Scaling of the geomagnetic secular variation time scales

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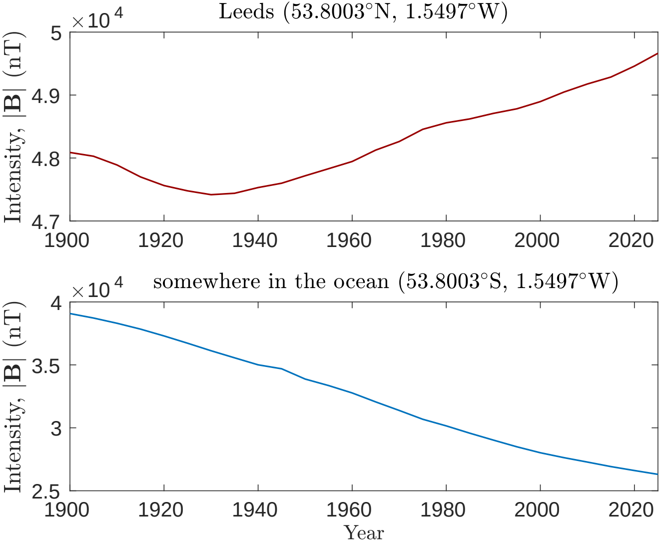
Time variation of the geomagnetic field



data from 13th generation International Geomagnetic Reference Field (IGRF-13) geomagnetic field model

Alken, P., Thébault, E., Beggan, C.D. et al., Earth Planets Space **73**, *49 (2021)*

The time-dependent Gauss coefficients

- Above the core–mantle boundary $r > r_{\text{cmb}}$: $\boldsymbol{j} = \boldsymbol{0}$ (assume mantle is electrically insulating)

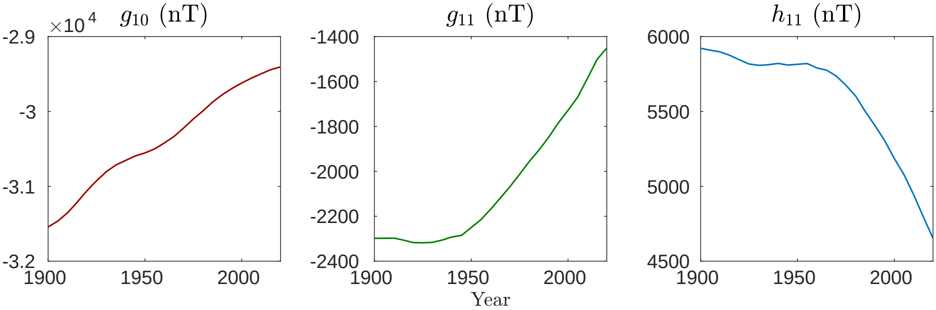
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} = \boldsymbol{0} \implies \boldsymbol{B} = -\nabla\Psi$$

$$\nabla \cdot \boldsymbol{B} = 0 \implies \nabla^2\Psi = 0$$

$$\Psi(r,\theta,\phi,t) = a \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left(\frac{a}{r}\right)^{l+1} \hat{P}_{lm}(\cos\theta)\left[g_{lm}(t)\cos m\phi + h_{lm}(t)\sin m\phi\right]$$

$\hat{P}_{lm}$: *Schmidt's semi-normalised associated Legendre polynomials;* $a =$ *Earth's radius*

- $\{g_{lm}(t), h_{lm}(t)\}$ and $\{\dot{g}_{lm}(t), \dot{h}_{lm}(t)\}$ are measured accurately by recent satellite missions



Spectral approach to time variation of the geomagnetic field

(1) Lowes spectrum (Mauersberger 1956, Lowes 1966)

$$\sum_{l=1}^{\infty} R(l,r,t) = \frac{1}{4\pi} \oint |\boldsymbol{B}(r,\theta,\phi,t)|^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$R(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[g_{lm}^2(t) + h_{lm}^2(t) \right], \qquad r \geqslant r_{\mathrm{cmb}}$$

