Atmospheric moisture transport: stochastic dynamics of the advection-condensation equation

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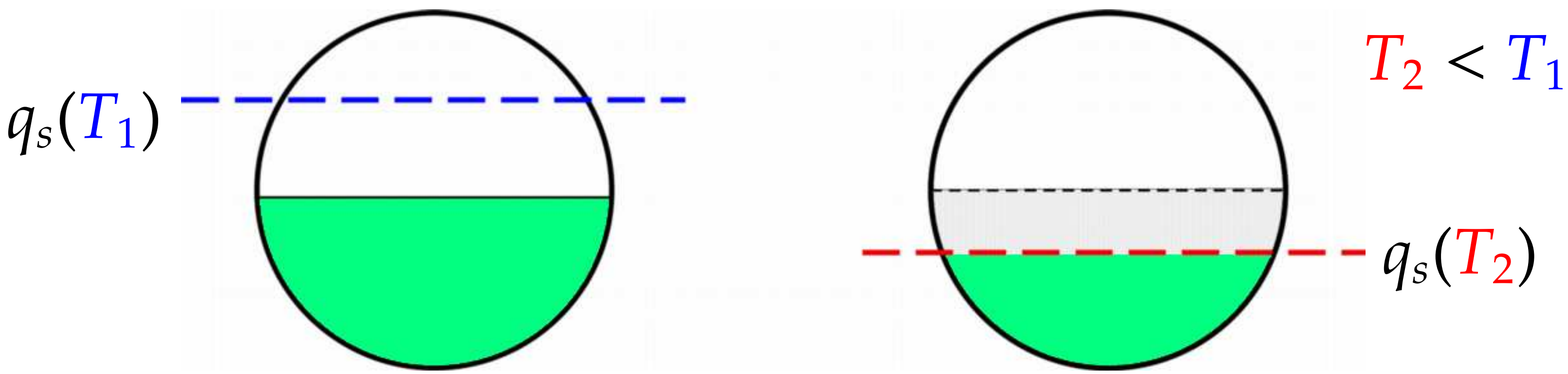
Moisture parameters

- specific humidity of an air parcel:

$$q = \frac{\text{mass of water vapor}}{\text{total air mass}}$$

- saturation specific humidity, $q_s(T)$
  - when $q > q_s$, condensation occurs
  - excessive moisture precipitates out, $q \rightarrow q_s$
  - $q_s(T)$ decreases with temperature T



Moisture field of the atmosphere

- y = latitude, temperature decreases with y
- model : $q_s(y) = q_0 \exp(-\alpha y)$
- moist air parcels being advected around in the troposphere

