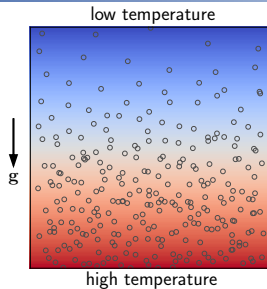


Oscillatory double-diffusive convection in a rotating spherical shell: effects of the Ekman number

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Double-diffusive convection



- Density variation comes from two different scalars: **temperature** and **composition**
- Consider a **heavy** element in the fluid, e.g. salt in seawater, He in H-He mixture
- Let C be the concentration of the heavy element (composition):

$$\rho(T, C) = \rho_m [1 - \alpha_T(T - T_m) + \alpha_C(C - C_m)], \quad \alpha_T, \alpha_C > 0$$

- What distinguishes T and C is their diffusivities: $\kappa_T \gg \kappa_C$
- Rayleigh numbers Ra_T and Ra_C : positive (destabilizing) or negative (stabilizing)

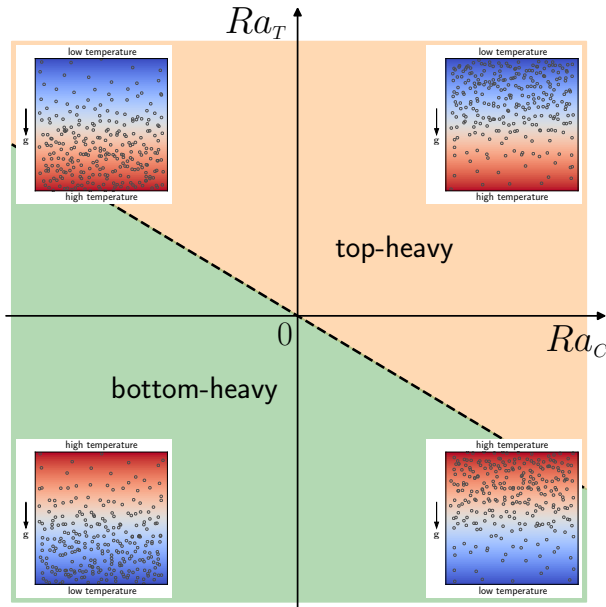
Different regimes of double-diffusive convection

● Oscillatory double-diffusive convection (ODDC)

● fast-diffusing temperature is destabilizing

● slow-diffusing composition is stabilizing

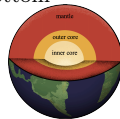
● stable



● e.g. Earth's outer core

● heavy iron solidifies onto the inner core

● excessive *light* elements at bottom



● e.g. upper ocean

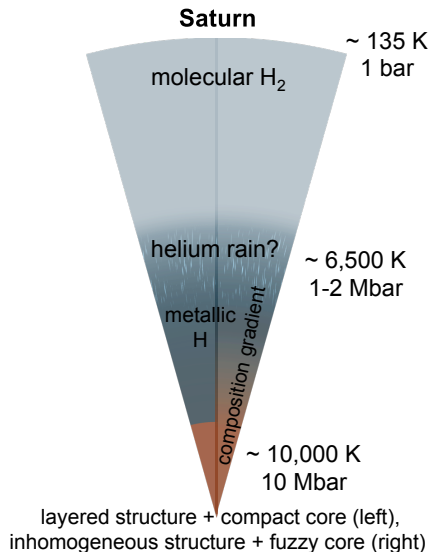
● instabilities in stably-stratified fluid

● salt fingers predicted by Stern(1960)



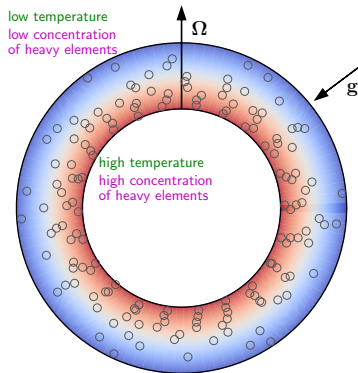
(Huppert & Turner 1981)

Possible scenarios of ODDC in planetary and stellar interiors



- Stabilising compositional gradient may form inside Saturn in two different ways:
 - **helium rain:** at some specific temperature and pressure, H and He become immiscible, heavy He droplets fall inwards and accumulate above the deep interior (*Salpeter 1973; Stevenson & Salpeter 1977*)
 - **dilute/fuzzy core:** recent observations suggest the existence of an extended stably stratified region created by heavy elements dissolving from a compact inner core (*Mankovich & Fuller 2021*)
- Composition gradient, and thus ODDC, may also exist in:
 - Jupiter
 - core-convective main sequence stars

