Ellipsoidal Vortices in Non-Hydrostatic Rotating Stratified Flows: Can They Survive?

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Elliptic and Ellipsoidal Vortex Patch

2D Euler flows

The vorticity equation:

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = \frac{\partial \zeta}{\partial t} + \mathbf{u}\cdot\nabla\zeta = 0$$

$$\zeta = \nabla^2\phi$$

$$(u, v) = (-\phi_y\,, \phi_x)$$

- Kirchhoff's elliptic vortex: an elliptic region (with semi-axes a and b) of uniform vorticity ζ_0 in an otherwise irrotational flow,

$$\text{rotation rate, } \Omega = \frac{a\,b\,\zeta_0}{(a+b)^2}$$

- steady (Moore & Saffman 1975) and unsteady (Kida 1981) elliptic vortex in a uniform shear flow

3D Rotating stably-stratified flows

The quasi-geostrophic approximation (with constant Coriolis frequency f and buoyancy frequency N):

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0$$

$$Q = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{f^2}{N^2}\frac{\partial^2\psi}{\partial z^2}$$

$$(u, v) = (-\psi_y\,, \psi_x)$$

$$Q = \text{quasi-geostrophic potential vorticity}$$

- ellipsoidal vortex of uniform QGPV in a quiescent flow (Meacham 1992),

$$\Omega = \mu\,\frac{\lambda^{-1}E(\mu^2,\lambda,\lambda^{-1}) - \lambda E(\mu^2,\lambda^{-1},\lambda)}{3(\lambda^{-1}-\lambda)}$$

(Laplace 1784), E=elliptic integral of the 2nd kind

– horizontal semi-axes: a and b ($a < b$)

– vertical semi-axis: c

– aspect ratios λ and μ:

$$\lambda = \frac{a}{b}$$

$$\mu = \frac{c}{\sqrt{ab}}$$

- QG ellipsoidal vortex in a flow with uniform horizontal strain and vertical shear (Meacham et al. 1994, Hashimoto et al. 1999)

Ellipsoidal vortices beyond QG

Non-hydrostatic Boussinesq equations

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\hat{\boldsymbol{z}}\times\boldsymbol{u} = -\frac{1}{\rho_0}\nabla\Phi - b\hat{\boldsymbol{z}}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} + N^2 w = 0$$

$$\nabla\cdot\boldsymbol{u} = 0$$

where $\boldsymbol{u} = (u, v, w)$

b = buoyancy

Φ = geopotential

take $f/N = 0.1$

Define the potential vorticity:

$$q = \left(\frac{\boldsymbol{\omega}}{f} + \hat{\boldsymbol{z}}\right)\cdot\nabla\mathcal{Z}$$

where $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ (vorticity)

$\mathcal{Z} = z + \dfrac{b}{N^2}$ (isopycnal ref. height)

- We study numerically the evolution of an ellipsoid of uniform potential vorticity q_0.
- Such an ellipsoidal vortex is not a solution of the non-hydrostatic Boussinesq system and will in general emit internal gravity waves (IGW).

Numerical methods

- Recasting the prognostic variables

$$(u, v, b) \to (\boldsymbol{A}_h, q)$$

$$\text{where } \boldsymbol{A} = (\boldsymbol{A}_h\,, A_3) = \frac{\boldsymbol{\omega}}{f} + \frac{\nabla b}{f^2}$$

$$\frac{\mathrm{D}\boldsymbol{A}_h}{\mathrm{D}t} = \mathcal{N}(\boldsymbol{A}_h, \boldsymbol{u}, b)$$

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \quad \text{(PV conservation)}$$

The PV equation is solved using the Contour Advection Semi-Lagrangian (CASL) algorithm.

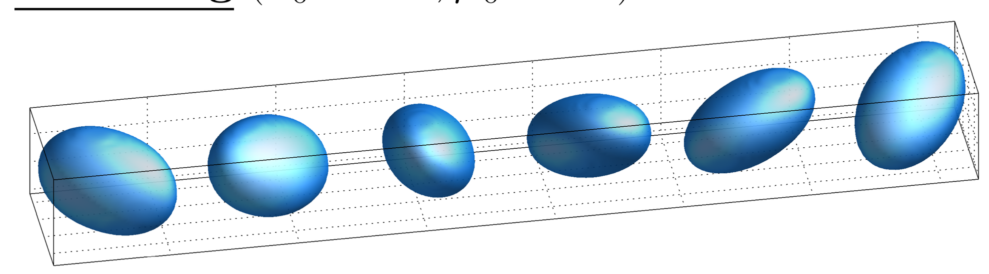
- Dynamic PV initialisation:

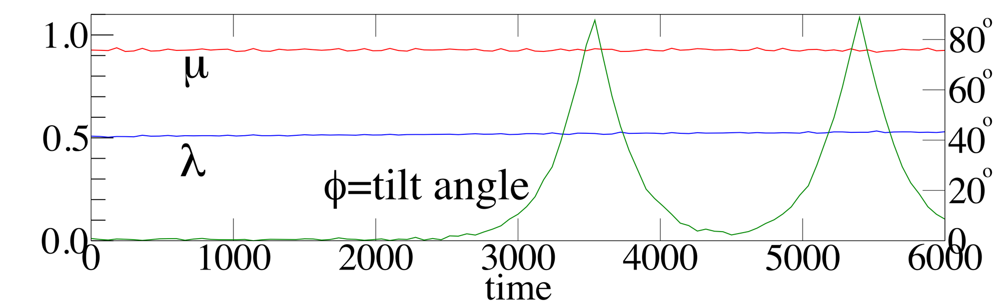
– need to generate the initial vortex in a near balance state to minimize IGW emission

– initialisation period: smoothly ramp up the PV inside the ellipsoid from 0 to q_0 while time stepping the equations of motion forward

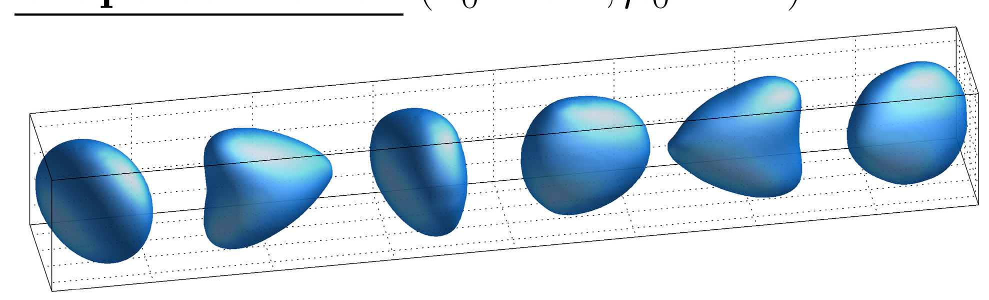
Nonlinear Evolution: Ro = 0.25

Tumbling ($\lambda_0 = 0.5\,, \mu_0 = 1.0$)

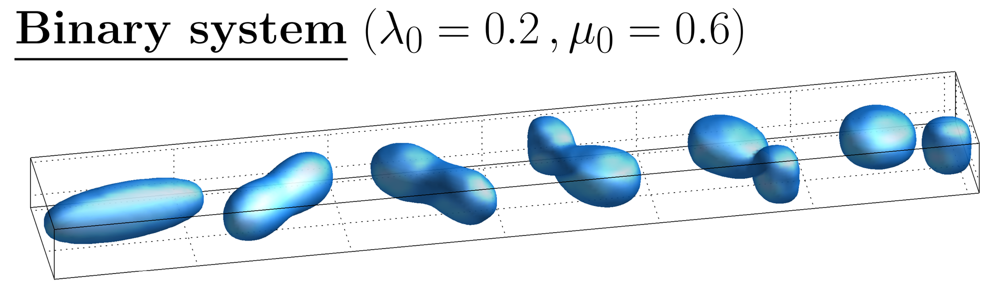




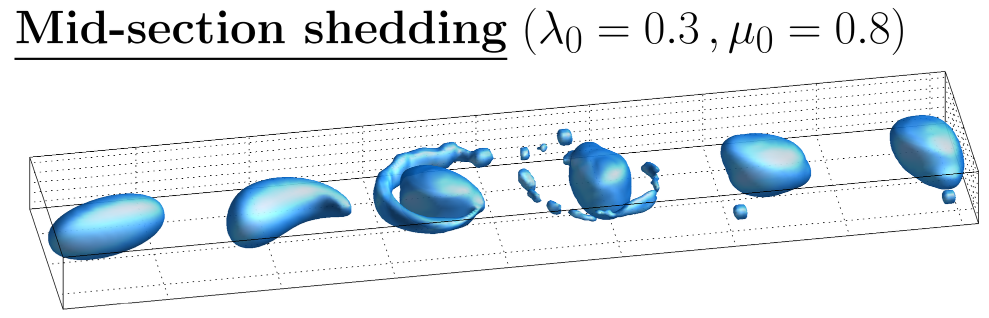
Shape oscillation ($\lambda_0 = 0.5\,, \mu_0 = 1.4$)



