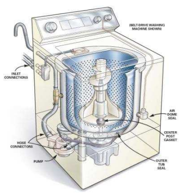
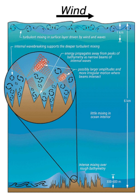
Scaling of Energy Injection Rate in Two-dimensional Turbulence with Drag

Yue-Kin Tsang and William R. Young

Scripps Institution of Oceanography, University of California, San Diego

Forced-Dissipative Systems



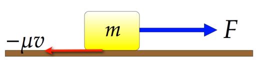


- Forcing (energy input)
- Nonlinear interaction (energy redistributed)
- Dissipation (energy removed)

$\varepsilon \equiv$ Power Input $=$ Force $\times$ Velocity

Question: *How does ε depend on the control parameters of the systems?*

Example 1: Block on a rough surface



Newton's second law,

$$F - \mu v = m\frac{dv}{dt}$$

Steady state velocity,

$$v = \frac{F}{\mu}$$

Energy injection rate (Power input),

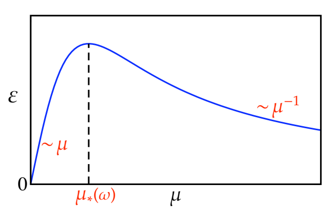
$$\varepsilon = Fv = F\left(\frac{F}{\mu}\right) \sim \mu^{-1}$$

Example 2: Forced Harmonic Oscillator

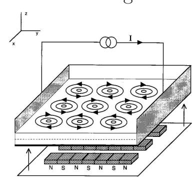
$$\ddot{x} + \omega_0^2 x = -\mu\dot{x} + A\cos\omega t$$

instantaneous : $\varepsilon_{int}(t) = \dot{x}(t)\, A\cos\omega t$

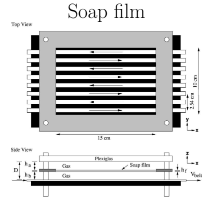
averaged : $\varepsilon = \frac{1}{T}\int_0^T \varepsilon_{int}(t')\, dt'$



Two-dimensional Turbulence



Paret and Tabeling (1998)



Burgess et al. (1999)

Navier-Stokes Equation

$$\zeta_t + \boldsymbol{u}\cdot\nabla\zeta = f(\boldsymbol{x},t) - \mu\zeta + \nu\nabla^2\zeta\,, \quad \boldsymbol{u} = (u,v)$$

$$\zeta \equiv v_x - u_y = \nabla^2\psi$$

Energy injection rate

$$\varepsilon = -\langle\psi f\rangle$$

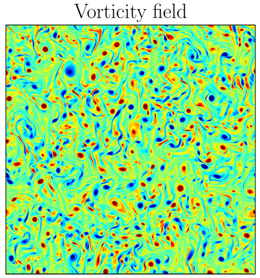
Conservation of energy

$$\varepsilon = \underbrace{\mu\langle u^2+v^2\rangle}_{\varepsilon_\mu} + \underbrace{\nu\langle\zeta^2\rangle}_{\varepsilon_\nu}$$

- Prescribed small-scale, steady forcing: $f = \cos x\,,$ box size $\gg 2\pi$
- Drag is the main dissipative mechanism: $\varepsilon_\mu \gg \varepsilon_\nu$