Mathematical model: ODDC in a rotating spherical shell



Boundary conditions

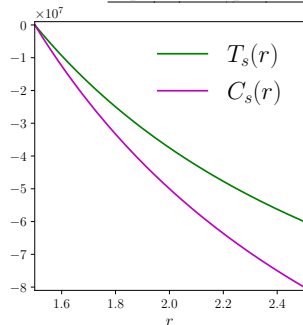
fixed T gradient:

$$\left. \frac{\partial T}{\partial r} \right|_{r_i}, \left. \frac{\partial T}{\partial r} \right|_{r_o} < 0$$

fixed C gradient:

$$\left. \frac{\partial C}{\partial r} \right|_{r_i}, \left. \frac{\partial C}{\partial r} \right|_{r_o} < 0$$

Equilibrium profiles (at $\mathbf{u} = 0$)



$$\frac{dT_s}{dr} < 0 \text{ } \textit{destabilising}$$

$$\frac{dC_s}{dr} < 0 \text{ } \textit{stabilising}$$

Consider a **Boussinesq** fluid in a rotating spherical shell of inner radius r_i and outer radius r_o

• Density: $\rho(T, C) = \rho_m [1 - \alpha_T (T - T_m) + \alpha_C (C - C_m)]$

• Buoyancy frequency: $N_0^2 = -\frac{g}{\rho_m} \frac{d}{dr} [\rho(T_s, C_s)] \implies N_0^2 = g\alpha_T \frac{dT_s}{dr} - g\alpha_C \frac{dC_s}{dr}$

Governing equations

Non-dimensional equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{2}{Ek} \hat{\mathbf{z}} \times \mathbf{u} = -\nabla \Pi + (\Theta - \xi) r \hat{\mathbf{r}} + \nabla^2 \mathbf{u},$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla \Theta = \frac{Ra_T}{Pr} \left(\frac{\Gamma}{1-\Gamma} \right)^2 \frac{u_r}{r^2} + \frac{1}{Pr} \nabla^2 \Theta, \quad \Theta(\mathbf{x}, t) = T(\mathbf{x}, t) - T_s(r)$$

$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi = \frac{|Ra_C|}{Sc} \left(\frac{\Gamma}{1-\Gamma} \right)^2 \frac{u_r}{r^2} + \frac{1}{Sc} \nabla^2 \xi, \quad \xi(\mathbf{x}, t) = C(\mathbf{x}, t) - C_s(r)$$

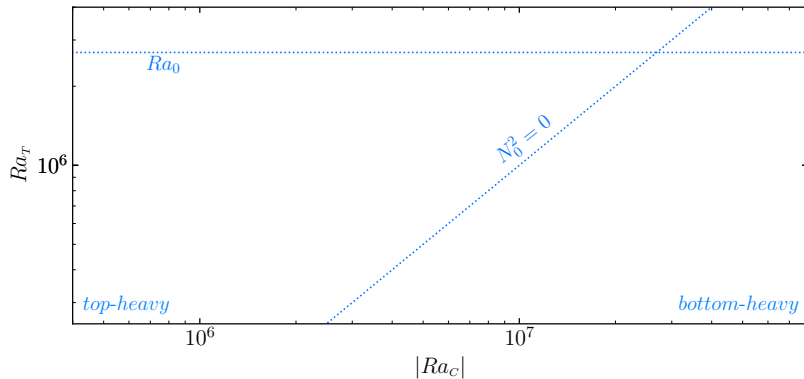
Dimensionless numbers:

$$\Gamma = \frac{r_i}{r_o} = 0.6, \quad Ek = \frac{\nu}{\Omega D^2} = 10^{-5}, \quad (\text{small}) \quad Pr = \frac{\nu}{\kappa_T} = 0.3, \quad Sc = \frac{\nu}{\kappa_C} = 3 \quad \left(\tau = \frac{\kappa_C}{\kappa_T} = 0.1 \right)$$

$$Ra_T = \frac{g_o \alpha_T D^5}{r_o \nu \kappa_T} |T'_s(r_i)| \quad \text{and} \quad Ra_C = -\frac{g_o \alpha_C D^5}{r_o \nu \kappa_C} |C'_s(r_i)|$$

Numerical simulations using XSHELLS by Nathanaël Schaeffer (Université Grenoble Alpes)

