Ageostrophic effects on the evolution of ellipsoidal vortices

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Elliptic and Ellipsoidal Vortices

- 2D Euler flows: elliptic patch of constant vorticity
  - in an ambient potential flow, the Kirchhoff vortex (1876)
  - steady and unsteady solutions in an external strain and shear flow (Chaplygin 1899, Moore & Saffman 1971, Kida 1981)
- 3D rotating stratified flows: ellipsoid of constant potential vorticity
  - Meacham 1992: quasi-geostrophic approximation
  - Meacham *et al.* 1994: unsteady solutions in shear flows
  - Miyazaki *et al.* 1999: steady tilted vortex
- In the quasi-geostrophic limit (asymptotically rapid rotating, $\mathrm{Ro} \to 0$)
  - inertial-gravity wave being filtered out (balanced state)
  - much less computational expensive
  - ellipsoid is an exact solution
- We investigate numerically the stability and evolution of an ellipsoidal vortex in a non-hydrostatic system at moderate $\pm\,\mathrm{Ro}$

Rotating, Continuously Stratified Flows

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} + f_0\,\hat{k} \times \vec{u} = -\frac{1}{\rho_0}\nabla\Phi + b\hat{k} \qquad \rho(\vec{x}) = \rho_0 + \bar{\rho}(z) + \rho'(\vec{x})$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} + N^2 w = 0 \qquad b = -\rho_0^{-1} g\,\rho'$$

$$\vec{\omega} = \nabla \times \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

- rotating: f-plane approximation, $\vec{\Omega} = \frac{1}{2} f_0\,\hat{k}$
- stratified: constant buoyancy frequency, $N^2 = -\rho_0^{-1} g\,\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}$

$$\sigma = \frac{f_0}{N} = 0.1$$

- potential vorticity anomaly:

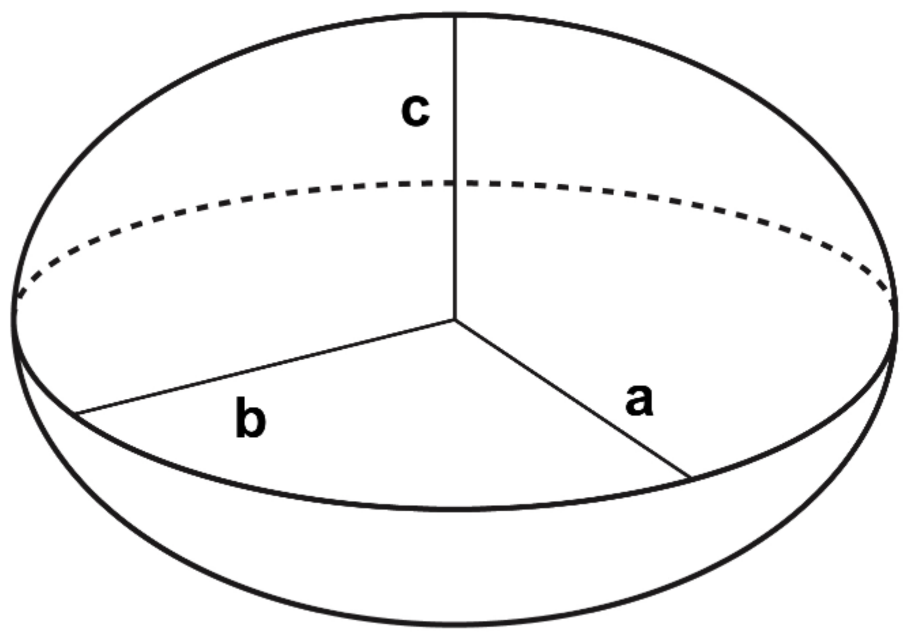
$$Q \equiv \frac{\vec{\omega} + f_0\hat{k}}{f_0} \cdot \frac{\nabla b + N^2\hat{k}}{N^2} = 1 + q \;\sim (\vec{\omega} + 2\vec{\Omega}) \cdot \nabla\rho$$

Computational Domain and Initial Conditions

- periodic three-dimensional domain: $L \times L \times H$
- small aspect ratio: $H = \sigma L$
- initial conditions: an ellipsoidal volume of constant PV anomaly q_0 in a near balanced state $(\mathrm{Ro} \sim q_0)$
- vortex aspect ratios λ and μ :

$$\lambda = \frac{a}{b} \quad (a < b)$$

$$\mu = \frac{c}{\sigma \sqrt{ab}}$$



Numerical methods

- change of variables: $\{b, \vec{u}_h\} \longrightarrow \{q, \vec{A}_h\}$

$$\vec{A}_h = \frac{\vec{\omega}_h}{f_0} + \frac{\nabla_h b}{f_0^2}$$

- equations of motion:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0$$

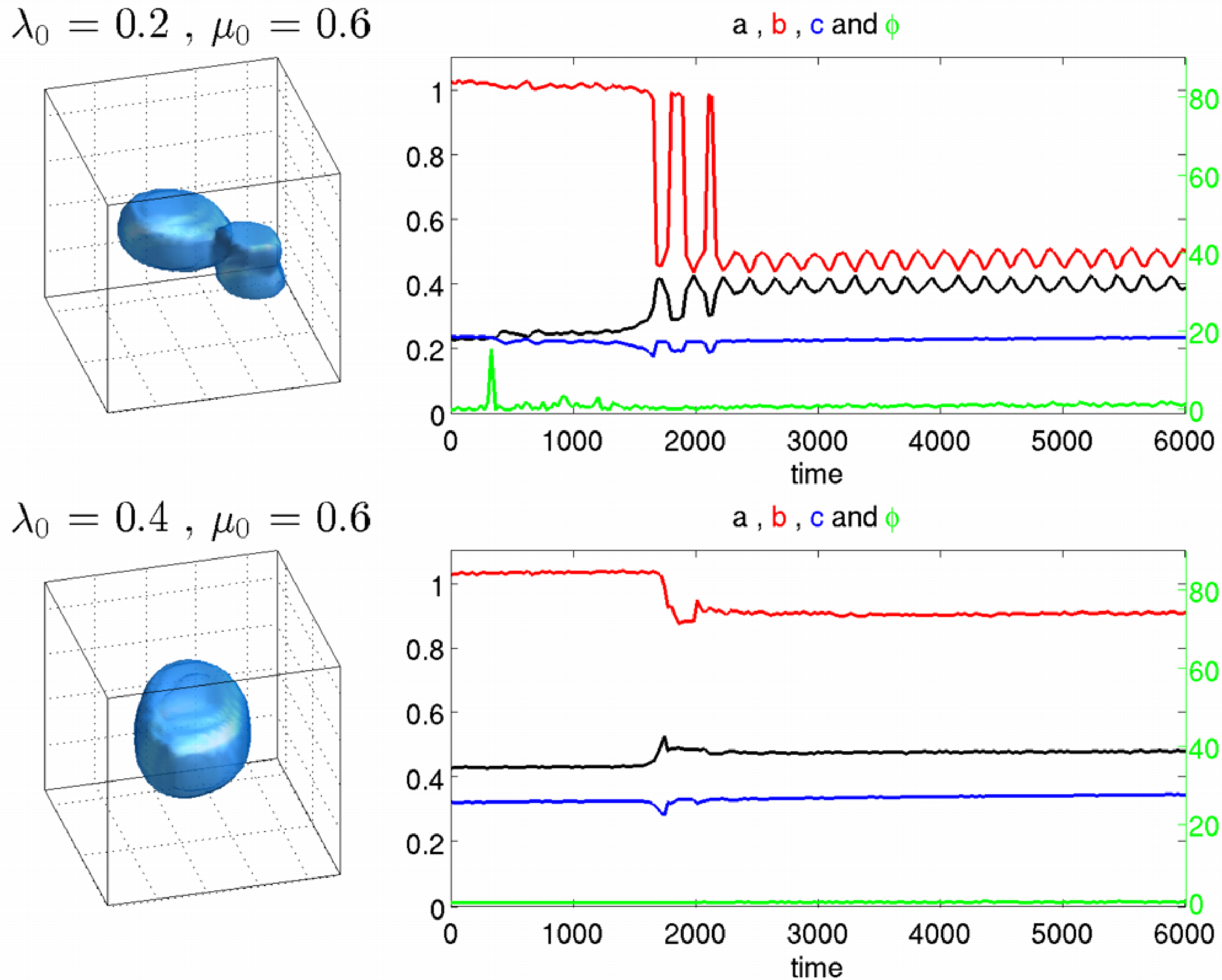
$$\frac{\mathrm{D}\vec{A}_h}{\mathrm{D}t} + f_0\,\hat{k} \times \vec{A}_h = \mathcal{N}(\vec{A}_h, q)$$

- Thermal wind (geostrophic + hydrostatic) balance: $\vec{A}_h = 0$

- q is materially conserved $\Rightarrow$ equations solved efficiently by the Contour-Advective Semi-Lagrangian (CASL) algorithm

  (Dritschel & Viúdez 2003 , JFM **488**, 123-150)

Nonlinear evolution: $q_0 = 0.5$



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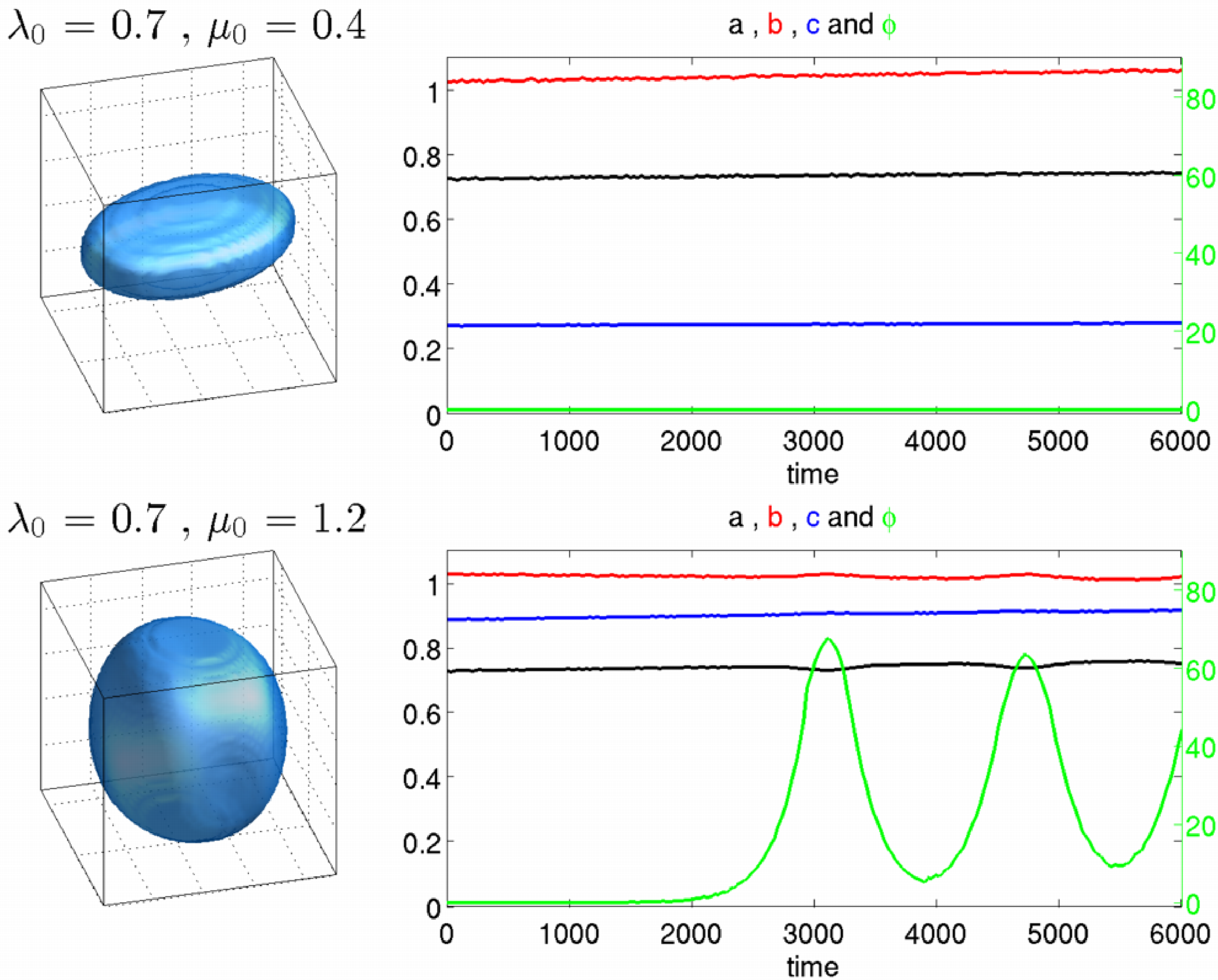