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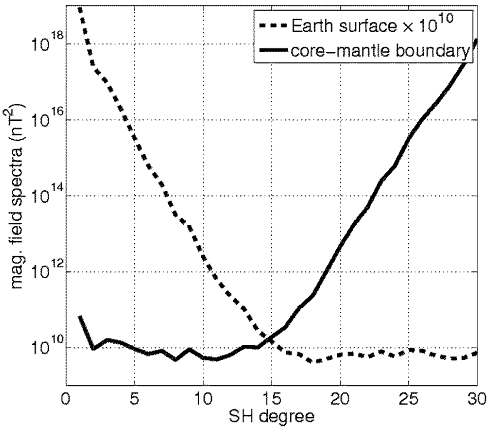
(2) Secular variation spectrum (Lowes 1974)

$$\sum_{l=1}^{\infty} R_{\mathrm{sv}}(l,r,t) = \frac{1}{4\pi}\oint |\dot{\boldsymbol{B}}(r,\theta,\phi,t)|^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi\,, \qquad \dot{\boldsymbol{B}} = \frac{\partial \boldsymbol{B}}{\partial t}$$

$$R_{\mathrm{sv}}(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[\, \dot{g}_{lm}^2(t) + \dot{h}_{lm}^2(t) \,\right], \qquad r \geqslant r_{\mathrm{cmb}}$$

Spectral approach to time variation of the geomagnetic field

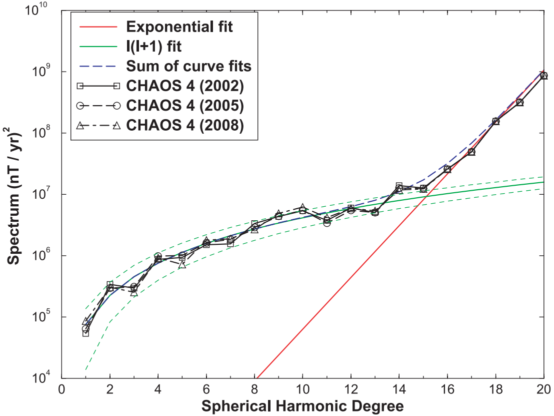
$R(l)$



Gillet, Lesur and Olsen (2010)

$$R(r = r_{\rm cmb}) \sim l^0 \quad (l \lesssim 13)$$

$R_{\rm sv}(l)$ at the CMB



Holme, Olsen and Bairstow (2011)

$$R_{\rm sv}(r = r_{\rm cmb}) \sim l(l+1) \text{ or } l^2 \quad (l \lesssim 13)$$

(3) Secular variation time-scale spectrum (Booker 1969)

$$\tau_{\mathrm{sv}}(l,t) = \sqrt{\frac{R}{R_{\mathrm{sv}}}} = \sqrt{\frac{\sum_{m=0}^{l}\left(g_{lm}^2 + h_{lm}^2\right)}{\sum_{m=0}^{l}\left(\dot{g}_{lm}^2 + \dot{h}_{lm}^2\right)}}, \qquad r \geqslant r_{\mathrm{cmb}}$$

- characteristic time scale of magnetic field structures with spatial scale characterised by l
- by definition, τ_{sv} is independent of r ($\geqslant r_{\mathrm{cmb}}$)
- dependence on l has been investigated using satellite data and numerical simulations
- power-law relation with exponent γ (excluding $l = 1$):

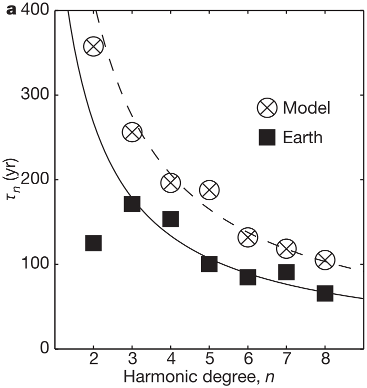
$$\tau_{\mathrm{sv}}(l) \sim \tau_* \cdot l^{-\gamma}$$

What is the value of γ ?

Scaling of $\tau_{\rm sv}(l)$: observations and numerical models

Christensen and Tilgner (2004)

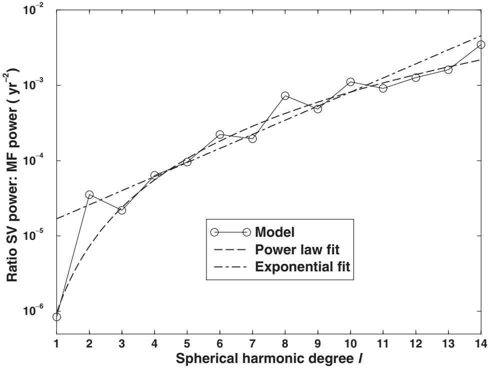
observation data 1840 – 1990
and numerical dynamo models



$$\tau_{\rm sv} \sim l^{-1}$$

Holme and Olsen (2006)