ODDC at low Rayleigh numbers



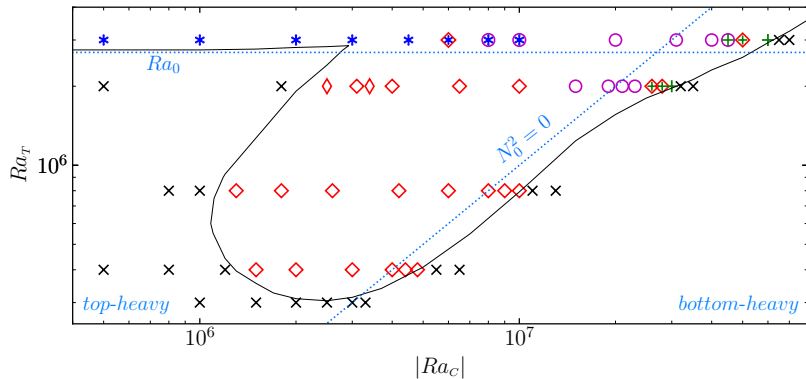
● (non-dimensional) $N_0^2 = -\frac{Ra_T}{Pr} \frac{r_i^2}{r} + \frac{|Ra_C|}{Sc} \frac{r_i^2}{r}$

$N_0^2 < 0$: unstable stratification, $N_0^2 > 0$: stably stratified

● Ra_0 = critical Rayleigh number for pure thermal convection

pure thermal convection happens when $N_0^2 < 0$ and $Ra_T > Ra_0$

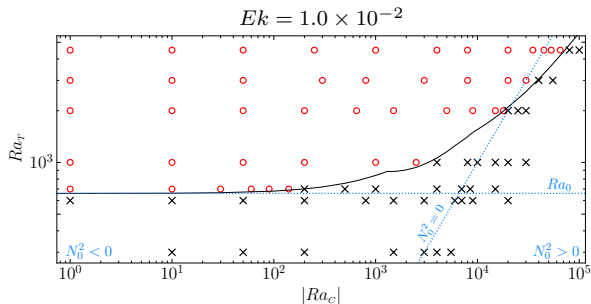
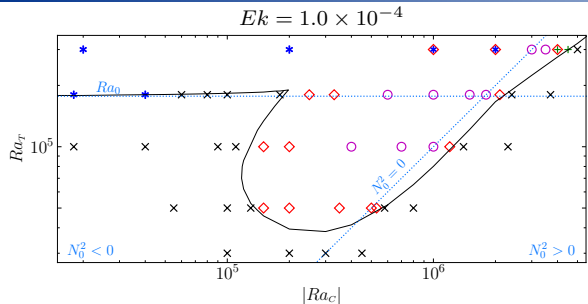
Stability diagram: $Ek = 10^{-5}$



black crosses ($\times$) $\Rightarrow$ stable coloured symbols ($*$, $+$, $\diamond$, $\diamond$, $\circ$) $\Rightarrow$ unstable

- Despite the equilibrium compositional gradient dC_s/dr is stabilising:
 - convection occurs for some $Ra_T < Ra_0$ (= critical Rayleigh number for thermal convection)
 - system can become unstable even when $N^2 > 0$
- This counter-intuitive phenomenon only happens for an intermediate range of $|Ra_C|$

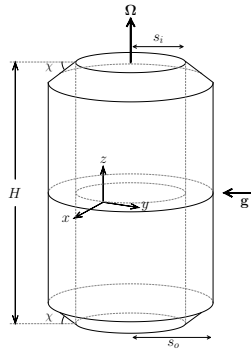
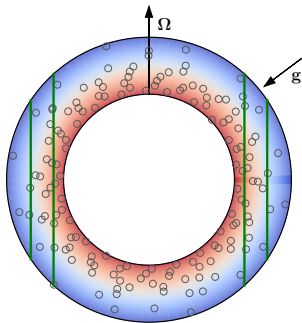
Effects of the Ekman number



- ODDC instability tongue below $Ra_T = Ra_0$ disappears at weak rotation (large Ek)
- Strong rotation required to sustain the instability at $Ra_T < Ra_0$

Thin cylindrical annulus model (Busse 1986)