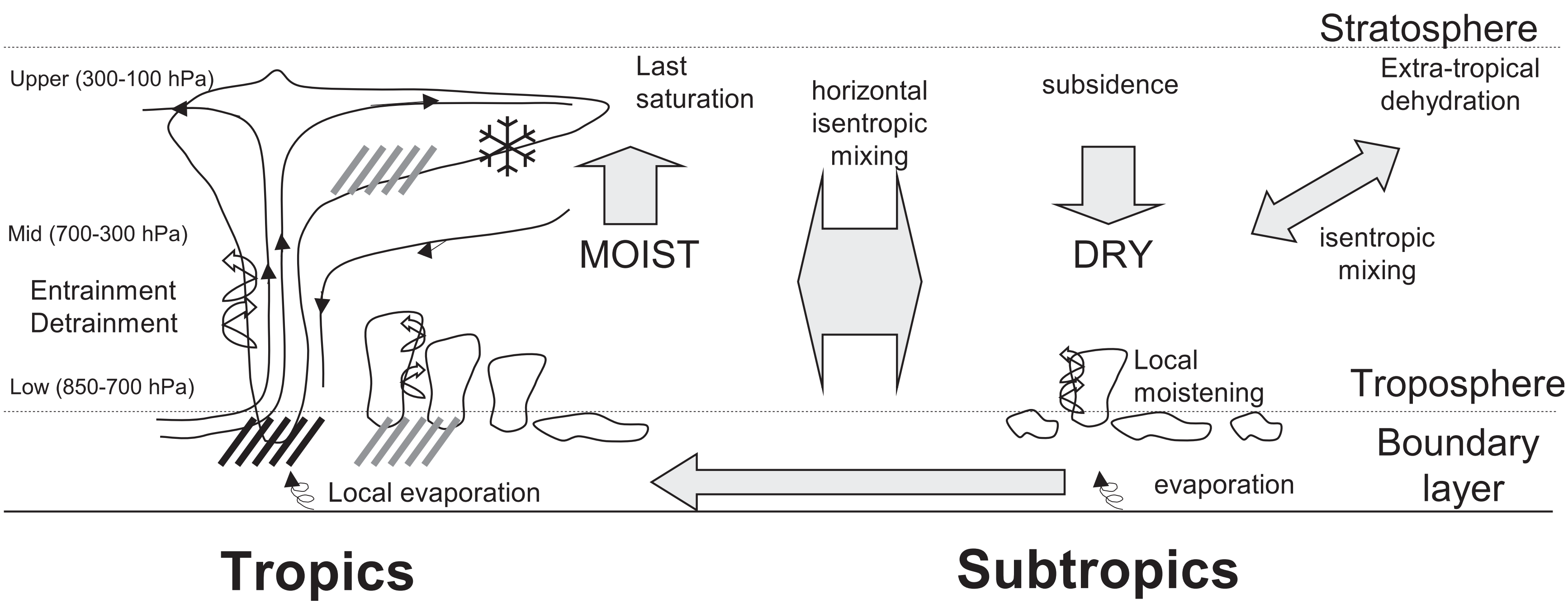


Figure 3. Schematic of the overturning circulation with emphasis on the mechanism controlling the humidity distribution in the subtropics.

(Sherwood et al., Reviews of Geophysics, 2010)

Atmospheric moisture and climate

- Earth's radiation budget:
  - absorption of incoming shortwave radiation generates heat
  - heat carried away by outgoing longwave radiation (OLR)
- water vapor is a greenhouse gas that traps OLR
- $\mathrm{OLR} \sim -\langle \log q \rangle$
- $\mathrm{OLR} \sim -\langle \log[\langle q \rangle + q'] \rangle \approx -\log \langle q \rangle + \frac{1}{2\langle q \rangle^2} \langle q'^2 \rangle$
- how fluctuation q' is generated?
- what is the probability distribution of water vapor in the atmosphere?

Advection-condensation model

$$\frac{\partial q}{\partial t} + \vec{u} \cdot \nabla q = S - C \, , \quad \vec{u} = (u, v)$$

- S = moisture source (evaporation)
- C = condensation sink
  - saturation profile: $q_s(y) = q_0 \exp(-\alpha y)$
  - rapid condensation limit:

$$C : q(x, y, t) = \min \left[q(x, y, t) \, , \, q_s(y) \right]$$

- **Initial-value problem:** $S = 0$
  - entire domain saturated at $t = 0 : q(x, y, 0) = q_s(y)$
  - what is the PDF of q at location (x, y) and time t?
  - how fast does the total moisture content decay?