Simulation Results

Vorticity field



$\nabla^2\zeta \to -\nabla^8\zeta$

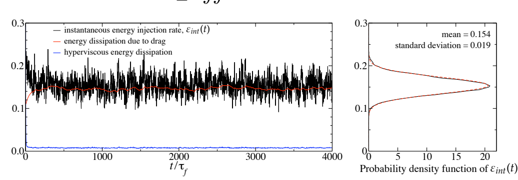
$L = 32(2\pi)$

$N = 1024^2$

$\mu = 0.007$

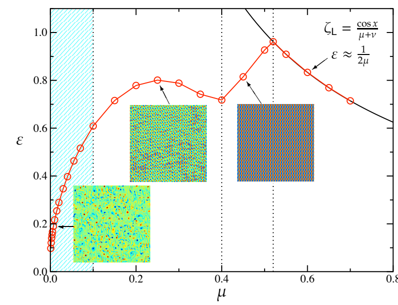
$\nu = 10^{-5}$

$$\varepsilon_{int}(t) = \frac{1}{L^2}\iint \zeta(x,y,t)\cos x\ dxdy$$

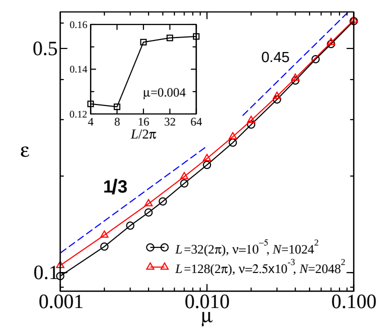


Energy Injection Rate vs. Drag

$$\varepsilon = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \varepsilon_{int}(t')\, dt'$$



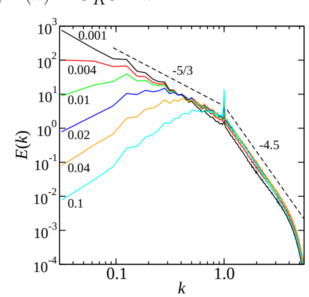
Scaling Law in the Small-drag Regime



Hence as the drag coefficient $\mu \to 0$,

$$\varepsilon \propto \mu^{1/3}$$

- Weak drag affects only the largest eddies
- $\varepsilon \propto \mu^{1/3} \Rightarrow$ drag controls energy injection at small scales: spectral nonlocality?
- Yet, $E(k) \approx C_K \varepsilon^{2/3} k^{-5/3}$!



A Closure Model

$$\zeta(x,y,t) = \underbrace{A(t)\cos x + B(t)\sin x}_{\text{forced mode, } \hat{\zeta}(x,t)} + \tilde{\zeta}(x,y,t)$$

$$\varepsilon = \langle\hat{\zeta}\cos x\rangle$$

Random Sweeping Model

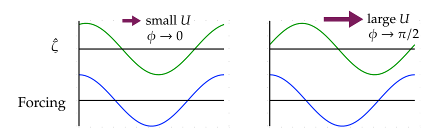
$$\hat{\zeta}_t + U\hat{\zeta}_x + V\hat{\zeta}_y = \cos x - \mu\hat{\zeta} - \eta\hat{\zeta} \qquad (1)$$

$$\varepsilon \approx \mu\langle U^2+V^2\rangle \approx 2\mu(U_{rms})^2 \qquad (2)$$

- advection by large-scale eddies (U,V)
  - isotropic: $\langle U^2\rangle = \langle V^2\rangle = (U_{rms})^2$
  - slowly varying in space and time
- nonlinear energy transfer out of the forced mode: $\eta \gg \mu \gg \nu$

Solution of the Model

$$\hat{\zeta} \approx \frac{\cos(x-\phi)}{\sqrt{\eta^2+U^2}}\ , \quad \tan\phi = \frac{U}{\eta}$$



Scaling of $\varepsilon(\mu)$

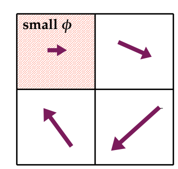
Taking ensemble average over U,

$$\varepsilon = \frac{1}{2}\int \frac{\eta}{\eta^2+U^2}\mathcal{P}(U)\, dU$$

Assume $\mathcal{P}(U) = \mathcal{P}(U/U_{rms})/U_{rms}$ and $U_{rms} \gg \eta$,

$$\varepsilon \approx \frac{\pi\mathcal{P}(0)}{2}(U_{rms})^{-1} \propto \mu^{1/3}$$

Energy injection is mostly due to "slow" regions.



- Nonlocal *interaction* between the large-scale (U,V) and the small-scale forced mode $\hat{\zeta}$
- No nonlocal *energy transfer* between the large and small scales as $\langle U\hat{\psi}\hat{\zeta}_x\rangle = 0$

Table 1:

Fig. 1.—

Fig. 2.—

Fig. 3.—