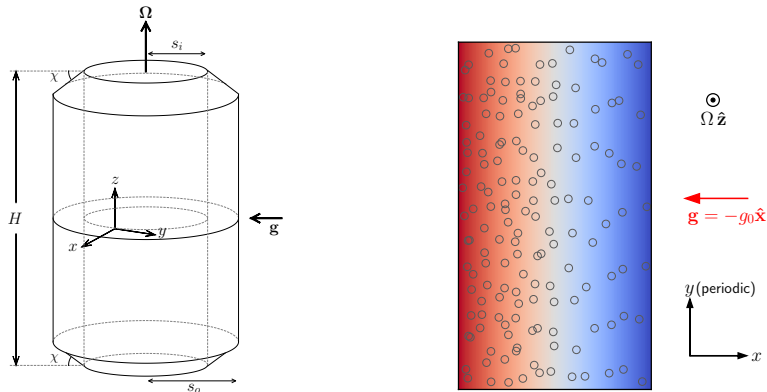
Trim the spherical shell into a cylindrical annulus ...



- Top and bottom of the cylindrical annulus tilted at a **constant angle χ**
- Gravity g points inwards and perpendicular to Ω
- Captures two pieces of physics of the spherical shell:
 1. rapid rotation
 2. curvature of the spherical geometry

Thin cylindrical annulus model (Busse 1986)

Flatten the cylindrical annulus into a plane ...



- **rapid** rotation $\Rightarrow$ **geostrophic balance** at leading order (columnar structures)
- integration over height + **tilted boundaries** $\Rightarrow$ two-dimensional system with **β -effect**
- **thin** annulus $\Rightarrow$ **Cartesian** coordinate: radial $\rightarrow x$, azimuthal $\rightarrow y$ (periodic)

ODDC on a two-dimensional β -plane

2D-velocity (u, v) in terms of a streamfunction ψ :

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

Define the Jacobian:

$$J(A, B) = \partial_x A \partial_y B - \partial_y A \partial_x B$$

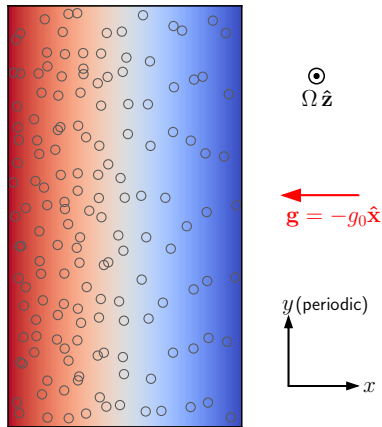
Governing equations for (ψ, Θ, ξ) :

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) - \beta \frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y} (\Theta - \xi) + \nabla^4 \psi$$

$$\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) = -\frac{Ra_T}{Pr} \frac{\partial \psi}{\partial y} + \frac{1}{Pr} \nabla^2 \Theta$$

$$\frac{\partial \xi}{\partial t} + J(\psi, \xi) = -\frac{|Ra_C|}{Sc} \frac{\partial \psi}{\partial y} + \frac{1}{Sc} \nabla^2 \xi$$

$$\beta = \frac{4\Gamma \tan \chi}{Ek} \quad \left(\Gamma = \frac{H}{s_o - s_i} \right)$$



Busse, *Geophysical Research Letters* **29**, 1105 (2002)

Simitev, *Physics of the Earth and Planetary Interiors* **186**, 183 (2011)

Linear stability analysis

Linearized equations (+ boundary conditions):

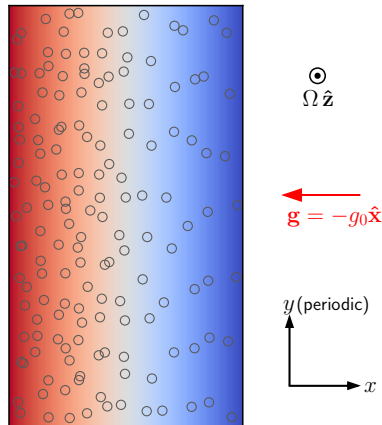
$$\begin{aligned}\frac{\partial}{\partial t} \nabla^2 \psi - \beta \frac{\partial \psi}{\partial y} &= -\frac{\partial \Theta}{\partial y} + \frac{\partial \xi}{\partial y} + \nabla^4 \psi \\ \frac{\partial \Theta}{\partial t} &= -\frac{Ra_T}{Pr} \frac{\partial \psi}{\partial y} + \frac{1}{Pr} \nabla^2 \Theta \\ \frac{\partial \xi}{\partial t} &= -\frac{|Ra_C|}{Sc} \frac{\partial \psi}{\partial y} + \frac{1}{Sc} \nabla^2 \xi\end{aligned}$$