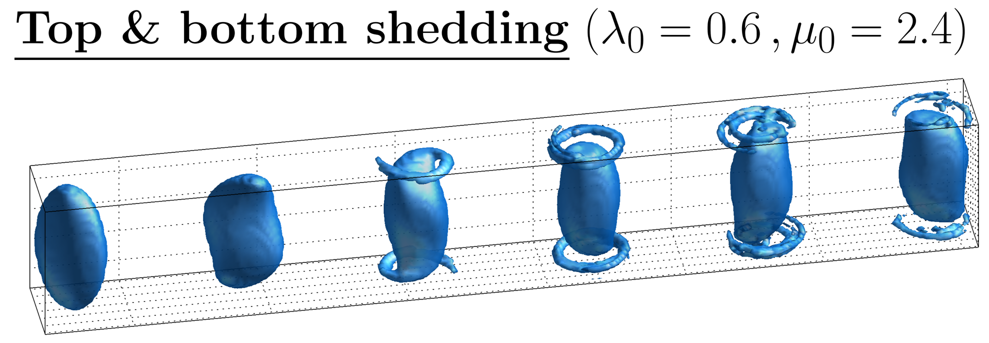
Binary system ($\lambda_0 = 0.2\,, \mu_0 = 0.6$)

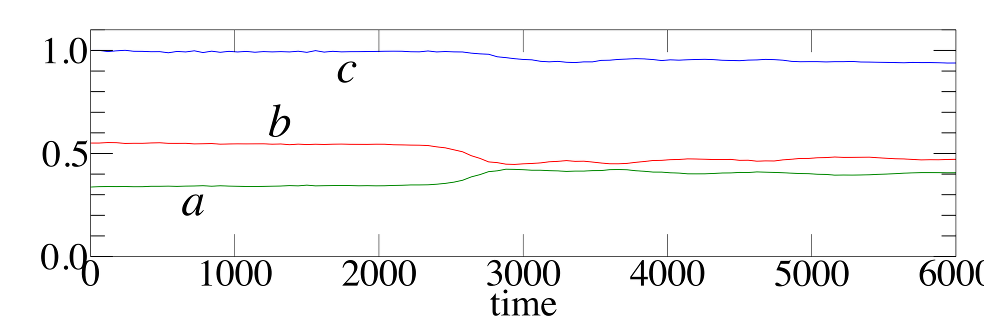


Mid-section shedding ($\lambda_0 = 0.3\,, \mu_0 = 0.8$)



Top & bottom shedding ($\lambda_0 = 0.6\,, \mu_0 = 2.4$)

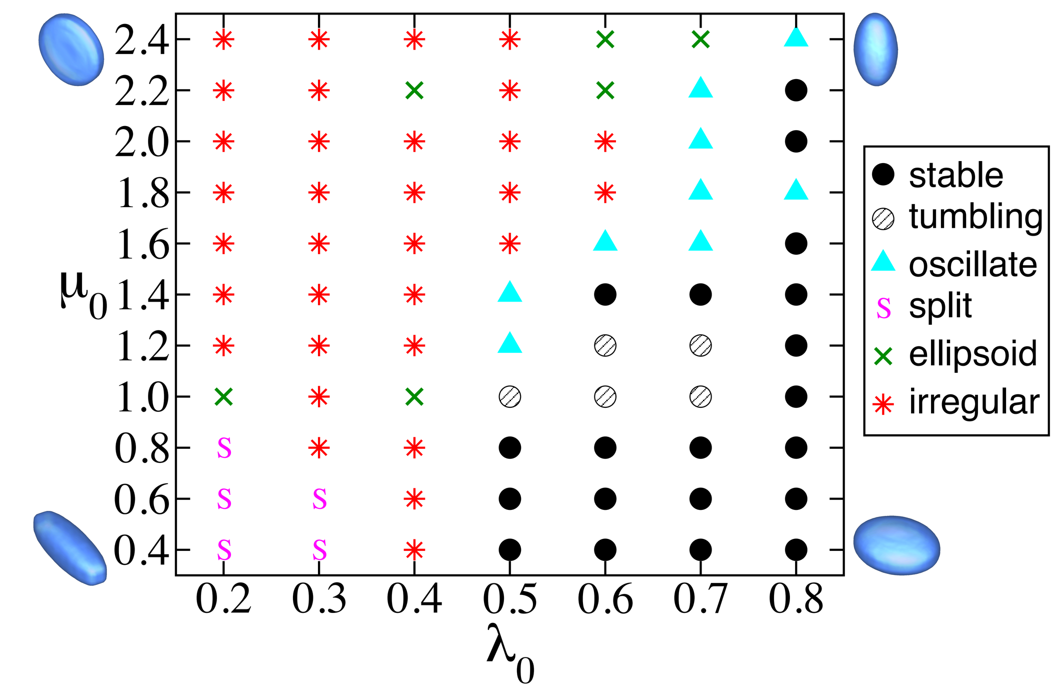




The Rossby number, Ro is define as the maximum value of the PV over space:

$$\mathrm{Ro} = \max_{\boldsymbol{x}}\{q\}$$

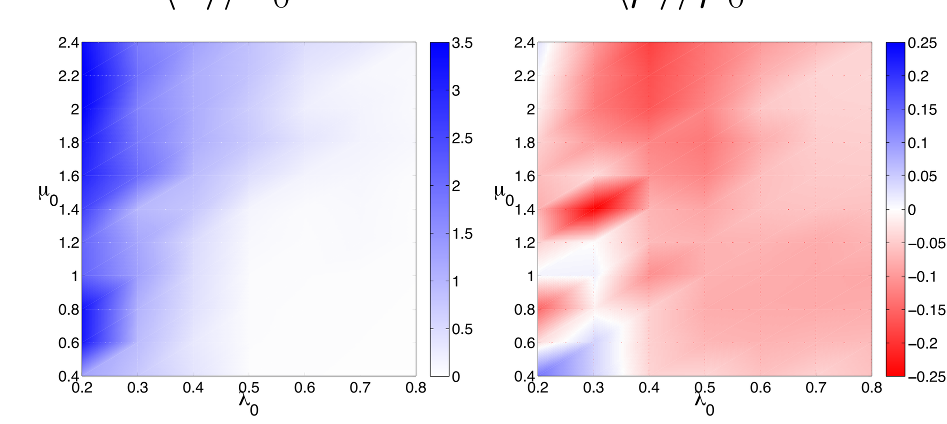
Preliminary Diagnostics: Ro = 0.25



- oblate ellipsoids ($\mu < 1$) with near circular cross sections ($\lambda \sim 1$) tend to be more stable

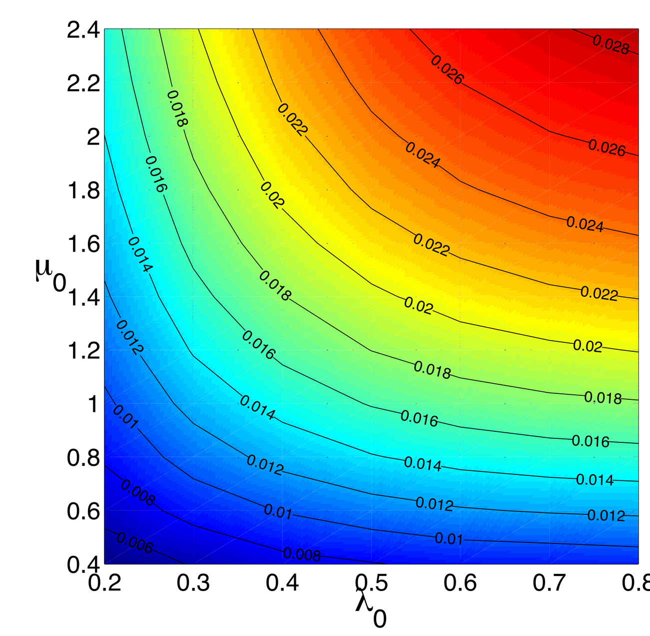
Fractional change of the aspect ratios

$\langle\lambda\rangle/\lambda_0 - 1$ $\quad$ $\langle\mu\rangle/\mu_0 - 1$

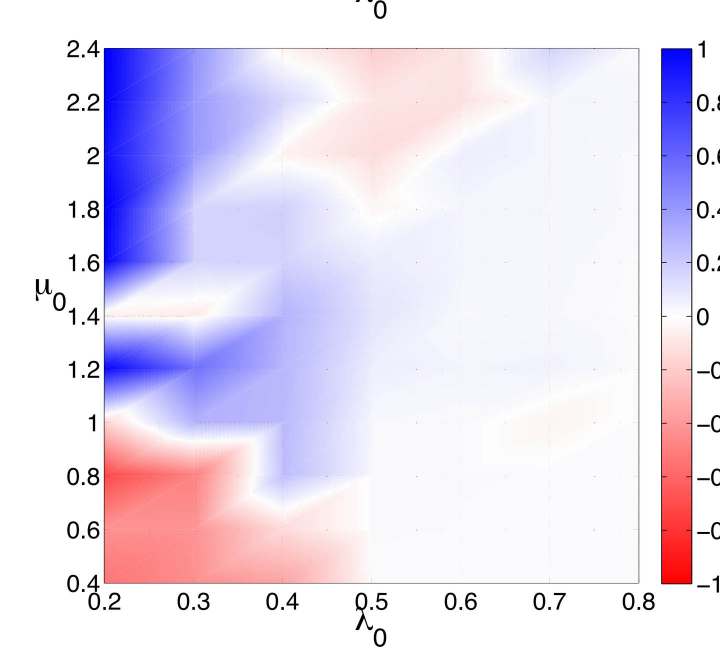


- $\langle\lambda\rangle \geqslant \lambda_0$ in all cases; $\langle\mu\rangle \leqslant \mu_0$ for most cases

Rotation rate



$\Omega_0 \equiv \Omega(t = 0)$



$\dfrac{\Omega_{\text{final}}}{\Omega_0} - 1$