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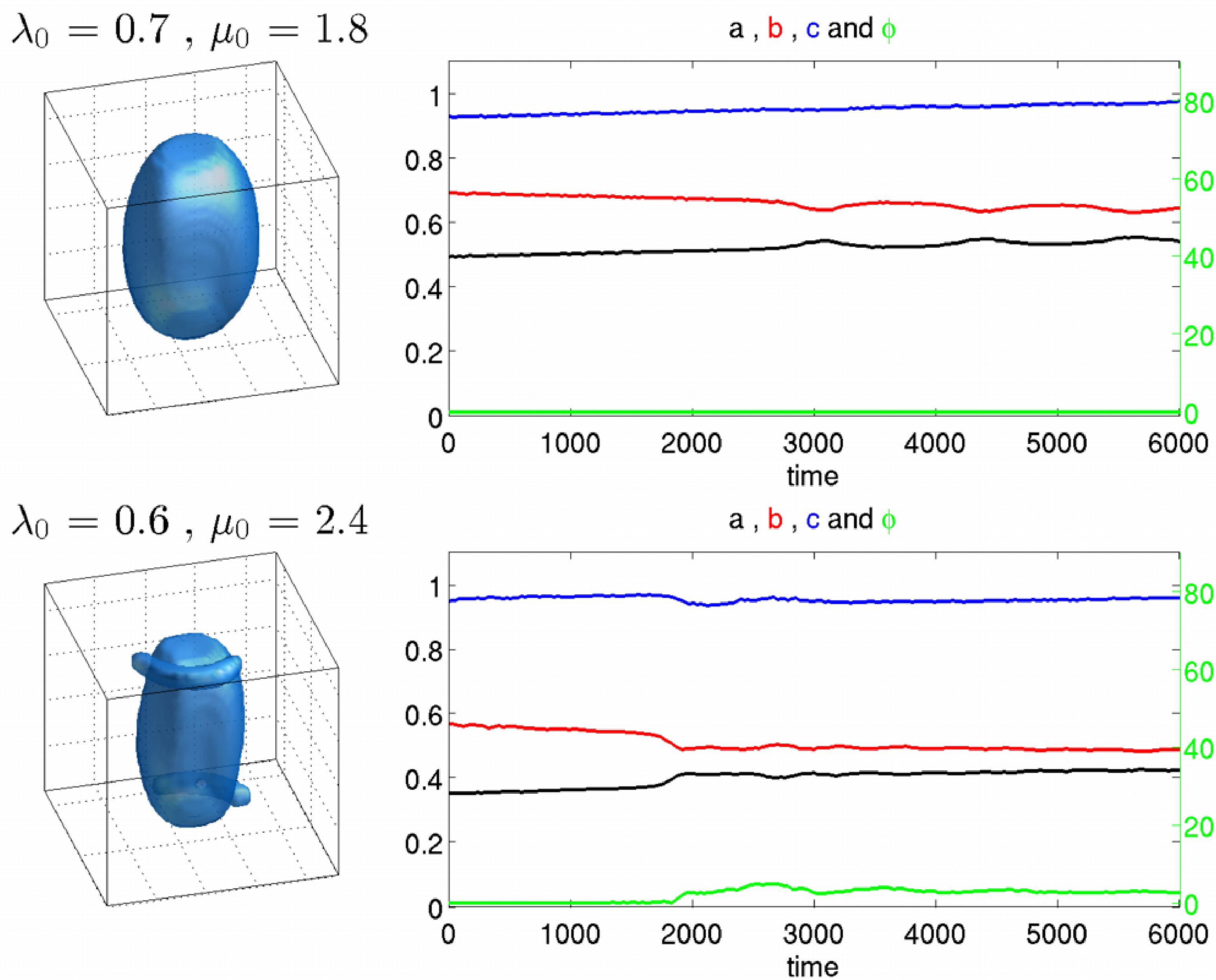
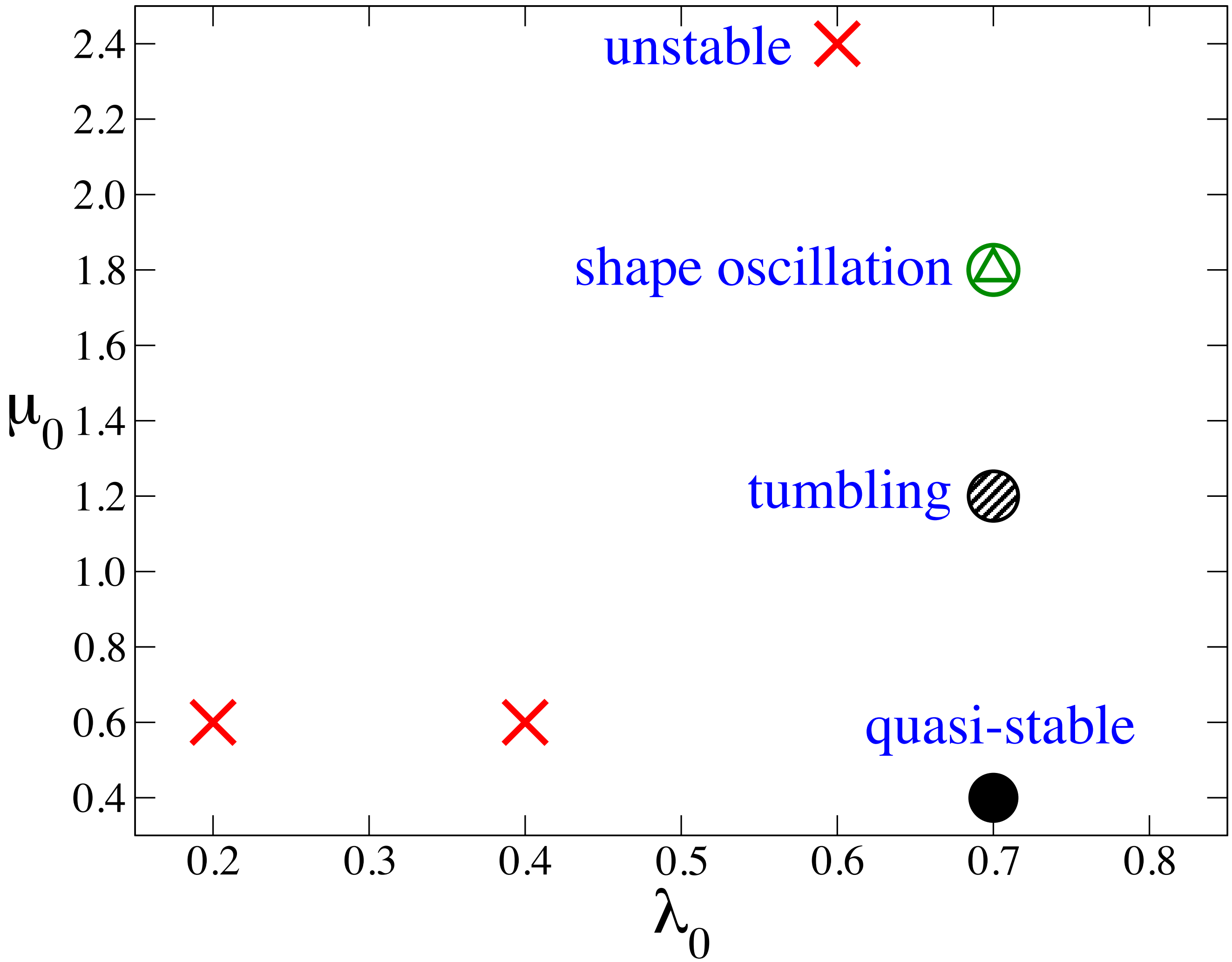
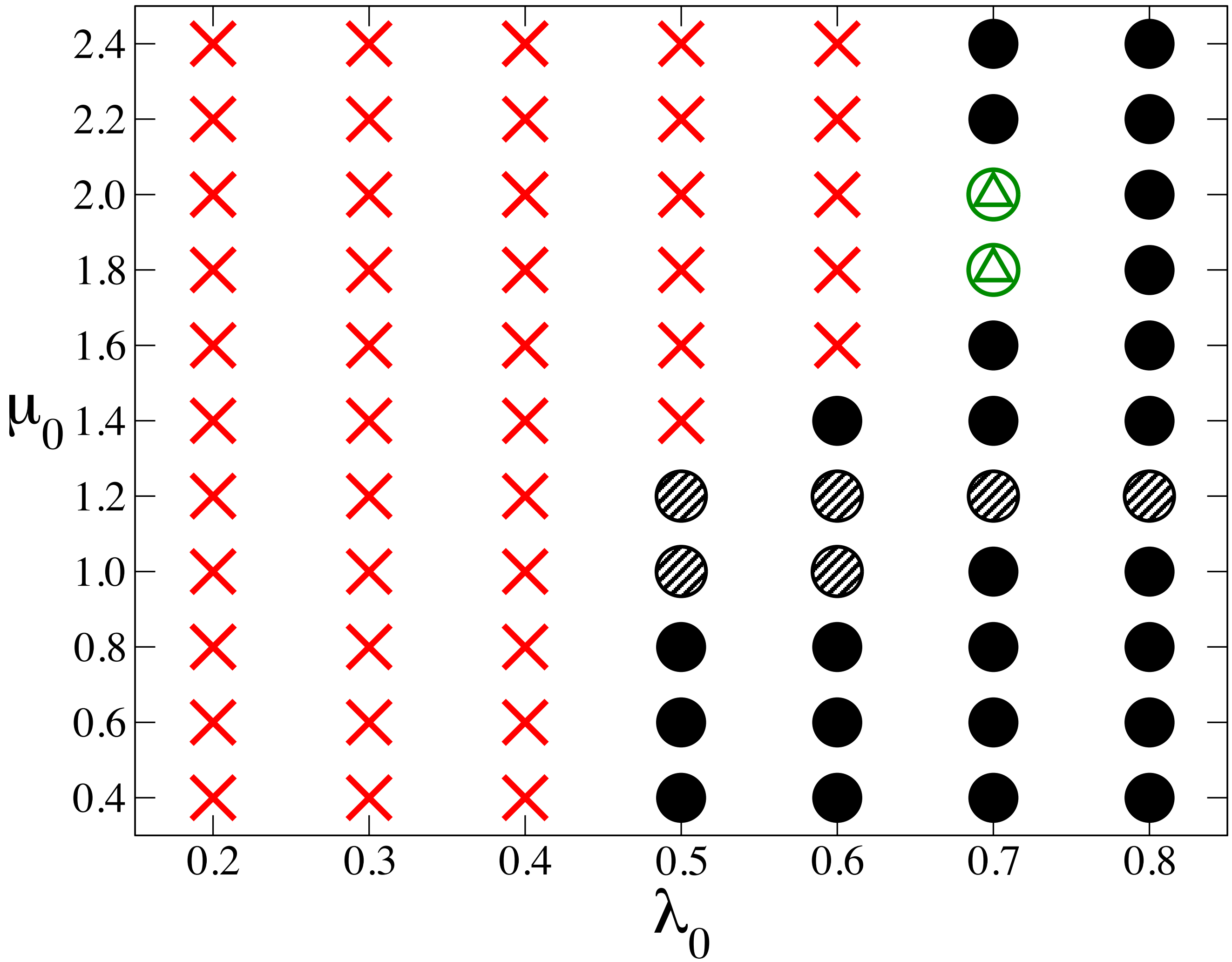