Eigenfunctions: modes with wavenumber (k, l) ,

$$\begin{aligned}\psi(x, y, t) &= \hat{\psi} \sin(kx) e^{ily} e^{\lambda t} \\ \Theta(x, y, t) &= \hat{\Theta} \sin(kx) e^{ily} e^{\lambda t} \\ \xi(x, y, t) &= \hat{\xi} \sin(kx) e^{ily} e^{\lambda t}\end{aligned}$$

Eigenvalues: complex growth rate,

$$\lambda = \sigma + i\omega \quad (\sigma, \omega \in \mathbb{R})$$



Maximum growth rate and neutral curve

Solvability condition gives the equation for the eigenvalue $\lambda = \sigma + i\omega$:

$$\lambda^3 + \left(\frac{1 + Pr + \tau}{Pr} k_h^2 + i \frac{\beta l}{k_h^2} \right) \lambda^2 + \left[\frac{Pr + \tau(1 + Pr)}{Pr^2} k_h^4 + \frac{\tau |Ra_C| - Ra_T}{Pr} \frac{l^2}{k_h^2} + i \frac{\beta(1 + \tau)}{Pr} l \right] \lambda + \frac{\tau}{Pr^2} [k_h^6 + (|Ra_C| - Ra_T) l^2 + i \beta k_h^2 l] = 0$$

$$(k, l) = (m\pi, \frac{n}{s_i}), \quad m, n \in \mathbb{Z}, \quad k_h^2 \equiv k^2 + l^2, \quad Pr = 0.3, \quad \tau = Pr/Sc = 0.1, \quad \beta \sim 10^5, 10^4, 10^2$$

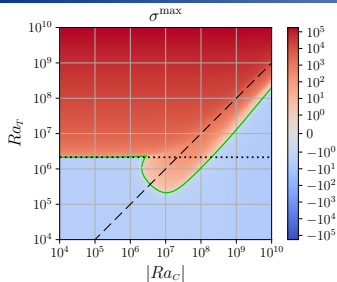
● A cubic equation $\implies$ three roots: $\lambda_\mu = \sigma_\mu + i\omega_\mu$, $\mu = 1, 2, 3$ ($\omega_1 < \omega_2 < \omega_3$)

1. For each (Ra_T, Ra_C) , maximum growth rate $\sigma^{\max}(Ra_C, Ra_T)$:

$$\sigma_{\mu}^{\max}(Ra_C, Ra_T) = \max_{k, l} \{ \sigma_{\mu}(k, l; Ra_C, Ra_T) \}, \quad \mu = 1, 2, 3$$
$$\sigma^{\max}(Ra_C, Ra_T) = \max_{\mu} \{ \sigma_{\mu}^{\max} \}$$

2. Setting $\sigma^{\max} = 0$, we can trace the stability boundary (neutral curve) on the $Ra_T - Ra_C$ phase plane

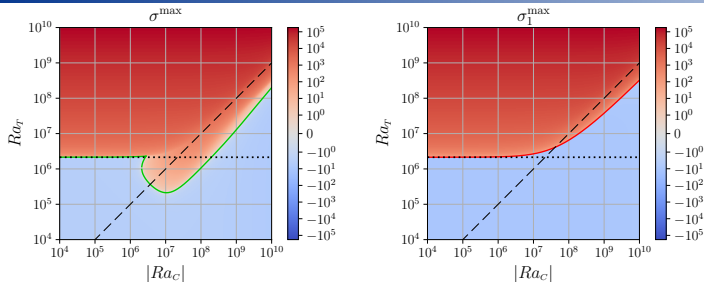
Results for strong rotation: $\beta = 1.5 \times 10^5$



$Ra_{0,\beta}$ = critical Ra of pure thermal convection on a β -plane (dotted line), $\beta = \frac{4\Gamma \tan \chi}{Ek}$

- Similar qualitative features as in the stability diagram for the spherical shell
- Investigate each $\sigma_\mu^{\max}$ individually: $\sigma^{\max}(Ra_C, Ra_T) = \max_q \{\sigma_\mu^{\max}\}$, $\mu = 1, 2, 3$
- There is always a decaying solution: $\sigma_2^{\max} < 0$

