# Transition in the geomagnetic secular variation timescale under the core-mantle boundary


**Yue-Kin Tsang**

*School of Mathematics, Statistics and Physics*
*Newcastle University*

Chris Jones, *Newcastle University*

# Spectra: to study properties at different spatial scales

For a potential field ($\boldsymbol{B} = -\nabla\Phi$) specified by the Gauss coefficients $(g_{lm}, h_{lm})$, the following spectra have been defined for $r \geqslant r_{\text{cmb}}$:

(1) **Lowes spectrum**

$$R(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[ g_{lm}^2(t) + h_{lm}^2(t) \right]$$

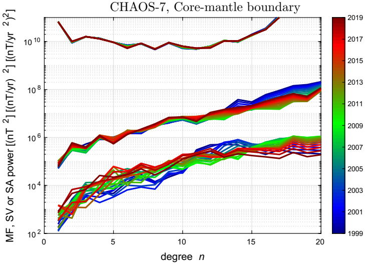
($a$ = Earth's radius)

(2) **Secular variation spectrum**

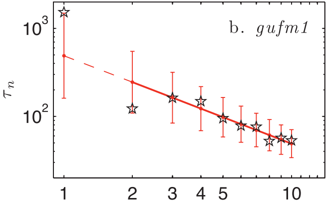
$$R_{\text{sv}}(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[ \dot{g}_{lm}^2(t) + \dot{h}_{lm}^2(t) \right]$$

(3) **Secular variation time-scale spectrum**

$$\tau_{\text{sv}}(l,t) = \sqrt{\frac{R}{R_{\text{sv}}}} = \sqrt{\frac{\sum_{m=0}^{l} \left(g_{lm}^2 + h_{lm}^2\right)}{\sum_{m=0}^{l} \left(\dot{g}_{lm}^2 + \dot{h}_{lm}^2\right)}}$$



Finlay et al. (2020)



Lhuillier et al. (2011)

# Secular variation time-scale spectrum $\tau_{\text{sv}}$

$$\tau_{\text{sv}}(l,t) = \sqrt{\frac{R}{R_{\text{sv}}}} \quad (r \geqslant r_{\text{cmb}})$$

- Characteristic time scale of magnetic field structures with spatial scale characterised by $l$
- $\tau_{\text{sv}}$ is sometimes interpreted as a characteristic time scale of the magnetohydrodynamics inside the outer core
- **Question:** $\tau_{\text{sv}}$ is defined using the Gauss coefficients derived from $\boldsymbol{B}$ observed at the Earth's surface. Do $\tau_{\text{sv}}$ describe the time variation of $\boldsymbol{B}$ inside the outer core?
- We investigate using numerical simulations

# Generalisation to inside the dynamo region (outer core)

For a magnetic field $\boldsymbol{B}$ (not necessarily potential) at some $r$, expand in a set of vector spherical harmonics basis,

$$\boldsymbol{B}(r,\theta,\phi,t) = \sum_{lm} \left[ q_{lm}(r,t)\hat{\boldsymbol{Y}}_{lm}(\theta,\phi) + s_{lm}(r,t)\hat{\boldsymbol{\Psi}}_{lm}(\theta,\phi) + t_{lm}(r,t)\hat{\boldsymbol{\Phi}}_{lm}(\theta,\phi) \right]$$

We define the magnetic energy spectrum $F(l,r,t)$ for any $r$:

$$\sum_{l=1}^{\infty} F(l,r,t) \equiv \frac{1}{4\pi}\oint |\boldsymbol{B}(r,\theta,\phi,t)|^2 \,\mathrm{d}\Omega = \sum_{l=1}^{\infty}\left[\frac{1}{(2l+1)}\sum_{m=0}^{l}\left(|q_{lm}|^2+|s_{lm}|^2+|t_{lm}|^2\right)(4-3\delta_{m,0})\right]$$

Similarly, define the time variation spectrum $F_{\dot{B}}(l,r,t)$:

$$\sum_{l=1}^{\infty} F_{\dot{B}}(l,r,t) \equiv \frac{1}{4\pi}\oint |\dot{\boldsymbol{B}}(r,\theta,\phi,t)|^2 \,\mathrm{d}\Omega = \sum_{l=1}^{\infty}\left[\frac{1}{(2l+1)}\sum_{m=0}^{l}\left(|\dot{q}_{lm}|^2+|\dot{s}_{lm}|^2+|\dot{t}_{lm}|^2\right)(4-3\delta_{m,0})\right]$$