Various phases for $q_0 = 0.5$

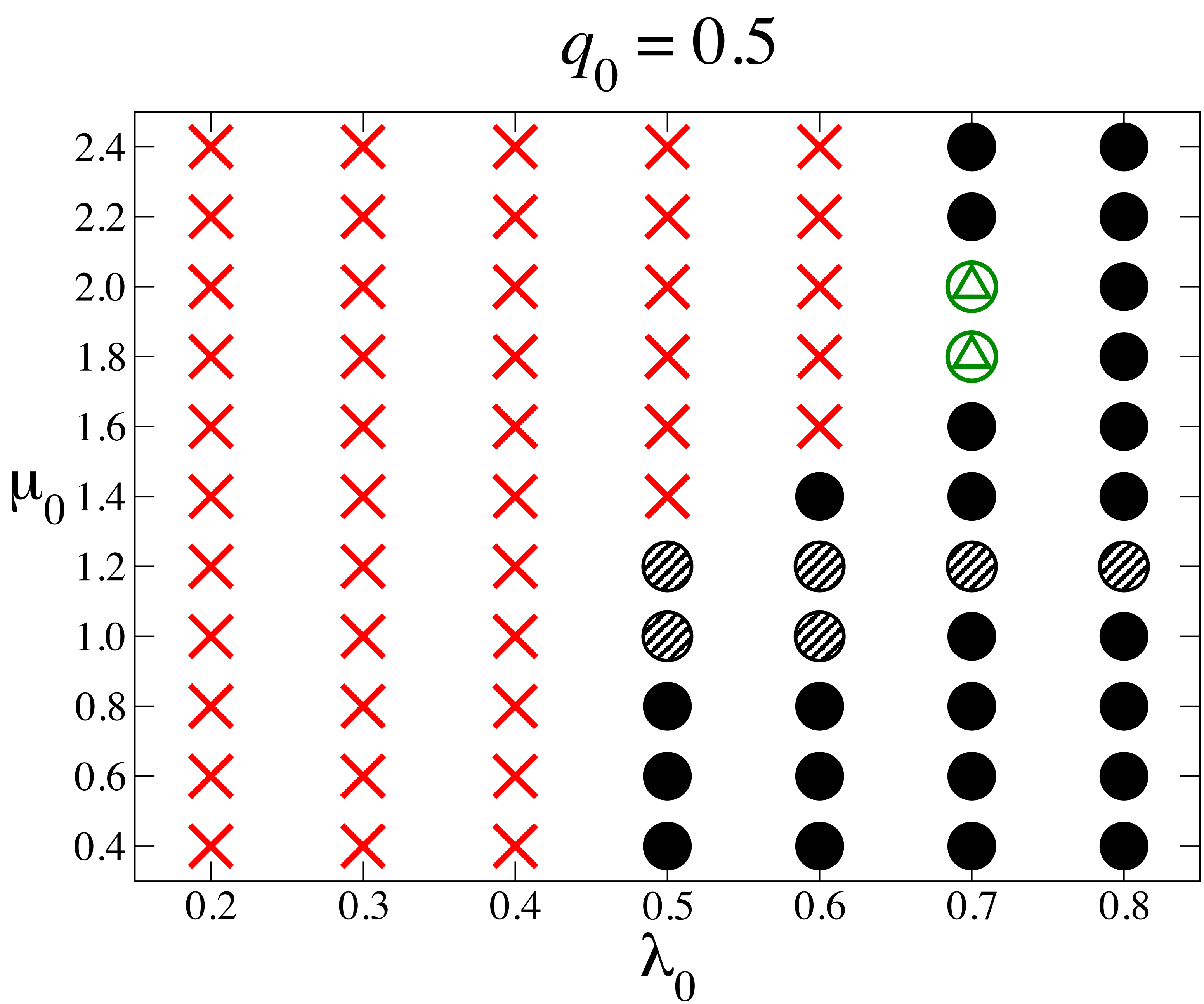


Stability and vortex geometry: $q_0 = 0.5$



- oblate (small $\mu = \sigma^{-1} c / \sqrt{ab}$) vortices are more stable
- vortices with close to circular cross-section (large $\lambda = a/b$) are more stable

Stability and the Rossby number q_0



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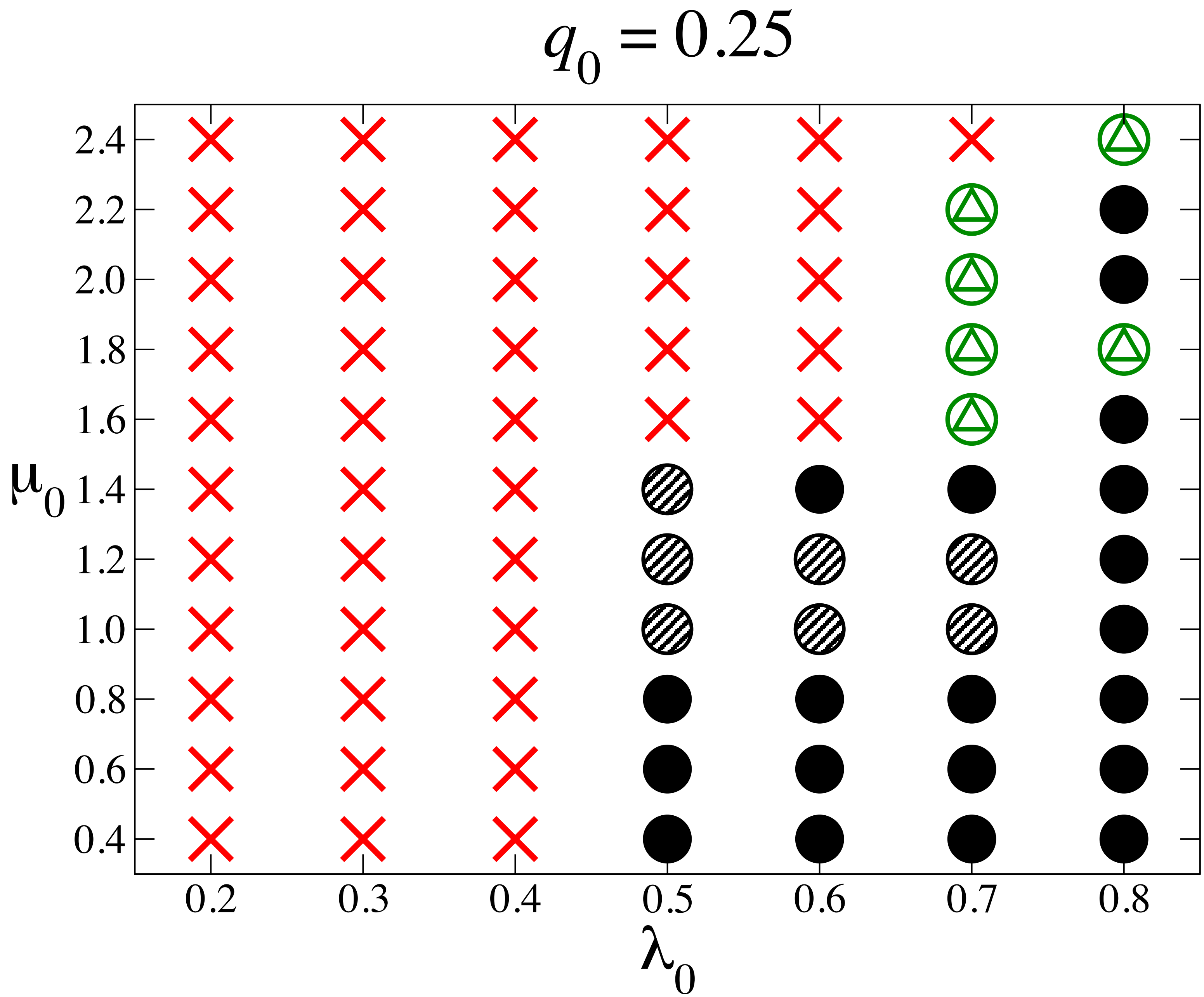

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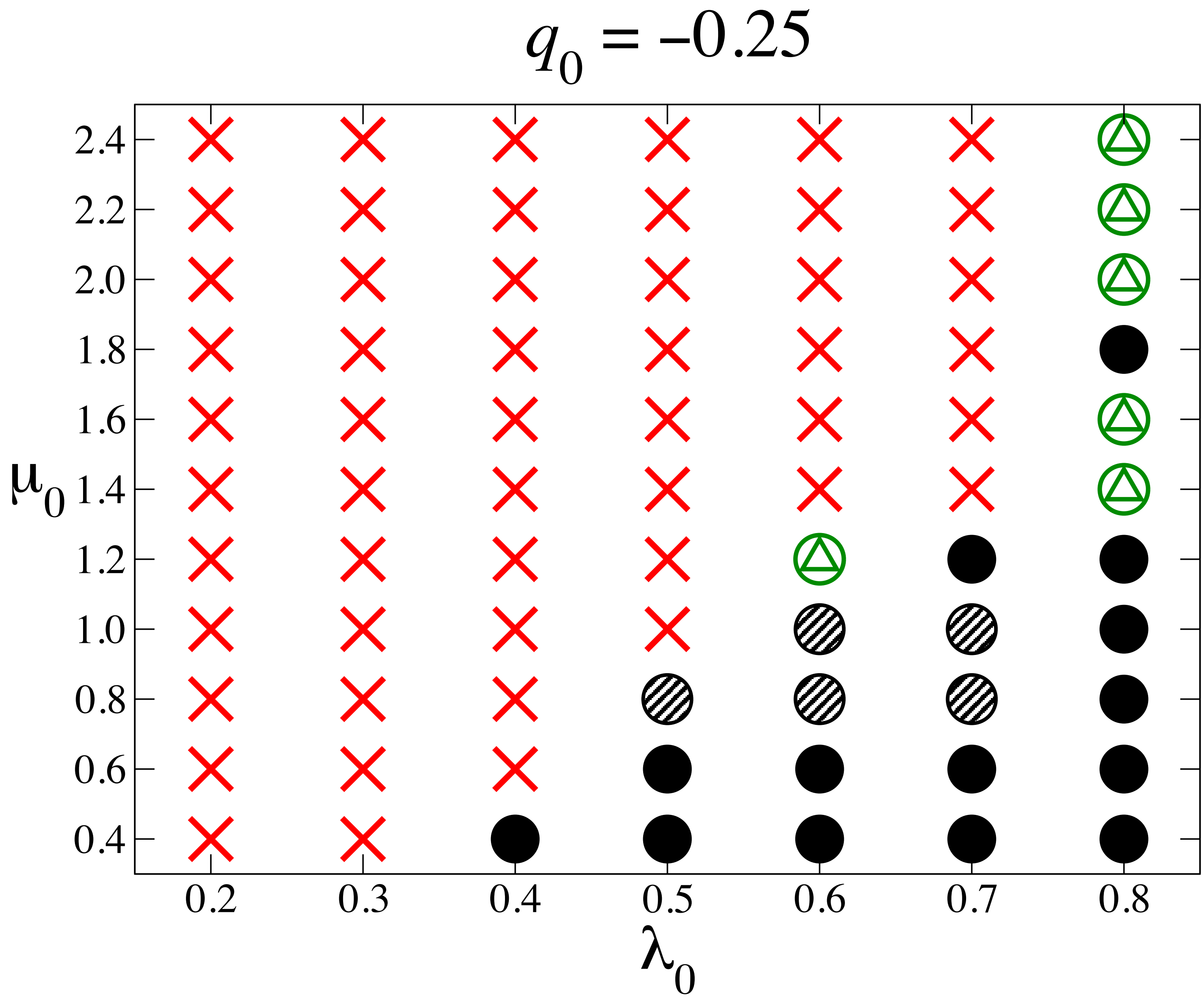