satellite data 1999 – 2003



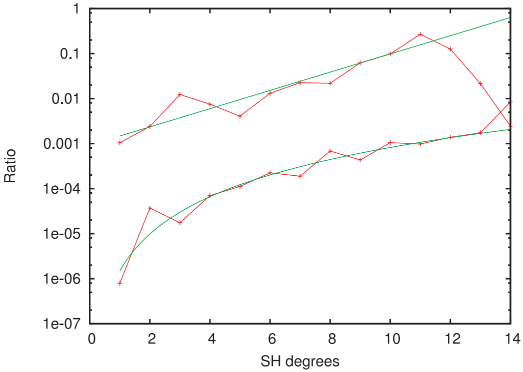
$$\frac{R_{\rm sv}}{R} \sim l^{2.9}$$

$$\Longrightarrow \quad \tau_{\rm sv} \sim l^{-1.45}$$

Scaling of $\tau_{\rm sv}(l)$: observations and numerical models

Lesur et al. (2008)

6yr CHAMP + 5yr observatory data

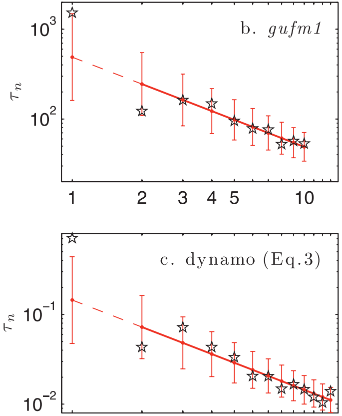


$$\frac{R_{\rm sv}}{R} \sim l^{2.75}$$

$$\Longrightarrow \tau_{\rm sv} \sim l^{-1.38}$$

Lhuillier et al. (2011)

'historical data' 1840−1990, satellite data (2005) and numerical dynamo models



$$\tau_{\rm sv} \sim l^{-1}$$

An argument for $\tau_{\text{sv}} \sim l^{-1}$?

- numerical models: $\tau_{\text{sv}} \sim l^{-1}$
- observations: mixed results, $1.32 < \gamma < 1.45$ and $\gamma = 1$
- time average vs. snapshot
- a crude argument by neglecting magnetic diffusion η (frozen flux hypothesis):

$$\dot{B}_r = -\nabla_h \cdot (\boldsymbol{u}_h B_r)$$

$$\nabla_h \sim \sqrt{l(l+1)} \sim l \quad \text{and} \quad \boldsymbol{u}_h \sim U$$

$$\tau_{\text{sv}} \overset{?}{\sim} B_r/\dot{B}_r \sim l^{-1}$$

Some questions:

1. η may be important near the boundaries. Is the frozen flux hypothesis valid?
2. What is the balance between different effects near the CMB?
3. Is τ_{sv} relevant to the time scale of $\dot{\boldsymbol{B}}$ *inside* the outer core?

Generalisation to inside the dynamo region (outer core)

Recall the definition of the Lowes spectrum for $r \geqslant r_{\text{cmb}}$,

$$\Psi(r,\theta,\phi,t) = a \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left(\frac{a}{r}\right)^{l+1} \hat{P}_{lm}(\cos\theta)\left[g_{lm}(t)\cos m\phi + h_{lm}(t)\sin m\phi\right]$$

$$\frac{1}{4\pi} \oint |\boldsymbol{B}(r,\theta,\phi,t)|^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \sum_{l=1}^{\infty} R(l,r,t)$$

Expand in vector spherical harmonics for any r,

$$\boldsymbol{B}(r,\theta,\phi,t) = \sum_{lm} \left[q_{lm}(r,t)\hat{\boldsymbol{Y}}_{lm}(\theta,\phi) + s_{lm}(r,t)\hat{\boldsymbol{\Psi}}_{lm}(\theta,\phi) + t_{lm}(r,t)\hat{\boldsymbol{\Phi}}_{lm}(\theta,\phi)\right]$$

We define the magnetic energy spectrum:

$$\frac{1}{4\pi} \oint |\boldsymbol{B}(r,\theta,\phi,t)|^2 \, \mathrm{d}\Omega = \sum_{l=1}^{\infty} \left[\frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|q_{lm}|^2 + |s_{lm}|^2 + |t_{lm}|^2\right)(4 - 3\delta_{m,0})\right] \equiv \sum_{l=1}^{\infty} F(l,r,t)$$

Generalisation to inside the dynamo region (outer core)

$$F(l,r,t) = \frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|q_{lm}|^2 + |s_{lm}|^2 + |t_{lm}|^2\right)(4 - 3\delta_{m,0})$$