Coherent circulation + random transport

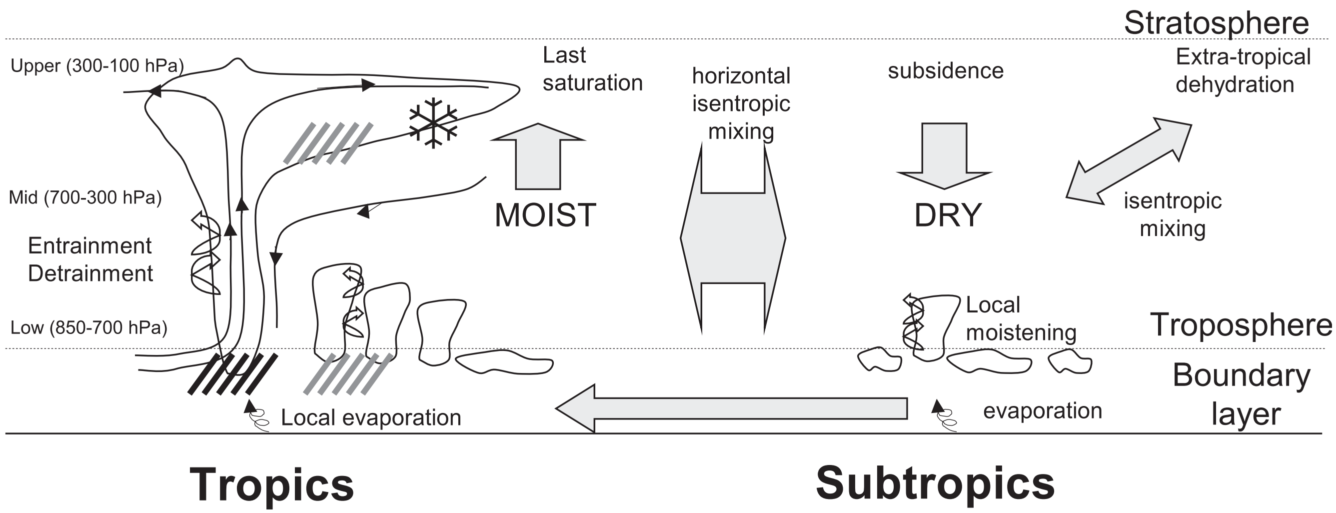


Figure 3. Schematic of the overturning circulation with emphasis on the mechanism controlling the humidity distribution in the subtropics.

- coherent circulating component:

$$u(X, Y) = -\Omega(R)\, Y$$

$$v(X, Y) = \Omega(R)\, X$$

where $R = \sqrt{X^2 + Y^2}$

- random component is δ-correlated in time (Brownain):