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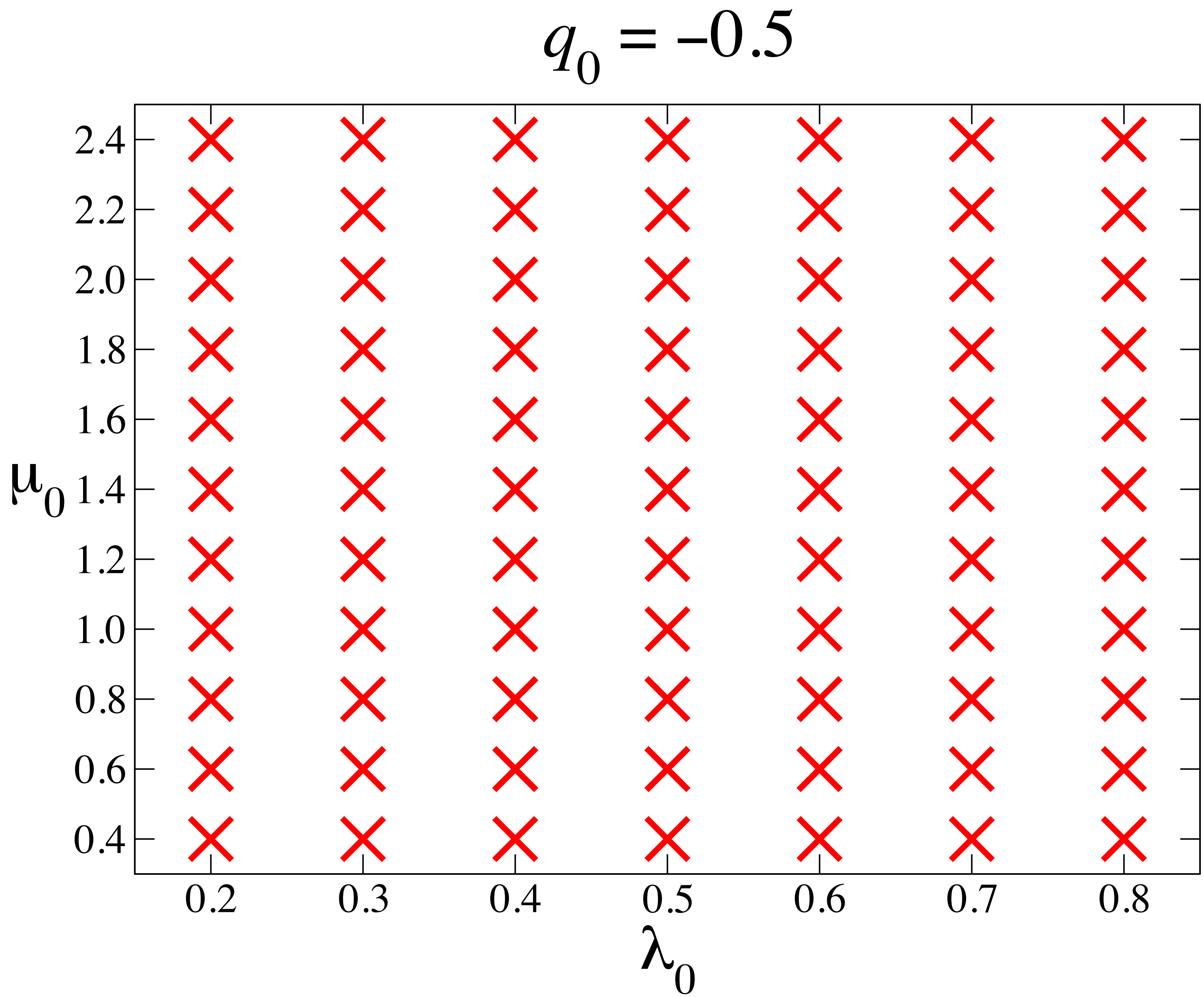

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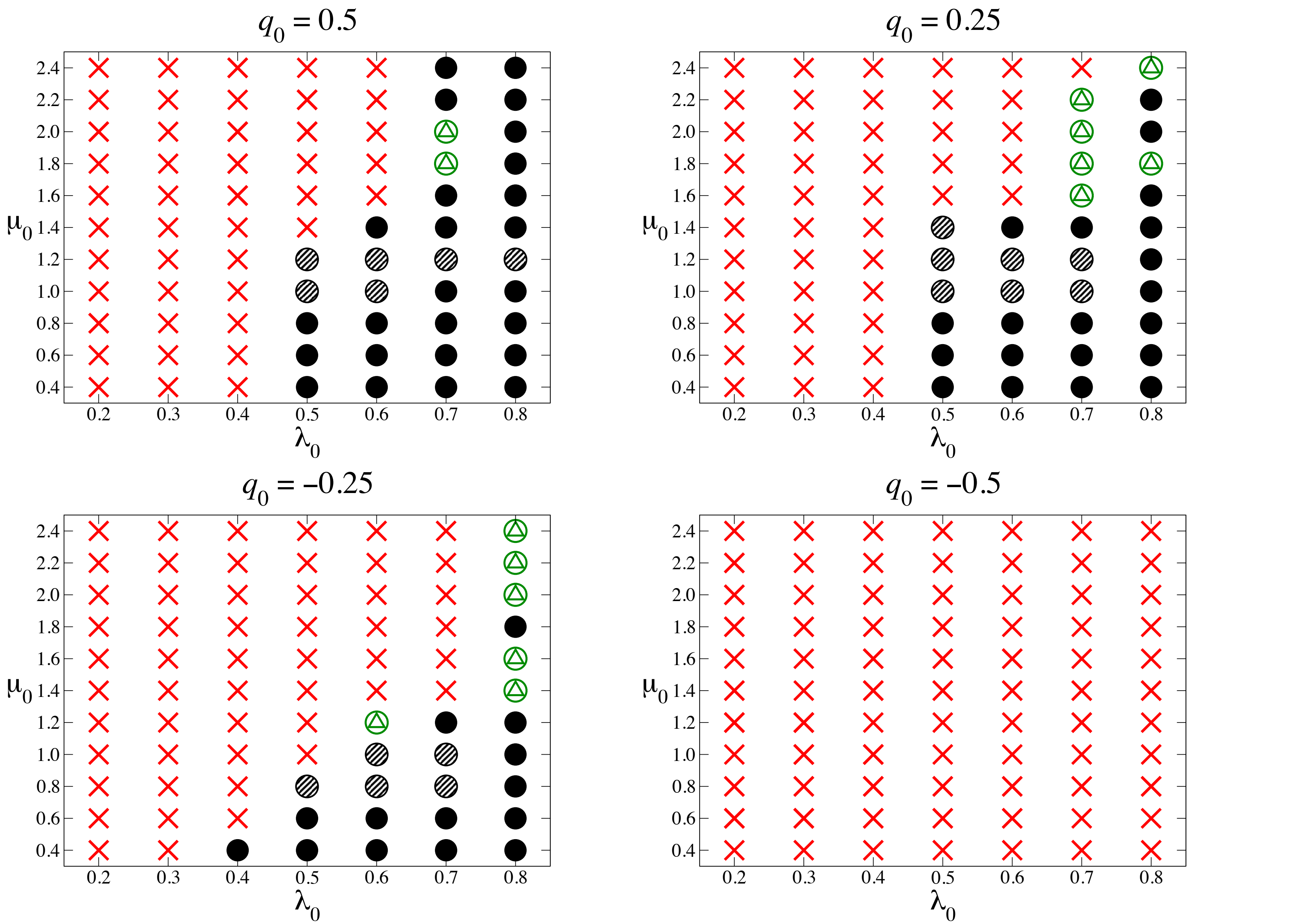
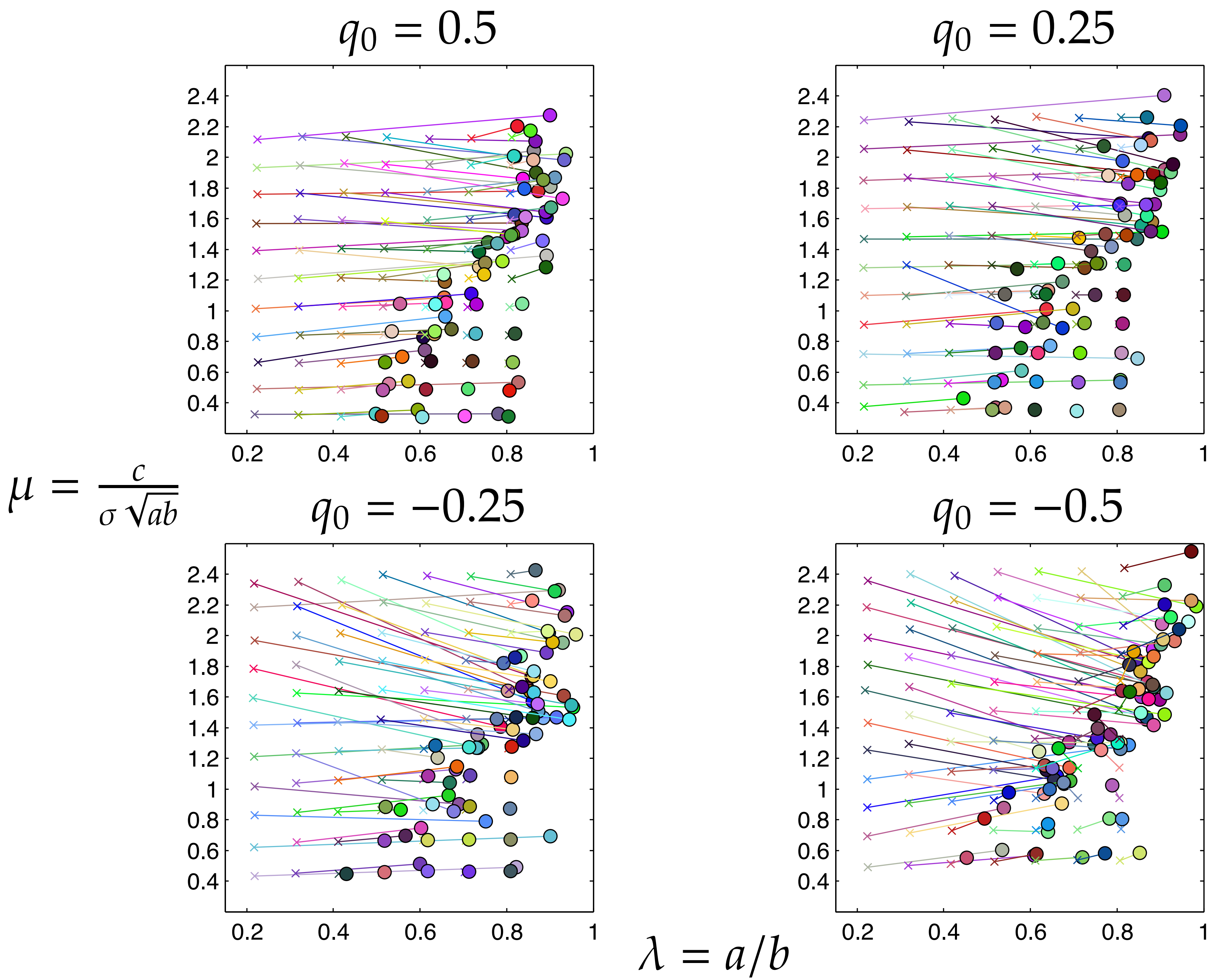
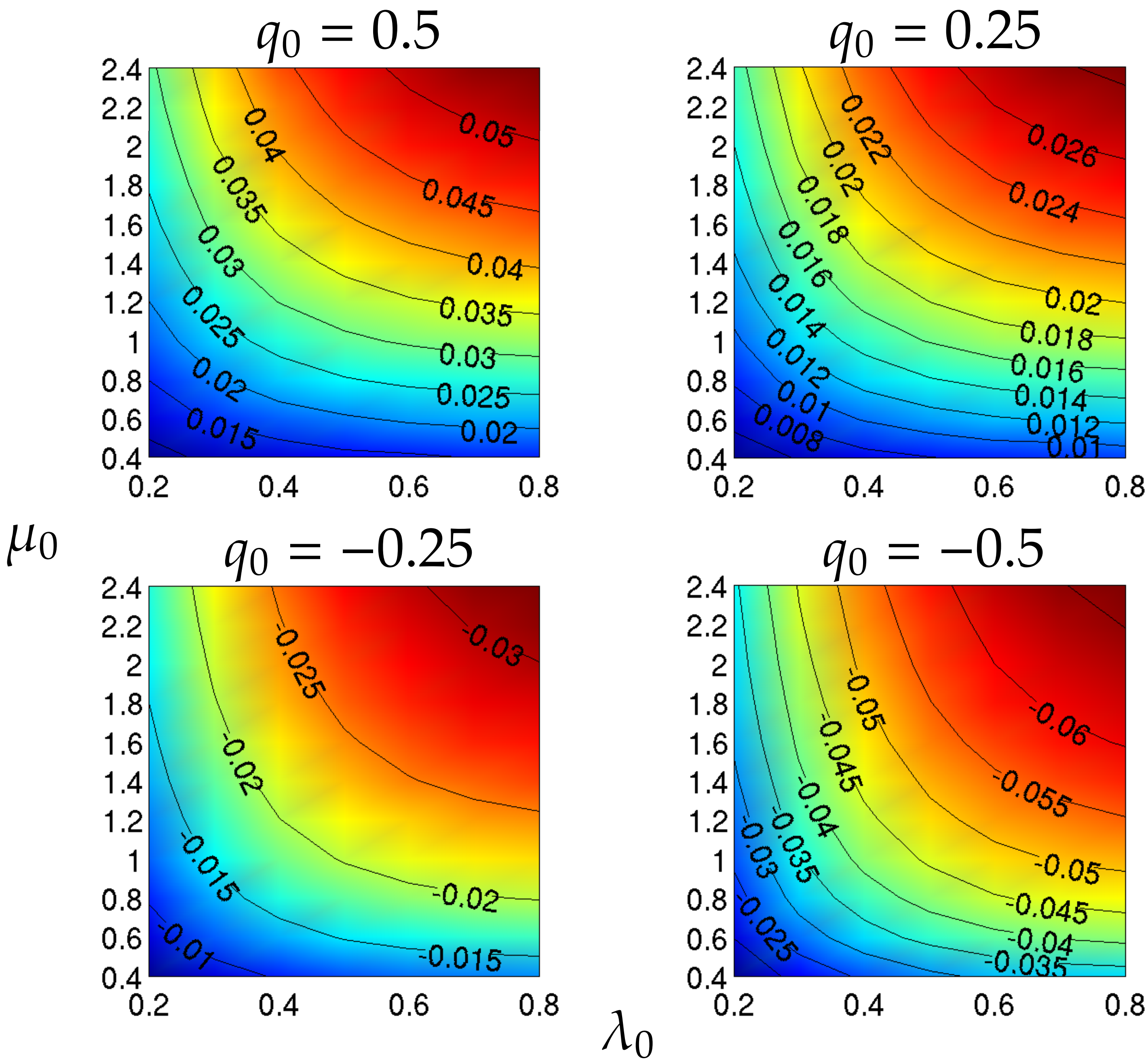


- For a given (λ_0, μ_0), stability generally decreases with q_0
- cyclones are more stable than anti-cyclones

Where do the unstable vortices go?