Similarly, define the time variation spectrum (spectrum of $\dot{\boldsymbol{B}}$):

$$\dot{\boldsymbol{B}}(r,\theta,\phi,t) = \sum_{lm} \left[\dot{q}_{lm}(r,t)\hat{\boldsymbol{Y}}_{lm}(\theta,\phi) + \dot{s}_{lm}(r,t)\hat{\boldsymbol{\Psi}}_{lm}(\theta,\phi) + \dot{t}_{lm}(r,t)\hat{\boldsymbol{\Phi}}_{lm}(\theta,\phi)\right]$$

$$\frac{1}{4\pi}\oint |\dot{\boldsymbol{B}}(r,\theta,\phi,t)|^2 \,\mathrm{d}\Omega = \sum_{l=1}^{\infty}\left[\frac{1}{(2l+1)}\sum_{m=0}^{l}\left(|\dot{q}_{lm}|^2 + |\dot{s}_{lm}|^2 + |\dot{t}_{lm}|^2\right)(4 - 3\delta_{m,0})\right] \equiv \sum_{l=1}^{\infty} F_{\dot{B}}(l,r,t)$$

Then, the magnetic time-scale spectrum is defined as:

$$\tau(l,r) = \left\langle \sqrt{\frac{F(l,r,t)}{F_{\dot{B}}(l,r,t)}} \right\rangle_t$$

Outside the dynamo region, $\boldsymbol{j} = \mathbf{0}$: $F = R$, $F_{\dot{B}} = R_{\rm sv}$, $\tau = \tau_{\rm sv}$

A numerical model of geodynamo

Boussinesq, compositional driven, rotating convection of a electrically conducting fluid:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\frac{Pm}{Ek}\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\frac{Pm}{Ek}\nabla\Pi' + \left(\frac{RaPm^2}{Pr}\right)C'\boldsymbol{r} + \frac{Pm}{Ek}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + Pm\nabla^2\boldsymbol{u},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla^2\boldsymbol{B}$$

$$\frac{\mathrm{D}C'}{\mathrm{D}t} = \frac{Pm}{Pr}\nabla^2 C' - 1$$

$$\nabla \cdot \boldsymbol{u} = 0$$

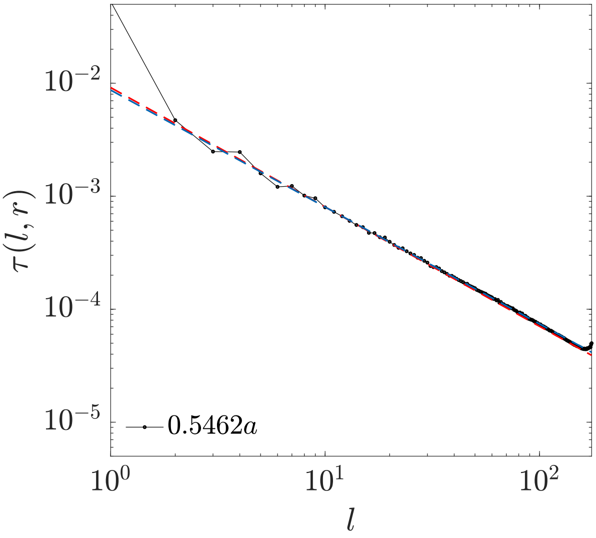
$$\nabla \cdot \boldsymbol{B} = 0$$

Boundary conditions: no-slip for $\boldsymbol{u}$, Neumann for C'

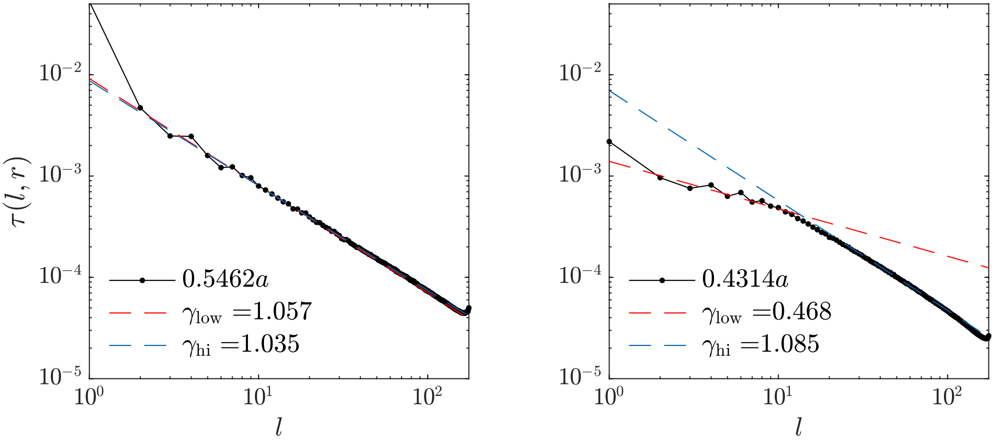
Domain: a spherical shell $0.1912\,a \leqslant r \leqslant 0.5462\,a$