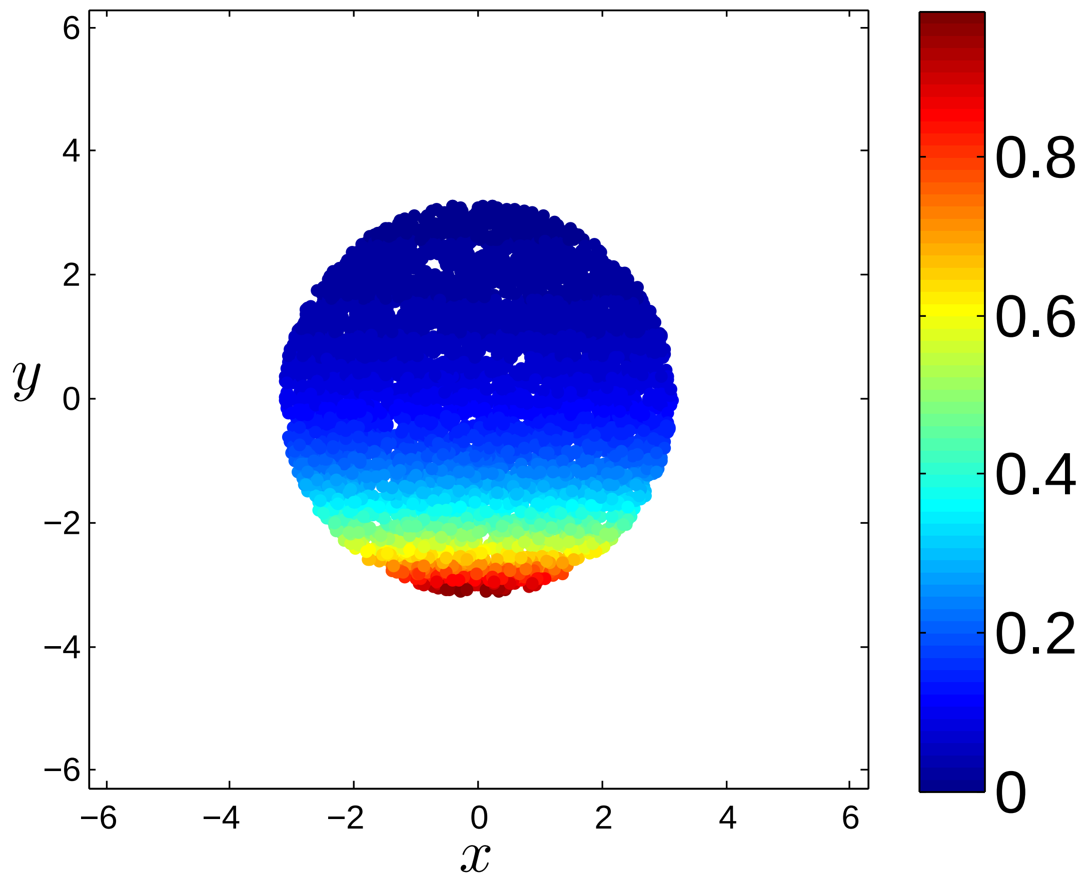
$$U \sim \dot{W}(t)$$

where $W(t)$ is a Wiener process ($\dot{W}(t) \sim$ white noise)

A stochastic transport model

$$\mathrm{d}X(t) = u(X,Y)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_1(t)$$
$$\mathrm{d}Y(t) = v(X,Y)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_2(t)$$
$$\mathrm{d}Q(t) = -C(Q,Y)\mathrm{d}t$$

$$u = -\Omega(R)Y$$
$$v = \Omega(R)X$$



$$\Omega_0 = 1$$
$$\kappa = 10^{-2}$$

Evolution of the moisture field

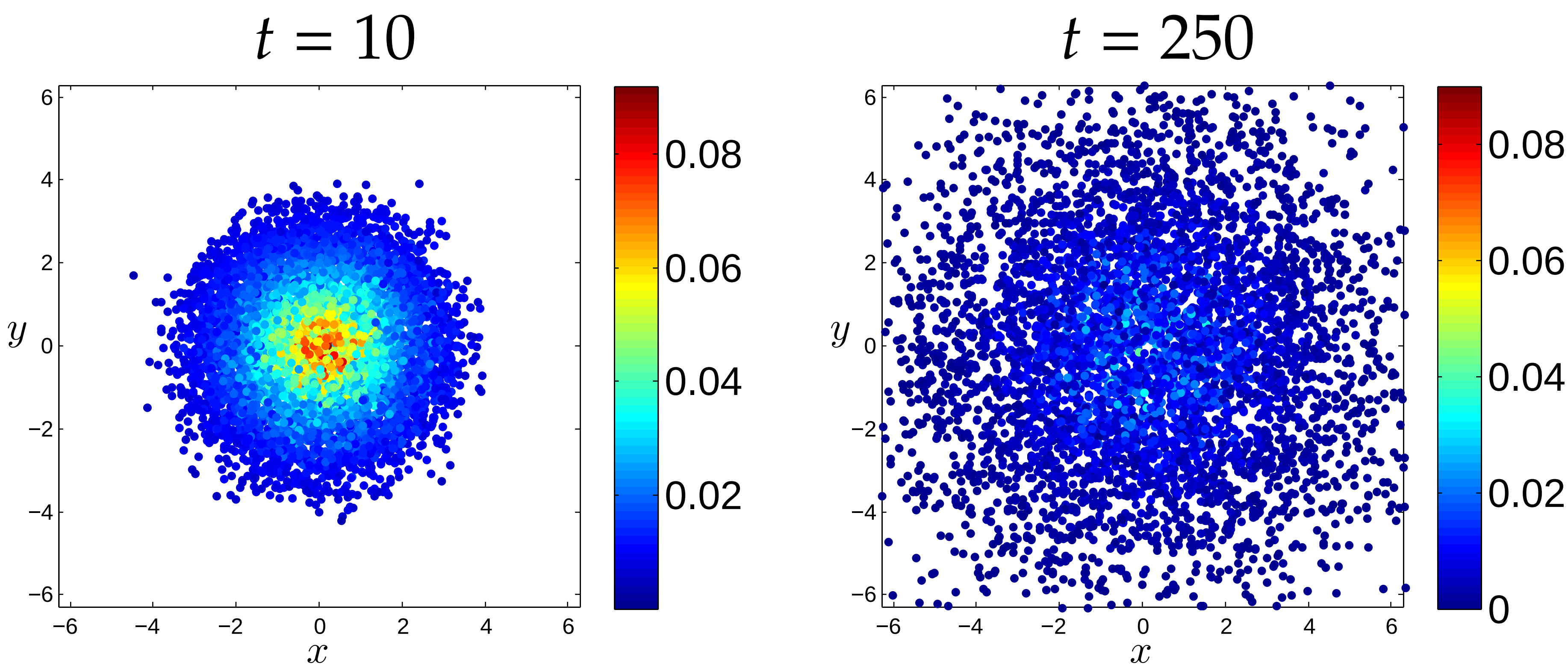
$$\mathrm{d}X(t) = u(X,Y)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_1(t)$$