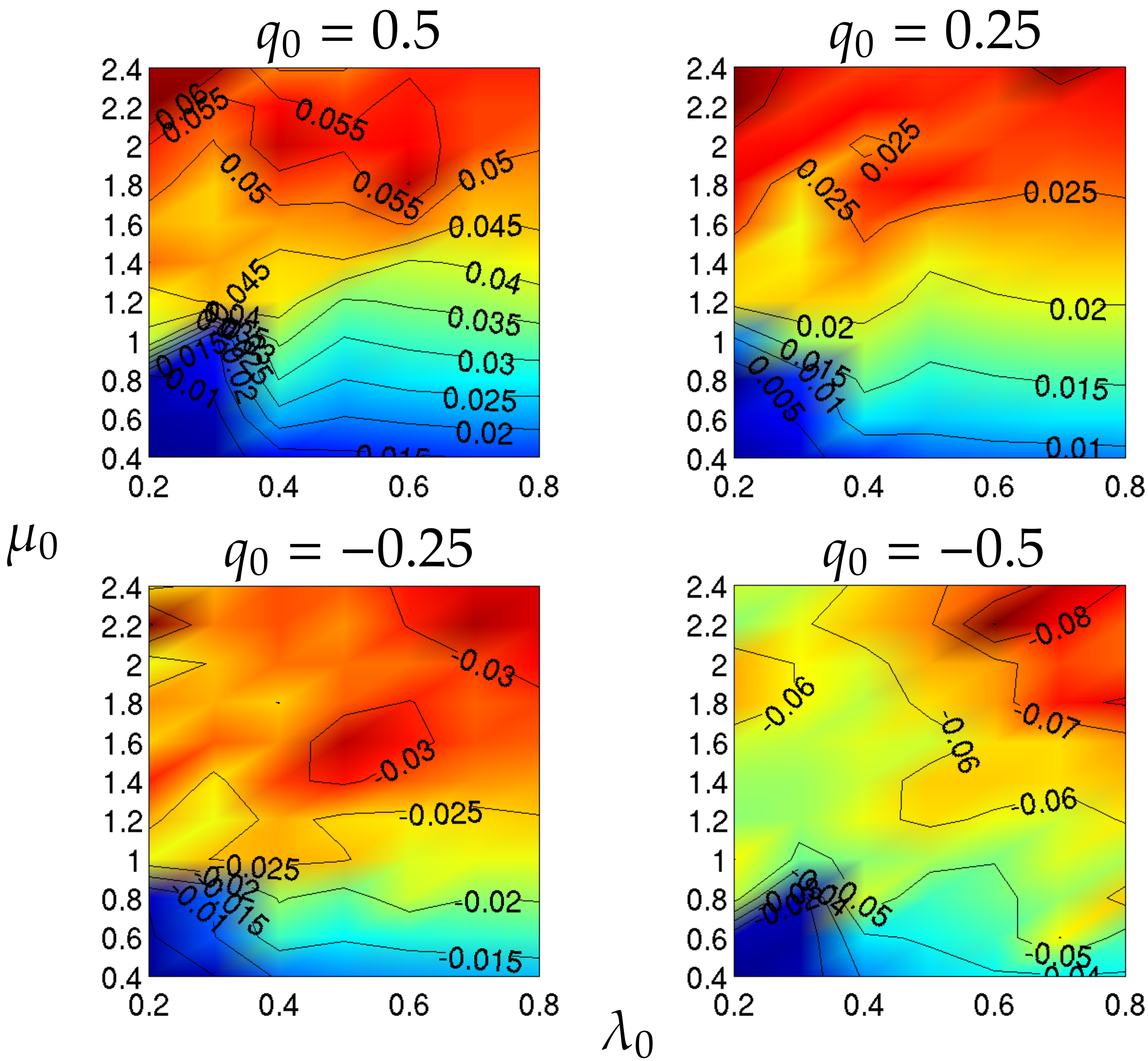


Initial vortex rotation rate at $t = 0 : \Omega_0$



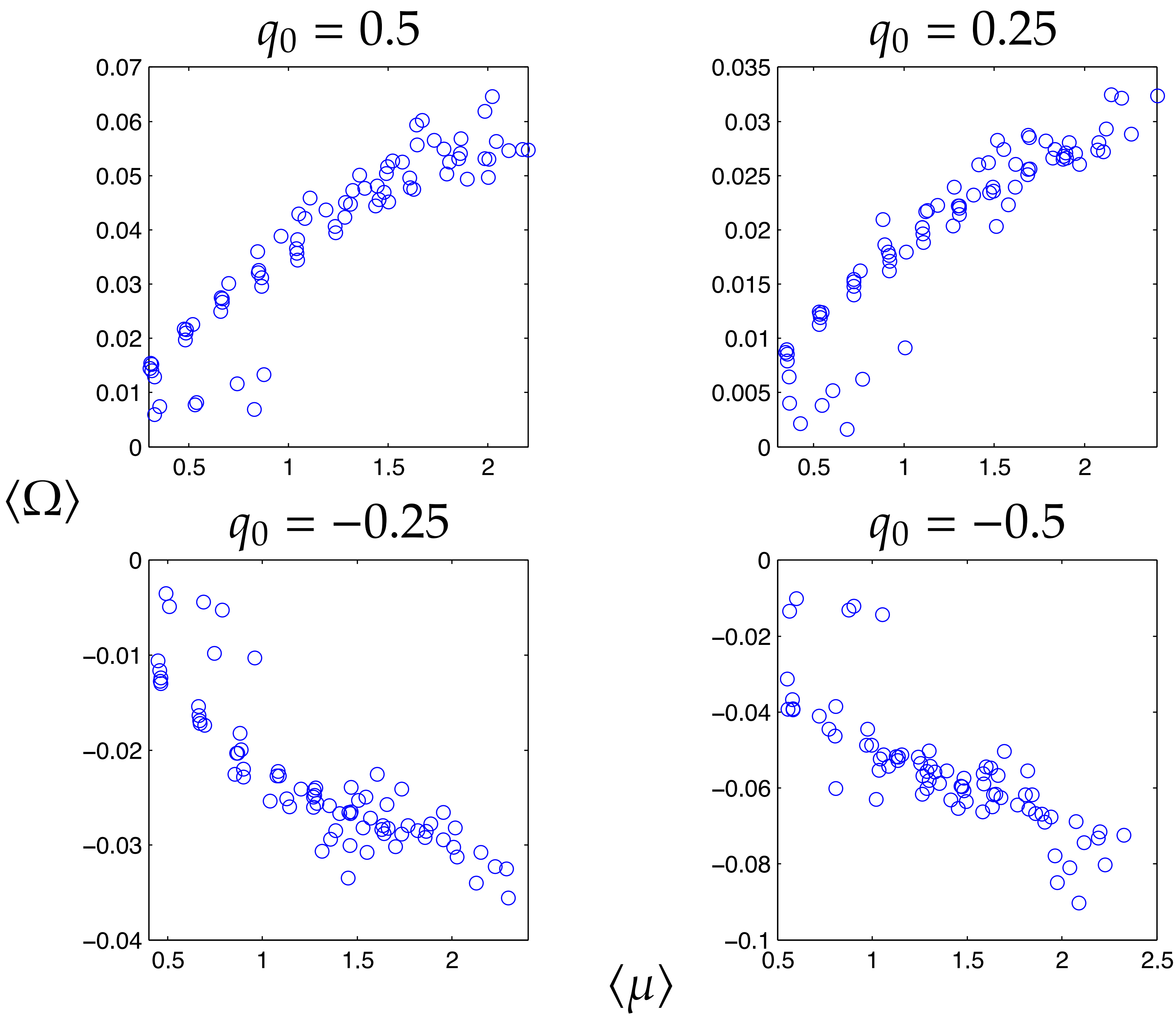
At $t = 0$, all vortices are ellipsoidal