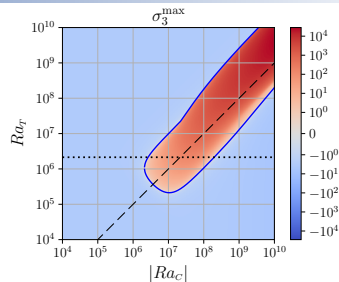
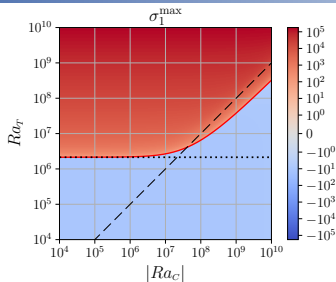
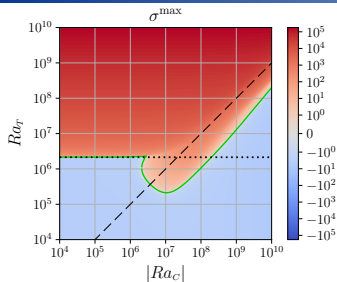
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- $\sigma_1^{\max}$: unstable only when $Ra_T > Ra_{0,\beta}$, small-scale and prograde
 $\Rightarrow \sim$ thermal-Rossby waves of pure thermal convection

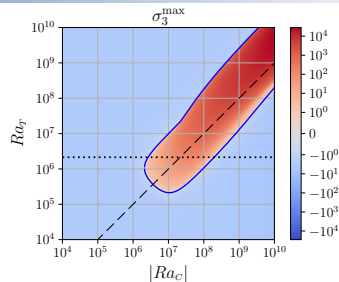
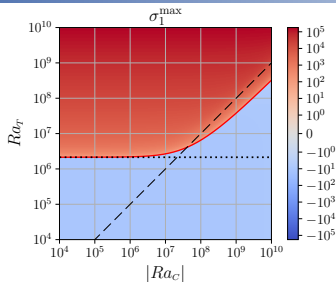
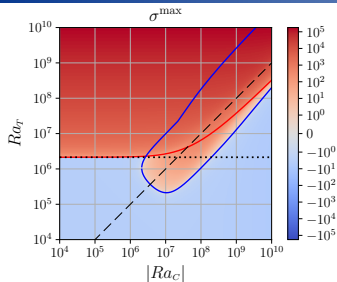
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- $\sigma_1^{\max}$: unstable only when $Ra_T > Ra_{0,\beta}$, small-scale and prograde
 $\Rightarrow \sim$ thermal-Rossby waves of pure thermal convection
- $\sigma_3^{\max}$: instability region extends below $Ra_{0,\beta}$, large-scale and retrograde
 $\Rightarrow$ a genuine double-diffusive mode

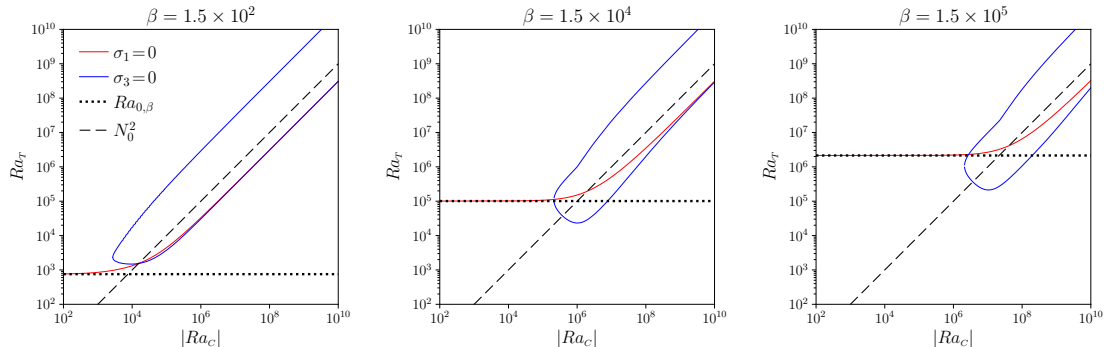
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Effects of rotation rate



- Rotation inhibits convection $\implies$ both neutral curves move upwards as β increases
- Thermal-Rossby mode* (σ_1) is more strongly inhibited than the *double-diffusive mode* (σ_3)
- For large β :

$$Ra_{0,\beta} = \beta^{\frac{4}{3}} \cdot 3 \left(\frac{1}{\sqrt{2}} \frac{Pr}{1 + Pr} \right)^{\frac{4}{3}} \quad (\text{Busse \& Or 1986}), \quad \text{tip of instability tongue: } Ra_T^\dagger \sim \beta \cdot \left(\frac{\tau Pr}{1 + Pr} \right)^{\frac{1}{2}} \quad (\text{small } \tau)$$

Summary