Then, the (time-averaged) **magnetic time-scale spectrum** is defined as:

$$\tau(l,r) = \left\langle \sqrt{\frac{F(l,r,t)}{F_{\dot{B}}(l,r,t)}} \right\rangle_t \quad \text{for any } r$$

- At the CMB $r = r_{\mathrm{cmb}}$: $F = R$ , $F_{\dot{B}} = R_{\mathrm{sv}}$ , $\tau(l, r_{\mathrm{cmb}}) = \tau_{\mathrm{sv}}(l)$

# A numerical simulation of strong-field geodynamo

Boussinesq, composition-driven, rotating convection of an electrically conducting fluid:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\frac{Pm}{Ek}\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\frac{Pm}{Ek}\nabla\Pi' + \left(\frac{RaPm^2}{Pr}\right)Cr\,\hat{\boldsymbol{r}} + \frac{Pm}{Ek}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + Pm\nabla^2\boldsymbol{u},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla^2\boldsymbol{B}$$

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{Pm}{Pr}\nabla^2 C - 1$$

$$\nabla \cdot \boldsymbol{u} = 0$$

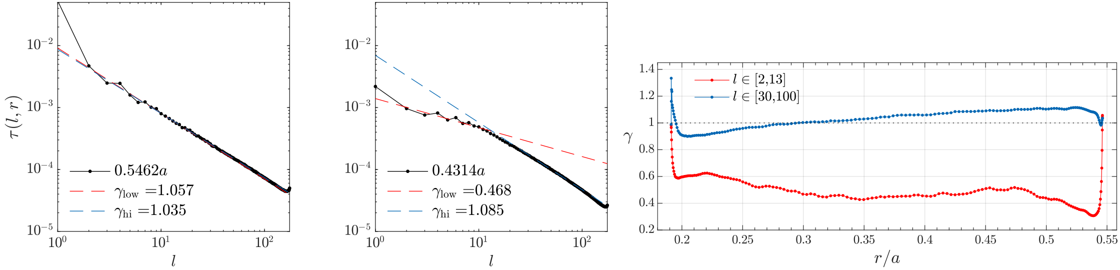
$$\nabla \cdot \boldsymbol{B} = 0$$

Boundary conditions: **no-slip** for $\boldsymbol{u}$, fixed-flux for $C$

Domain: a spherical shell $0.1912\,a \leqslant r \leqslant 0.5462\,a \equiv r_{\mathrm{cmb}}$ (or $r_{\mathrm{o}}$)

$Ra = 2.7 \times 10^8\,, Ek = 2.5 \times 10^{-5}\,, Pm = 2.5\,, Pr = 1$

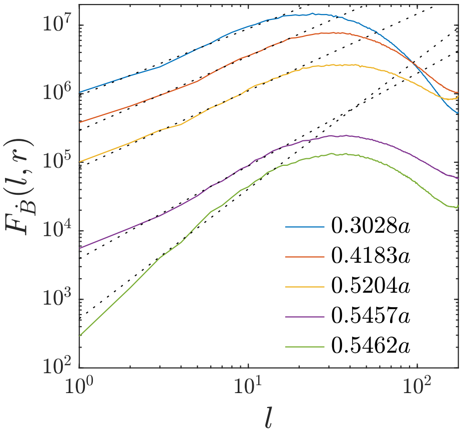
# Magnetic time-scale spectrum $\tau(l,r)$ at different depth

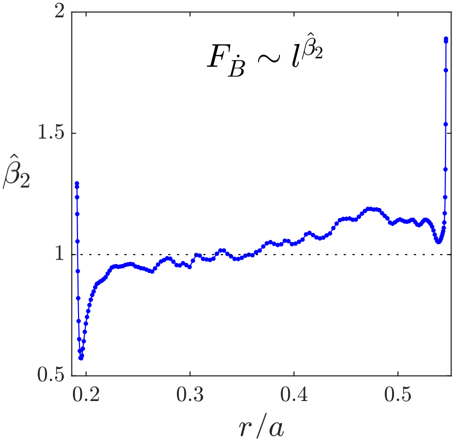



For the large-scale modes (small $l$),

- $\tau \sim l^{-1}$ at the CMB and $\tau \sim l^{-0.5}$ in the interior
- large-scale modes evolve faster in the interior!
- transition occurs inside a thin boundary layer under the CNB
- $\tau_{\text{sv}}(l)$ observed at the surface is different from the time scales inside the outer core