Average vortex rotation rate $\langle\Omega\rangle$

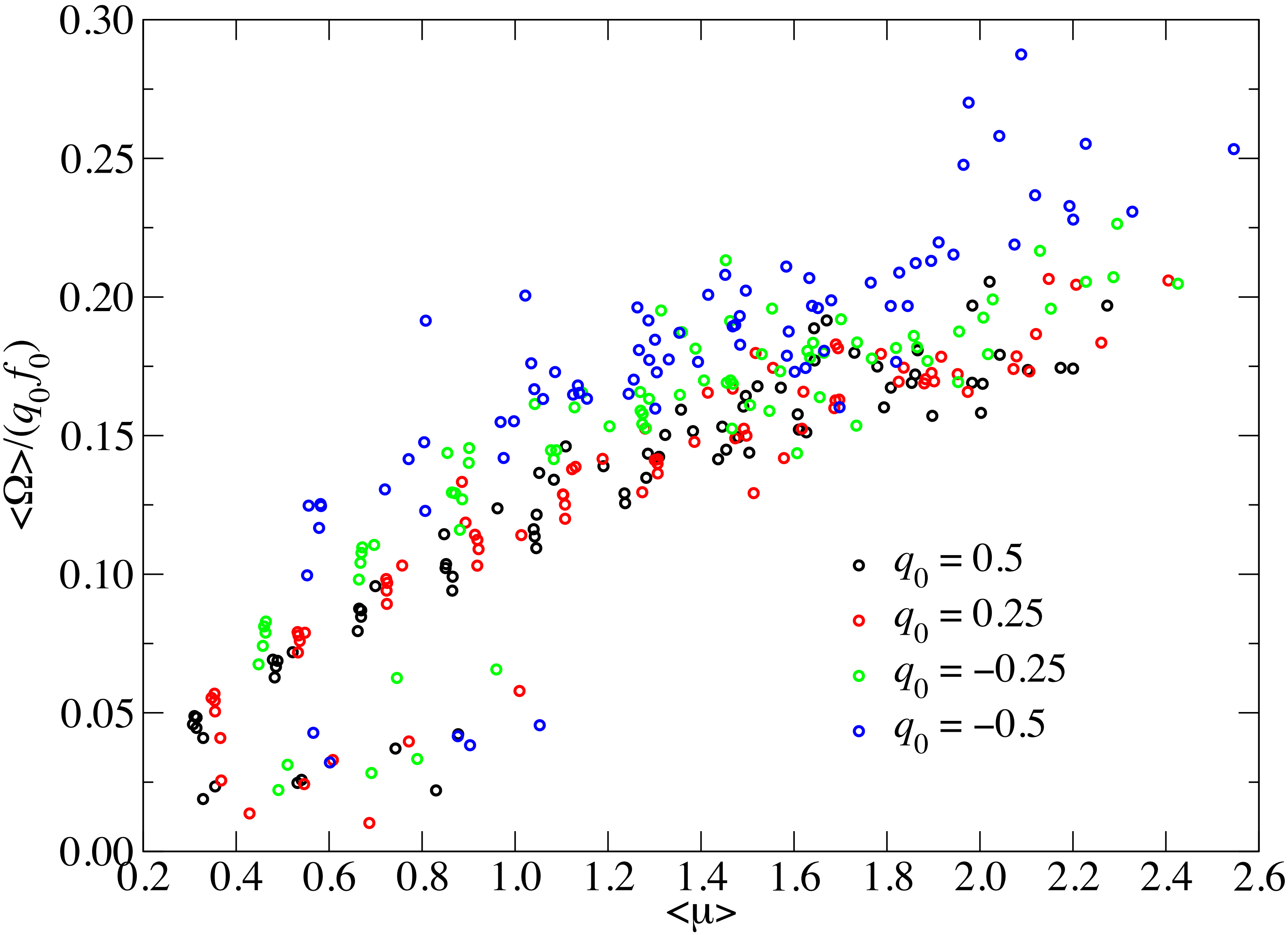


The time-averaged rotation rate $\langle\Omega\rangle$ depends only weakly on λ

Vortex rotation rate and the vertical aspect ratio

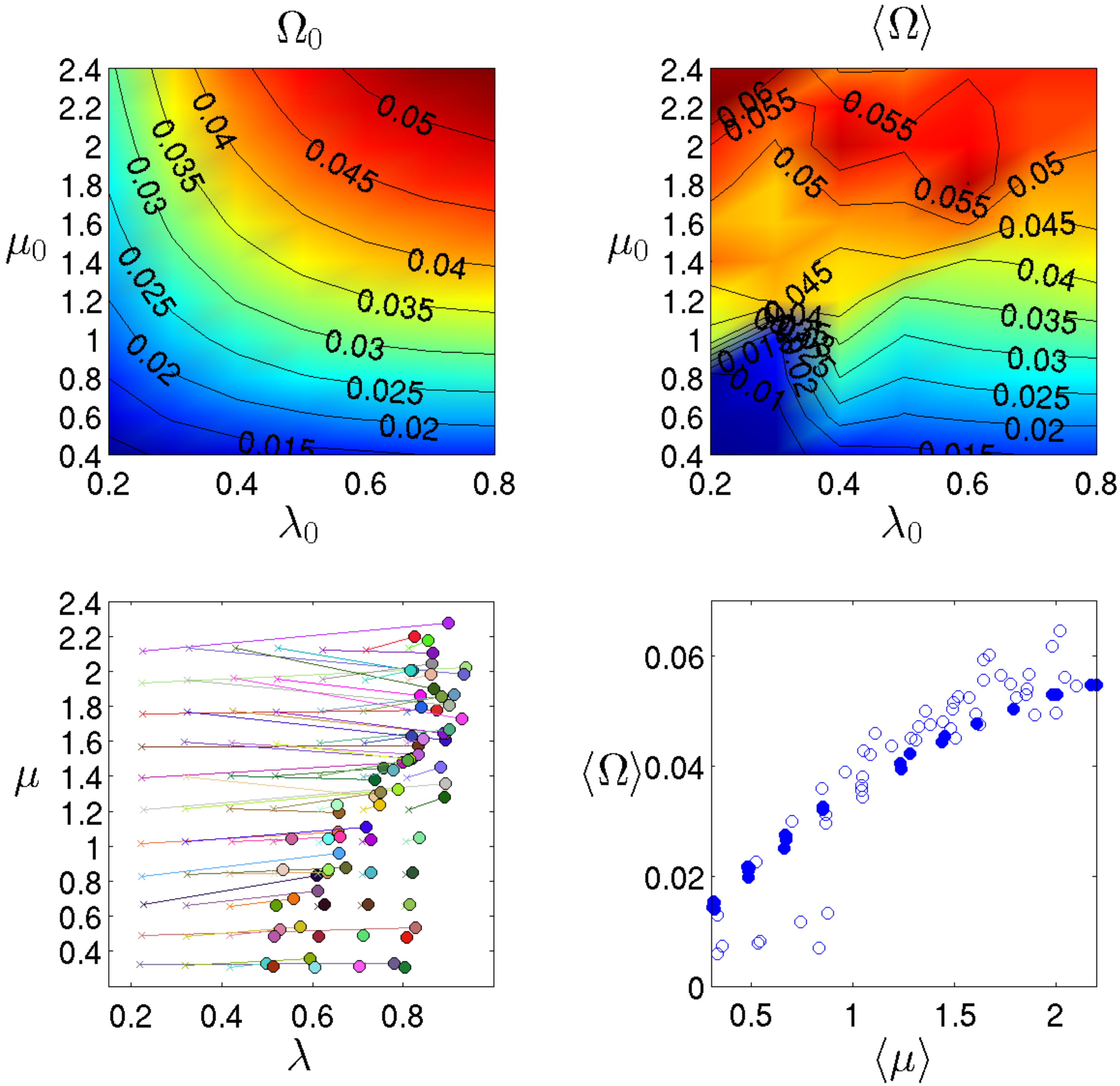


Vortex rotation rate and the vertical aspect ratio



A strong correlation between $\langle \Omega \rangle$ and $\langle \mu \rangle$

Correlation between $\langle\Omega\rangle$ and $\langle\mu\rangle$: $q_0 = 0.5$



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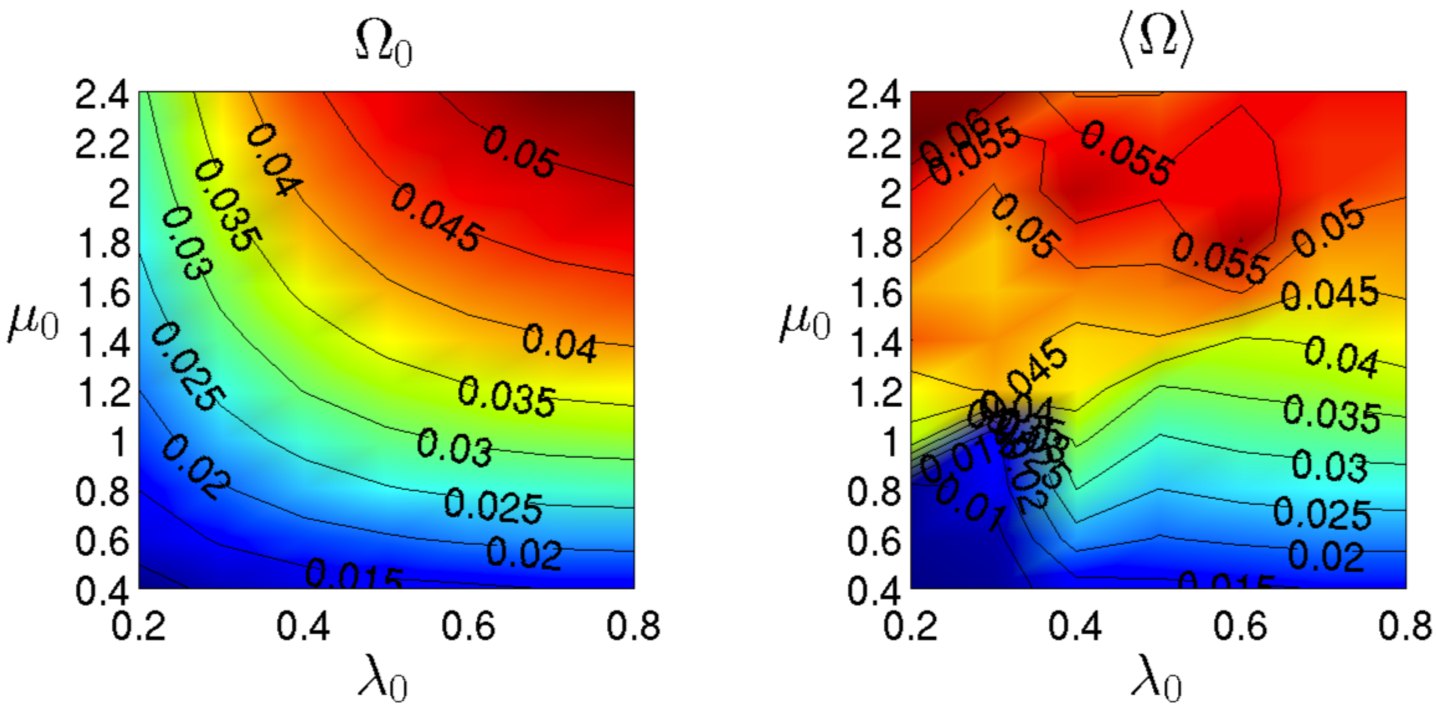
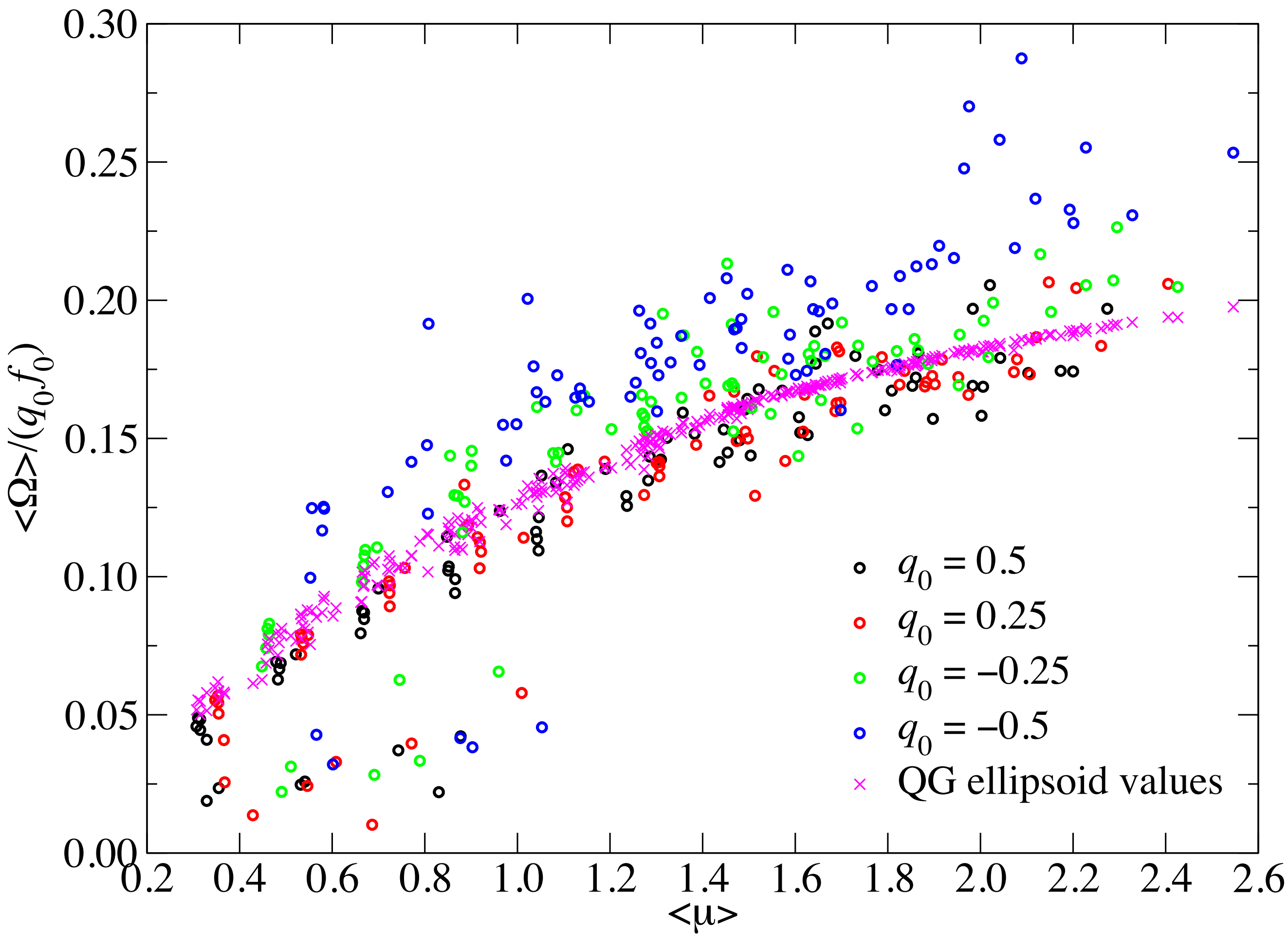


At time $t = 0$:

- the vortex is a perfect ellipsoid
- the vortex is in a near balanced state
- expect QG dynamics to be a good approximation
- $\Omega_0 \approx \Omega_{QG}$