$$\mathrm{d}Y(t) = v(X,Y)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_2(t)$$

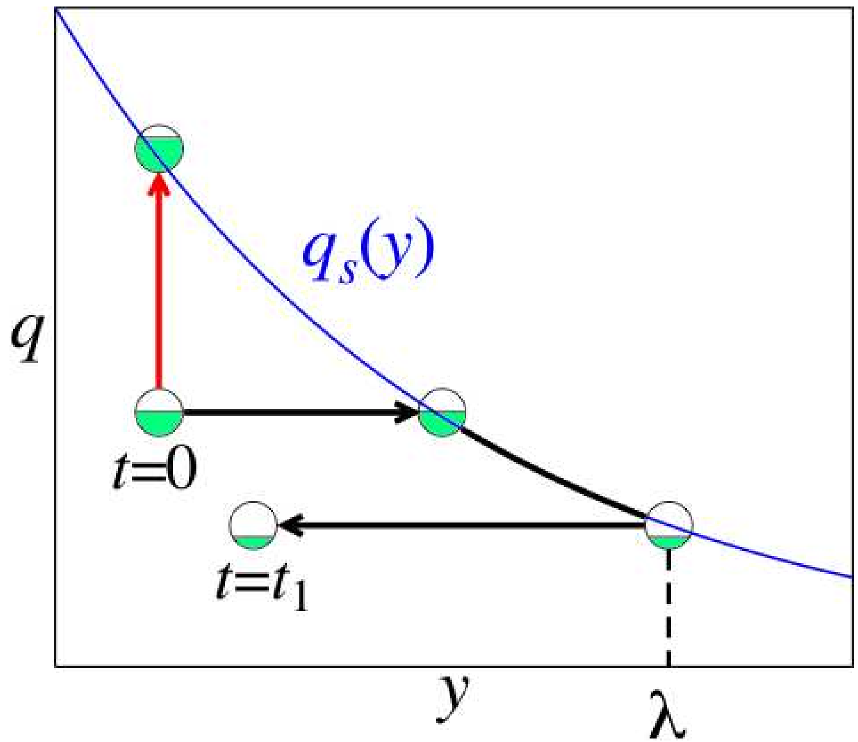
$$\mathrm{d}Q(t) = -C(Q,Y)\mathrm{d}t$$

$$u = -\Omega(R)Y$$

$$v = \Omega(R)X$$



Maximum excursion of an air parcel



- **maximum excursion**, $\lambda = \max_{t\in[0,t_1]} y(t)$

$$q(x, y, t_1) = q_s(\lambda)$$

- statistics of $q \Longleftrightarrow$ statistics of maximum excursion
- *Pierrehumbert, Brogniez & Roca 2007*:
  - obtain $P(q\,|\,y, t)$ for an ensemble of particles execute independent random walks in a 1D domain