# Change in the scaling of $\tau$: who causes it?




$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{l^{\hat{\beta}_2(r)}}} \sim F_{\dot{B}}{}^{-\hat{\beta}_2(r)/2}$$

$$F_{\dot{B}} \sim l \implies \tau \sim l^{-0.5} \quad \text{(interior)}$$

$$F_{\dot{B}} \sim l^2 \implies \tau \sim l^{-1} \quad \text{(CMB)}$$

# Transition in the scaling of $F_{\dot{B}}$: physical meaning

$$F_{\dot{B}}(l,r,t) = \frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|\dot{q}_{lm}|^2 + |\dot{s}_{lm}|^2 + |\dot{t}_{lm}|^2\right)(4 - 3\delta_{m,0})$$

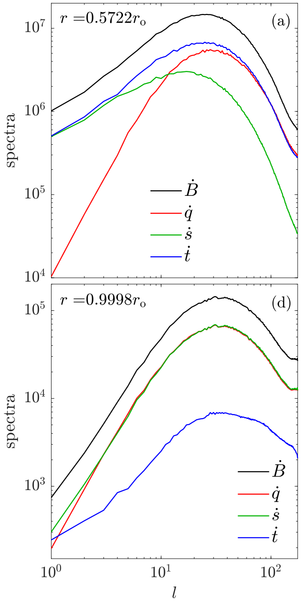
$$\equiv F_{\dot{q}} + F_{\dot{s}} + F_{\dot{t}}$$

$$B_r = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} q_{lm} Y_l^m,$$

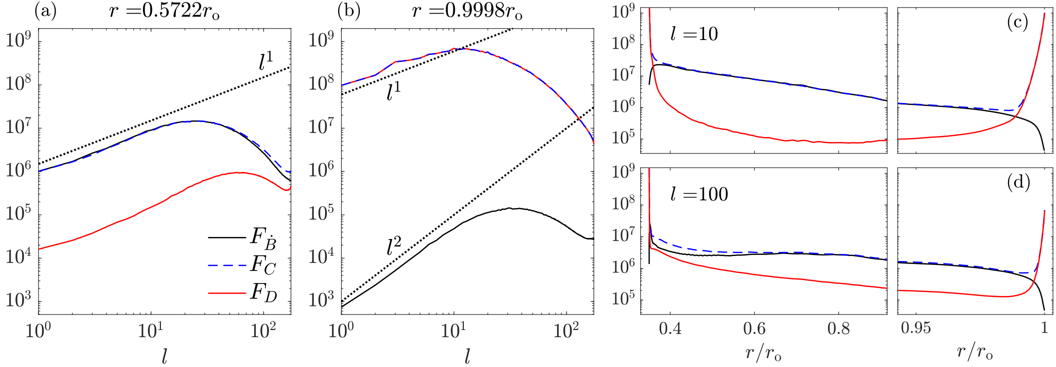
$$B_\theta = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{\sqrt{l(l+1)}} \left( s_{lm} \frac{\partial Y_l^m}{\partial \theta} - \frac{t_{lm}}{\sin\theta} \frac{\partial Y_l^m}{\partial \phi} \right),$$

$$B_\phi = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{\sqrt{l(l+1)}} \left( t_{lm} \frac{\partial Y_l^m}{\partial \theta} + \frac{s_{lm}}{\sin\theta} \frac{\partial Y_l^m}{\partial \phi} \right).$$