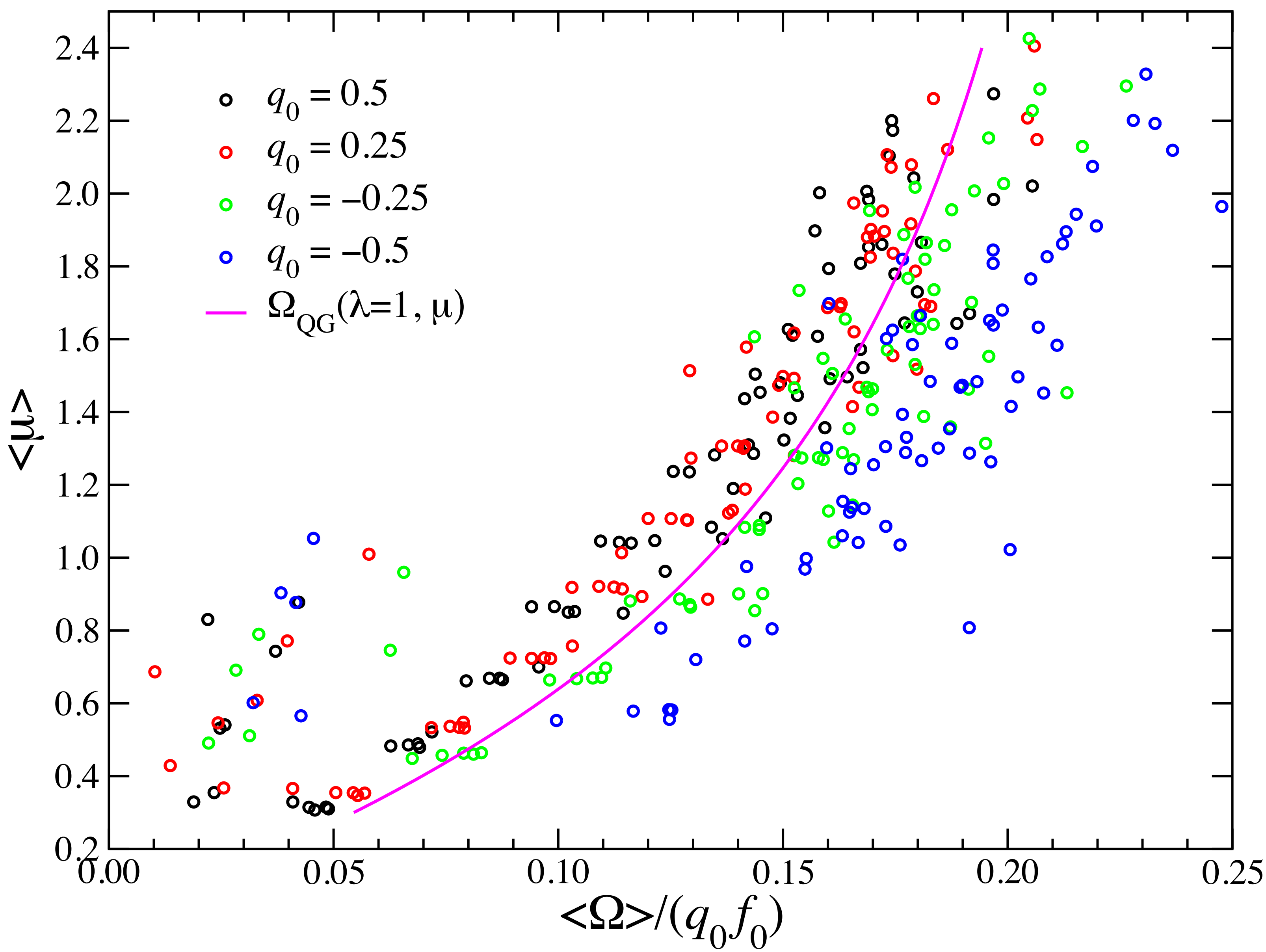
Correlation between $\langle \Omega \rangle$ and $\langle \mu \rangle$



$$\Omega_{QG}(\lambda, \mu) = \mu \, \frac{\lambda^{-1} R_D(\mu^2, \lambda, \lambda^{-1}) - \lambda \, R_D(\mu^2, \lambda^{-1}, \lambda)}{3(\lambda^{-1} - \lambda)}$$

Carlson's symmetric elliptic integral: $R_D(x, y, z) = \frac{3}{2} \int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t+x)(t+y)(t+z)^3}}$

Aspect ratio in terms of vortex rotation rate



$\lambda = 1\,(a = b)$: a spheroid

Balanced models and inertia-gravity waves

- Quasi-geostrophic model in the limit $\mathrm{Ro} \to 0$ (rapid rotation) and $\mathrm{Fr} \to 0$ (strong stratification):

$$\frac{\mathrm{D}q_{\mathrm{QG}}}{\mathrm{D}t} = 0 \quad , \quad q_{\mathrm{QG}} = (\nabla_h^2 + \sigma^2 \partial_z^2)\phi$$

$$u_{\mathrm{QG}} \sim -\partial_y \phi \quad , \quad v_{\mathrm{QG}} \sim \partial_x \phi \quad , \quad b_{\mathrm{QG}} \sim \partial_z \phi$$

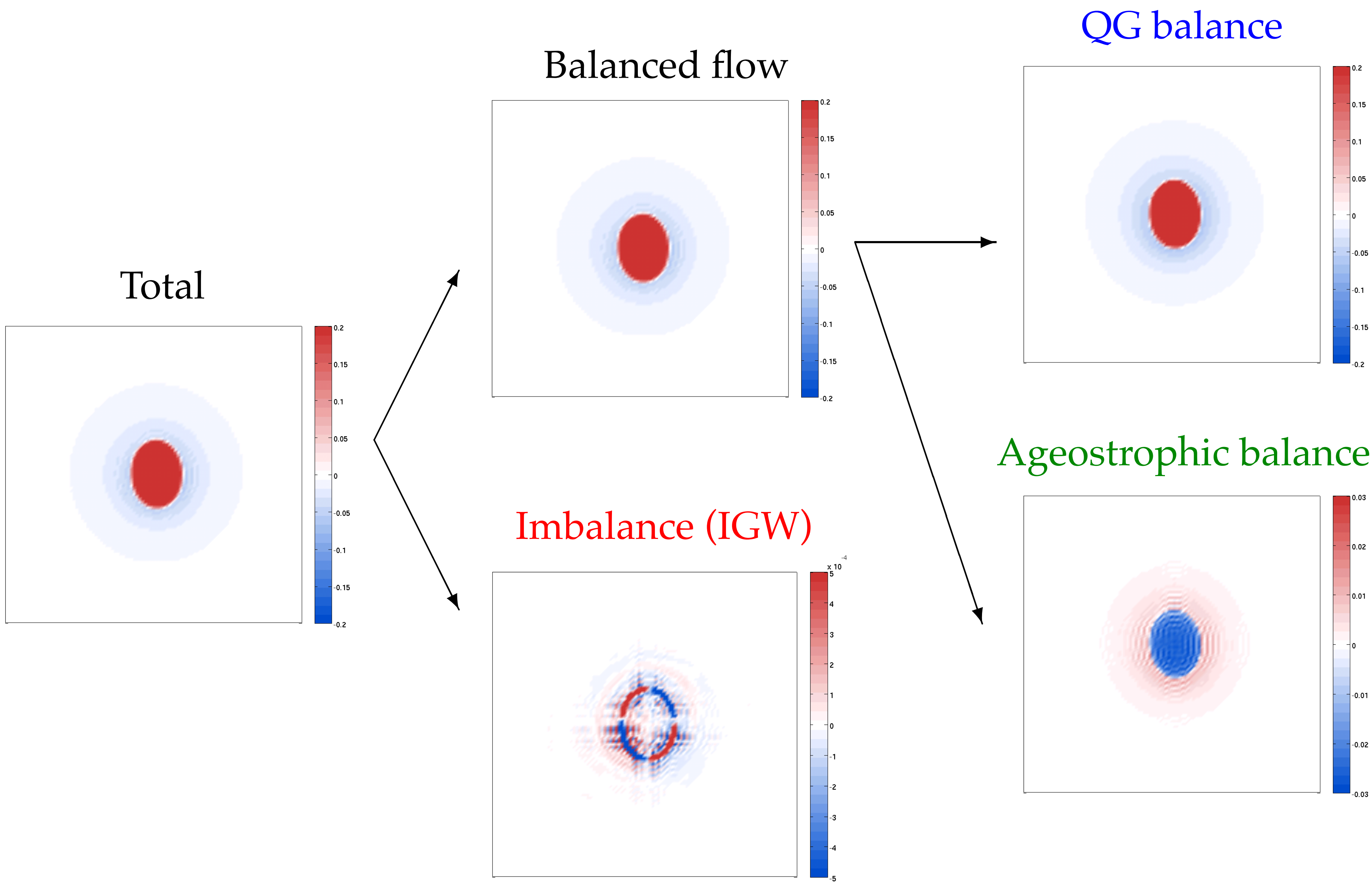
  Pick out the slow vortical component and filter out the fast inertia-gravity waves

- It is possible to derive higher-order balanced models by expansion beyond the lowest order in Ro:

$$\phi = \phi_0 + \mathrm{Ro}\,\phi_1 + \mathrm{Ro}^2\,\phi_2 + \cdots \,, \quad \mathrm{Ro} \ll 1$$

- Imbalance: inertial gravity waves

Balanced and imbalanced components



Balance-imbalance flow decomposition

$$u = \underbrace{u_{\mathrm{QG}} + u_{\mathrm{AGb}}}_{u_{\mathrm{bal}}} + u_{\mathrm{imb}}$$

- QG component:

$$q = (\nabla_h^2 + \sigma^2 \partial_z^2)\, \phi_{\mathrm{QG}}$$

$$u_{\mathrm{QG}} = -f_0 \frac{\partial \phi_{\mathrm{QG}}}{\partial y}$$

- balanced component: u_{bal} is obtained by "Optimal PV balance" — an iterative procedure to find $\vec{A}_h$ for a given q such that the flow $\{q, \vec{A}_h\}$ has minimal IGW emission

Then, one can obtain:

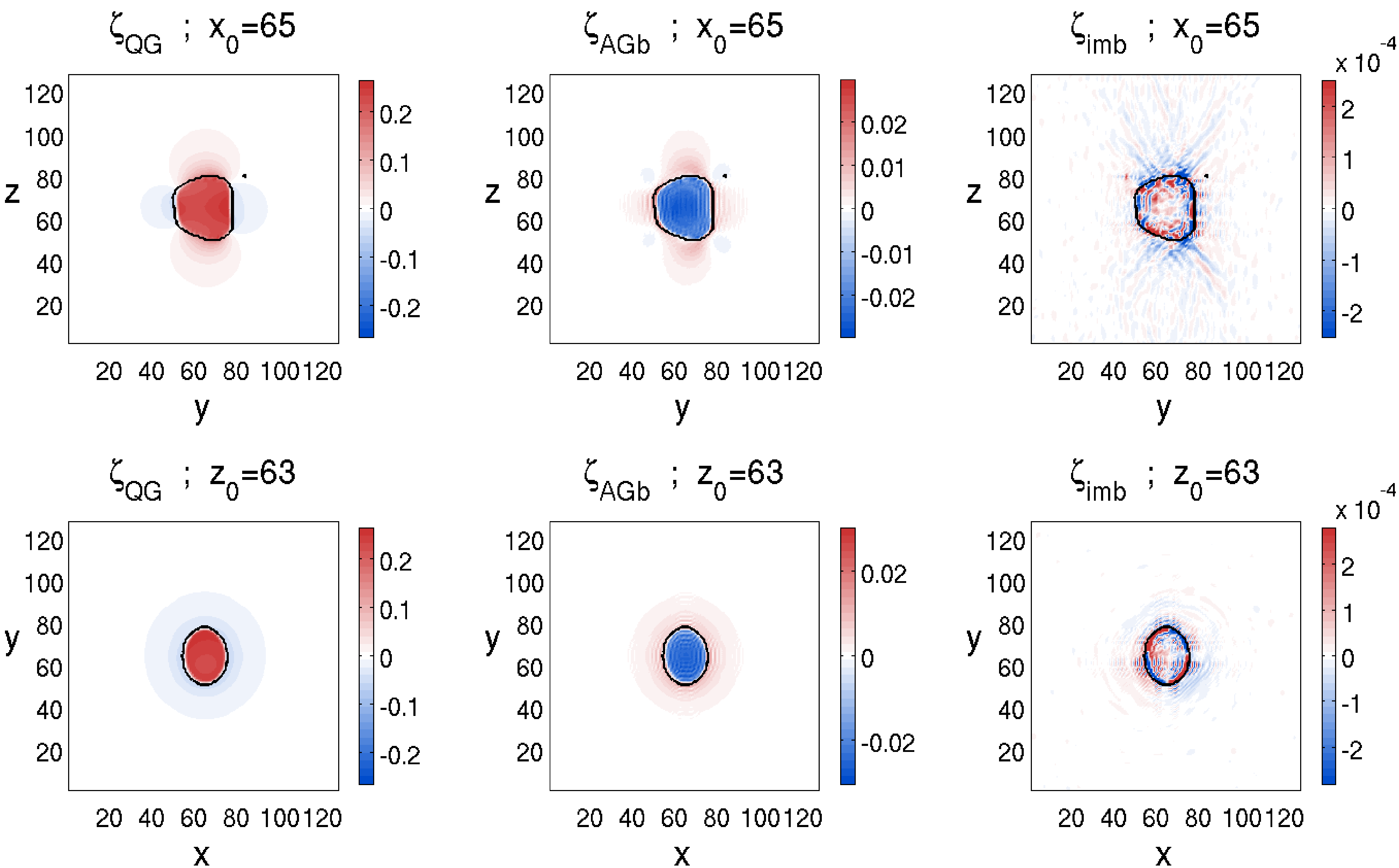
$$u_{\mathrm{AGb}} = u_{\mathrm{bal}} - u_{\mathrm{QG}}$$

$$u_{\mathrm{imb}} = u - u_{\mathrm{bal}}$$

Optimal PV balance: Viúdez & Dritschel, JFM 521, 343 (2004)