Theory: maximum excursion statistics

- The backward Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \vec{u} \cdot \nabla P + \kappa \nabla^2 P \,, \qquad P \equiv P(x', y', t \,|\, x, y, 0)$$

- **Boundary conditions**

$$y = \lambda \,, \; y = -\infty \text{ and } x = \pm\infty \; : \; P(x', y', t \,|\, x, y, 0) = 0$$

Initial conditions

$$P(x', y', 0 \,|\, x, y, 0) = \delta(x - x')\delta(y - y')$$

- For a parcel starts at (x, y) and time $\tau = 0$, the probability that the maximum excursion $\Lambda = \max_{\tau \in [0,t]} y(\tau) < \lambda$:

$$F(x, y, t; \lambda) \; = \; \int_{-\infty}^{\lambda} \mathrm{d}y' \; \int_{-\infty}^{\infty} \mathrm{d}x' \; P(x', y', t \,|\, x, y, 0)$$

- Probability density function of Λ:

$$P_\Lambda(\lambda, t \,|\, x, y) = \frac{\partial F}{\partial \lambda}$$

Asymptotics: fast advection limit

$$\frac{\partial F}{\partial t} = \vec{u} \cdot \nabla F + \kappa \nabla^2 F\,, \qquad F(x, y, t; \lambda)$$

$$\text{B.C.:} \quad F(x, y = \lambda, t; \lambda) = 0$$

$$\text{I.C.:} \quad F(x, y, t = 0; \lambda) = \begin{cases} 1 & \text{if } y < \lambda \\ 0 & \text{otherwise} \end{cases}$$

Fast advection limit : $\epsilon = \kappa / (\Omega_0 L^2) \ll 1$

Scaling : $x \to L\,x,\ t \to (L^2/\kappa)\,t,\ \vec{u} \to (\Omega_0 L)\,\vec{u}$

$$\frac{\partial F}{\partial t} = \epsilon^{-1}\,\vec{u} \cdot \nabla F + \nabla^2 F$$

Expand : $F = F_0 + \epsilon\,F_1\,,$

ϵ^{-1} : $\quad \vec{u}(r) \cdot \nabla F_0 = 0 \;\Rightarrow\; F_0 = F_0(r, t; \lambda)$ $\quad$ axisymmetric

PDF of specific humidity

$$\epsilon^{-1}: \quad \frac{\partial F_0}{\partial t} = \vec{u} \cdot \nabla F_1 + \nabla^2 F_0 \,, \qquad F_0(r,t;\lambda)\,, F_1(r,\theta,t;\lambda)$$

Averaging over θ with $\langle \vec{u} \cdot \nabla F_1 \rangle_\theta = 0$, we get

$$\frac{\partial F_0}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F_0}{\partial r}\right)$$

