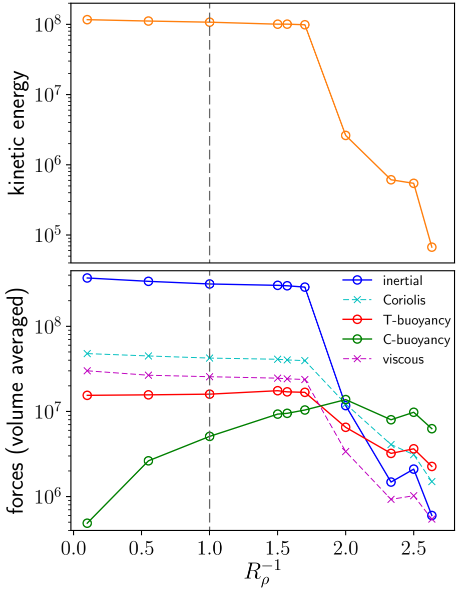




$$N_0^2 = g\alpha_T \frac{\mathrm{d}T_s}{\mathrm{d}r} - g\alpha_C \frac{\mathrm{d}C_s}{\mathrm{d}r}\,, \qquad N_0^2 > 0 \text{ for both cases}$$

$$\langle N^2 \rangle = g\alpha_T \frac{\mathrm{d}\langle T\rangle}{\mathrm{d}r} - g\alpha_C \frac{\mathrm{d}\langle C\rangle}{\mathrm{d}r}\,, \qquad \langle f \rangle = \int \left[\frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi} f(r,\theta,\phi,t)\,\mathrm{d}\theta\mathrm{d}\phi\right]\mathrm{d}t$$

Force balance $(Ra_T = 3 \times 10^7)$



$$\underbrace{\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}}_{\text{inertia}} + \underbrace{\frac{2}{Ek}\hat{\boldsymbol{z}}\times\boldsymbol{u}}_{\text{Coriolis}} = -\nabla\Pi + \underbrace{\Theta\, r\,\hat{\boldsymbol{r}}}_{\text{thermal buoyancy}} - \underbrace{\xi\, r\,\hat{\boldsymbol{r}}}_{\text{compositional buoyancy}} + \underbrace{\nabla^2\boldsymbol{u}}_{\text{viscous}}$$

- transition occurs near $R_\rho^{-1} = R_*$ where the strengths of thermal buoyancy and compositional buoyancy are comparable

Summary

- $R_\rho^{-1} < 1$: thermal-like overturning convection
- $1 < R_\rho^{-1} < R_*$: (still) thermal-like overturning convection
- R_* : thermal buoyancy $\sim$ compositional buoyancy
- $R_\rho^{-1} > R_*$: oscillatory double-diffusive behavior before the system becomes stable at large R_ρ^{-1}
- Dynamo at $R_\rho^{-1} > 1$? *Yes!*