Examples of balanced and imbalanced fields

$$q_0 = 0.5\ ,\ \lambda_0 = 0.3\ ,\ \mu_0 = 1.6$$



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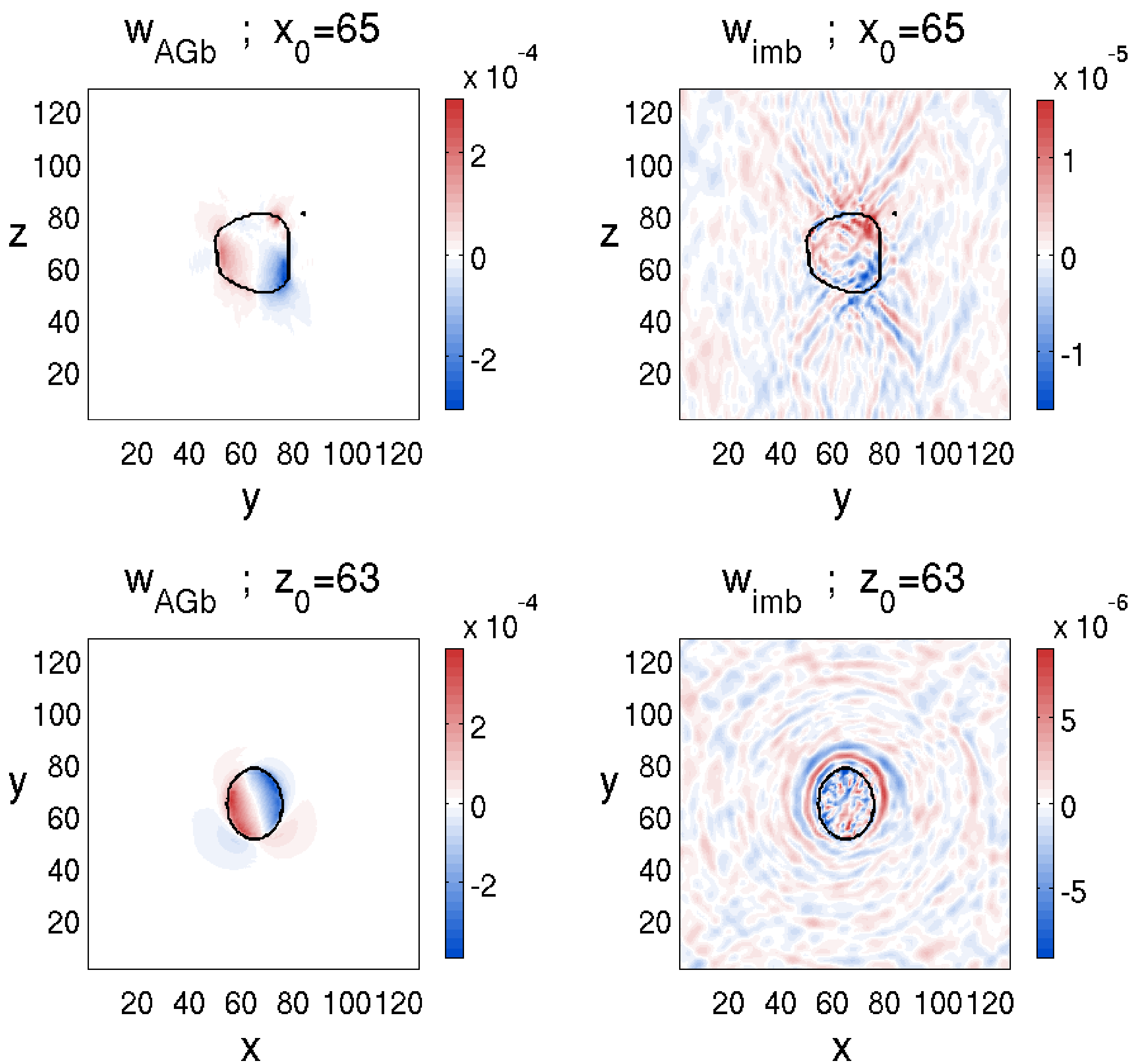
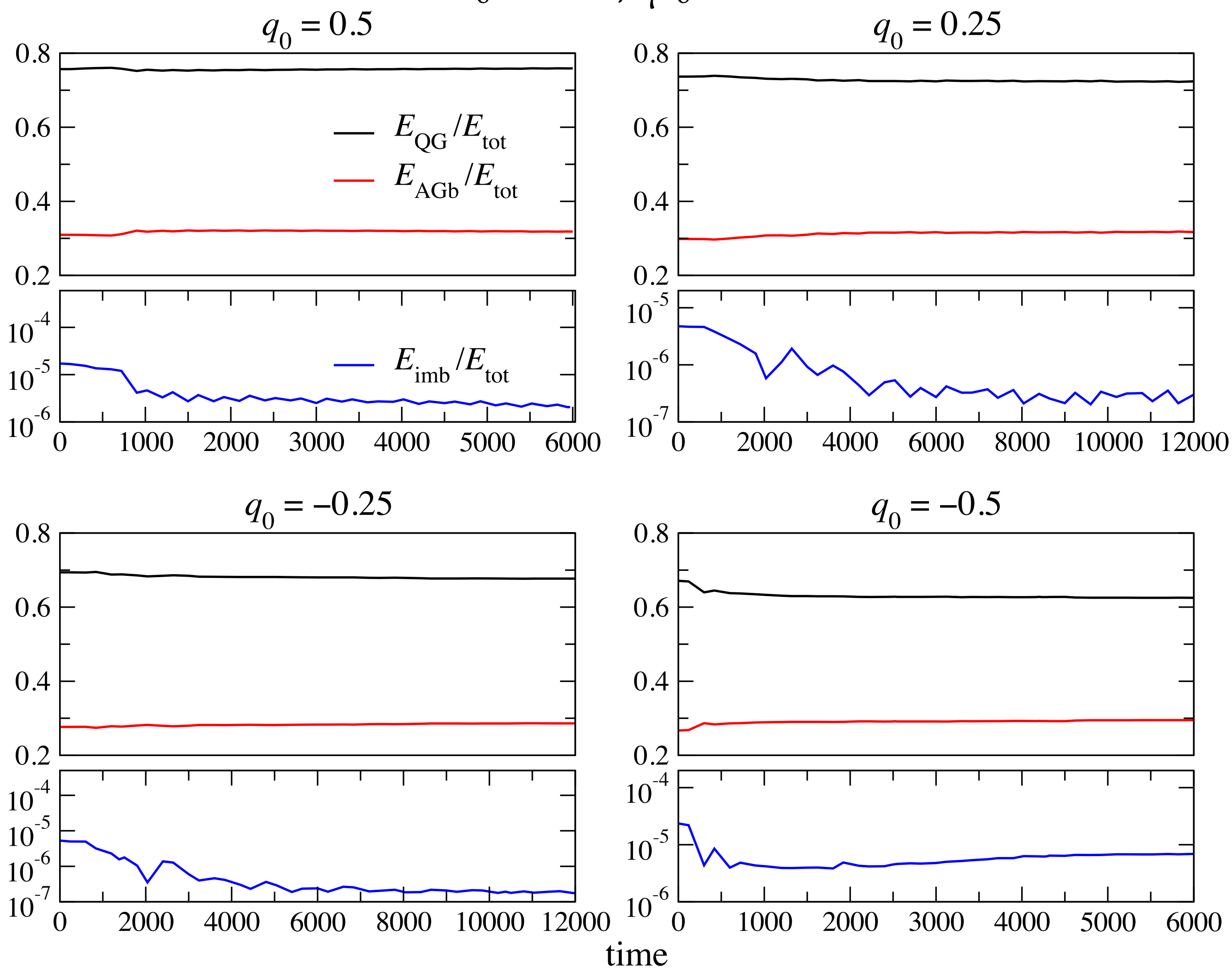


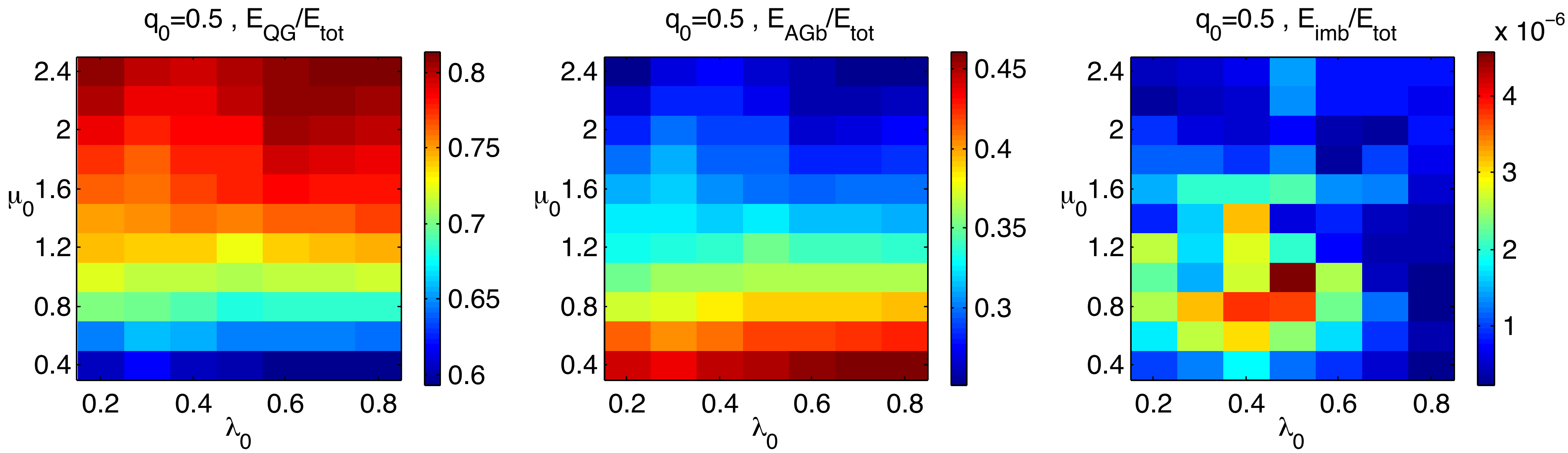
Quantifying balance-imbalance using energy norm

$\lambda_0 = 0.3\ ,\ \mu_0 = 1.6$



$$E_i = \frac{1}{2}\int (u_i^2 + v_i^2 + w_i^2 + \frac{b_i^2}{N^2})\, d\vec{x}\,, \quad i = \text{QG or AGb or imb}$$

Quantifying balance-imbalance using energy norm



- E_{QG} is the largest component in all cases
- E_{imb} is several orders of magnitude less than E_{QG} or E_{AGb} indicating the vortex is in an approximate balanced state
- oblate vortices ($\mu < 1$) tends to have the larger ageostrophic-balanced component E_{AGb}
- results are similar for other values of q_0

Summary

- oblate vortices are more stable
- vortices often attain close to circular cross-sections
- cyclones are more stable than anti-cyclones
- investigate the relation between vortex rotation rate and the aspect ratios
- vortices are in near-balanced state even for moderate Rossby number, $|Ro| \sim 0.5$