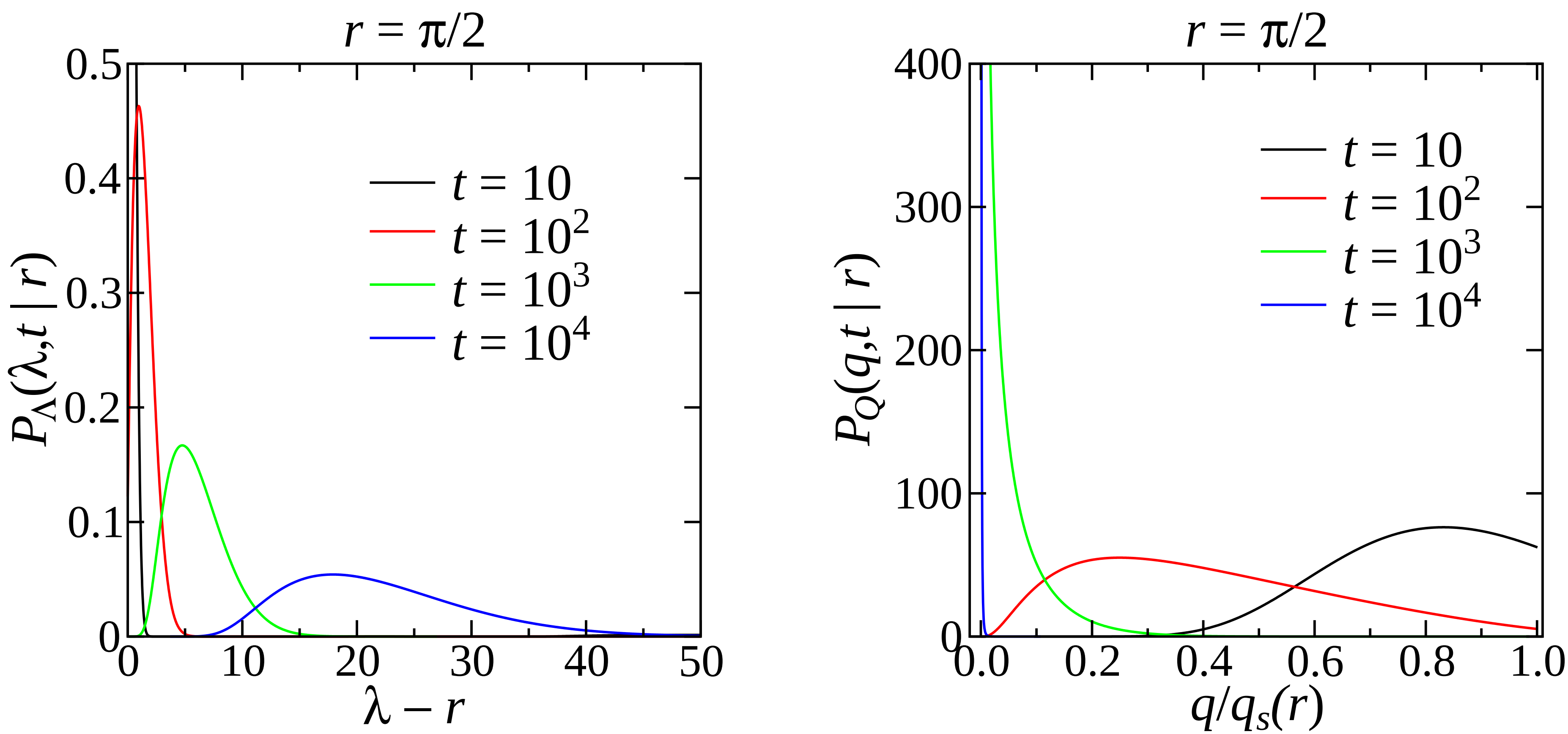
Boundary conditions: $F_0(r,t;\lambda) = 0$ at $r = \lambda$

$$P_\Lambda(\lambda, t\,|\,r) \approx \frac{\partial F_0}{\partial \lambda}$$

$$q(r,t) = q_s(\lambda) = q_0 \exp(-\alpha\lambda)$$

$$P_Q(q,t\,|\,r) = \left[P_\Lambda(\lambda,t\,|\,r) \left| \frac{\mathrm{d}\lambda}{\mathrm{d}q} \right| \right]_{\lambda=\alpha^{-1}\ln\frac{q_0}{q}}$$

Results: PDF of λ and q



$$P_Q(q,t\,|\,r) = \left[\frac{1}{\alpha q} P_\Lambda(\lambda, t\,|\,r) \right]_{\lambda=\alpha^{-1}\ln\frac{q_0}{q}}$$

Results: decay of total moisture content

$$\bar{Q}(t) = \frac{1}{A}\int \mathrm{d}A \int \mathrm{d}q\, q\, P_Q(q,t\,|\,r)$$