- At the CMB (for the large scales):
  - magnetic boundary condition ties $B_r, B_\theta, B_\phi$ together
  - $F_{\dot{B}} \sim F_{\dot{s}} \sim F_{\dot{t}} \sim l^2 \implies \tau \sim l^{-1}$
- In the interior (for the large scales):
  - $F_{\dot{s}}, F_{\dot{t}} \gg F_{\dot{q}} \implies \dot{\boldsymbol{B}}$ is dominated by $(\dot{B}_\theta, \dot{B}_\phi)$
  - $F_{\dot{s}}, F_{\dot{t}} \sim l$ and $F_{\dot{q}} \sim l^2 \implies (B_\theta, B_\phi)$ vary faster than $B_r$



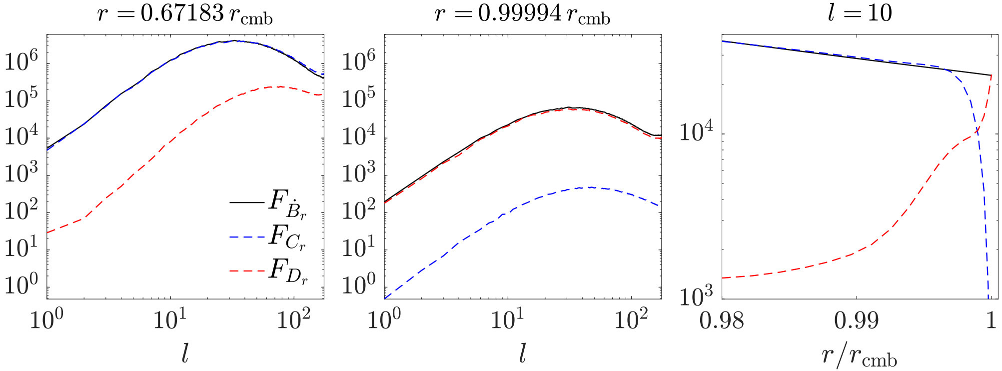
# Processes controlling the secular variation of (total) $B$




$$\dot{\boldsymbol{B}} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} = \boldsymbol{C} + \boldsymbol{D}$$

- Interior: $\dot{\boldsymbol{B}} \approx \boldsymbol{C} \gg \boldsymbol{D}$ , $F_{\dot{B}} \approx F_C \sim l$ (magnetic diffusion negligible)
- At the CMB:
  - $\boldsymbol{C} \approx \boldsymbol{D} \gg \dot{\boldsymbol{B}}$ ($\because$ no-slip)
  - $F_C \sim F_D \sim l$ but $F_{\dot{B}} \sim l^2$

# Processes controlling the secular variation of $B_r$




$$\dot{B}_r = \hat{\boldsymbol{r}} \cdot [\nabla \times (\boldsymbol{u} \times \boldsymbol{B})] + \eta(\hat{\boldsymbol{r}} \cdot \nabla^2 \boldsymbol{B}) = C_r + D_r$$

- Interior: $\dot{B}_r \approx C_r \gg D_r$ , $F_{\dot{B}_r} \approx F_{C_r} \sim l^2$ (magnetic diffusion negligible)
- At the CMB: $\dot{B}_r \approx D_r$ ($C_r \to 0$ for no-slip), $F_{\dot{B}_r} \approx F_{D_r} \sim l^2$
- Inside the CMB boundary layer: sharp transition for both $C_r$ and $D_r$ but $\dot{B}_r$ varies very weakly despite the switch in the controlling mechanism

## Summary

- For the large scales, scaling of $\tau(l,r)$ with $l$ at the CMB is different from that in the interior of the outer core:

$$\tau \sim l^{-0.5}, \quad \text{in the interior}$$
$$\tau \sim l^{-1}, \quad \text{at the CMB}$$

- The transition in scaling occurs within a thin boundary layer under the CMB.

- $F_{\dot{B}}$ is responsible for the transition ($\tau = \sqrt{F/F_{\dot{B}}}$)

- $F_{\dot{B}} \sim l$ in the interior to $F_{\dot{B}} \sim l^2$ near the CMB, meaning $\dot{\boldsymbol{B}} \sim (\dot{B}_\theta, \dot{B}_\phi) \gg \dot{B}_r$ in the interior

- mechanism controlling $\dot{B}_r$ changes "stealthily" from induction to diffusion as $r \to r_{\mathrm{cmb}}$ from below

Yue-Kin Tsang and Chris A. Jones, Geophysical Journal International **239**, 1 (2024)