$Ra = 2.7 \times 10^8\,, Ek = 2.5 \times 10^{-5}\,, Pm = 2.5\,, Pr = 1$

Magnetic time-scale spectrum $\tau(l, r)$ at different depth



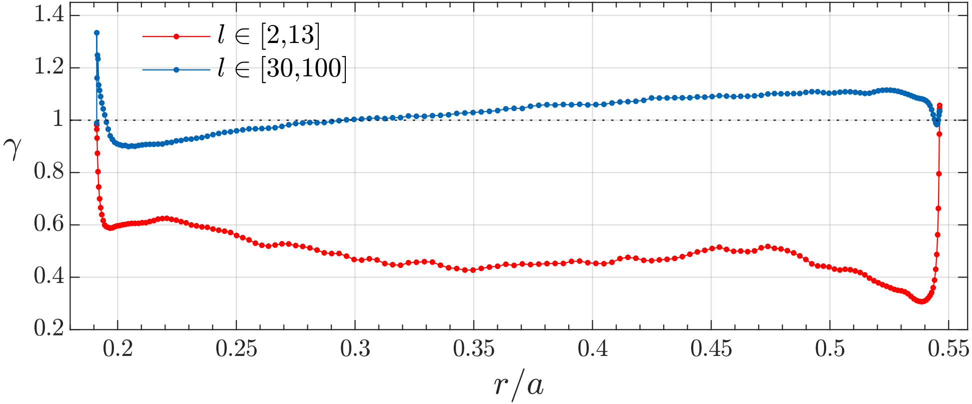
Change in the scaling of $\tau(l, r)$ at the large scales



For the large-scale modes (small l),

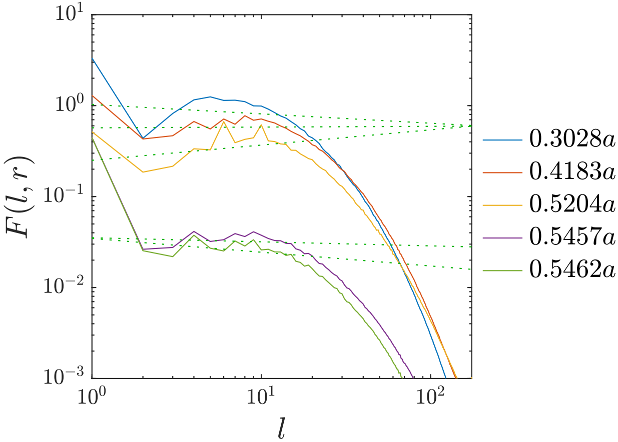
- in the interior: $\tau \sim l^{-0.5}$
- at the surface: $\tau \sim l^{-1}$, *the large-scale modes slow down near the CMB !*

Change in the scaling of τ: where does it occur?



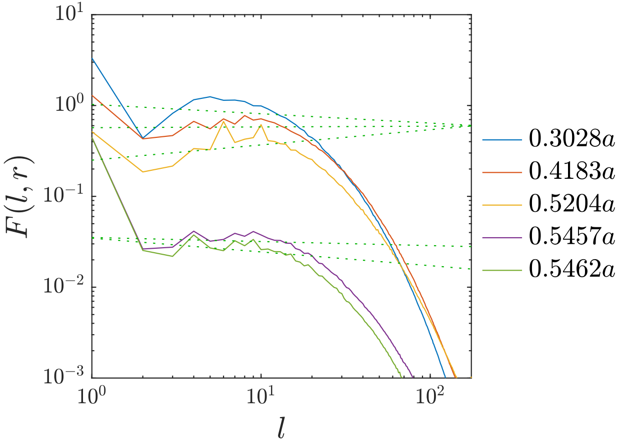
- γ for the large-scale modes increases sharply within a boundary layer under CMB
- $\tau \sim l^{-1}$ detected above CMB: a result of the magnetic boundary condition ($\boldsymbol{B}_P \to$ potential field, $\boldsymbol{B}_T \to \boldsymbol{0}$)?

Change in the scaling of τ: who causes it?



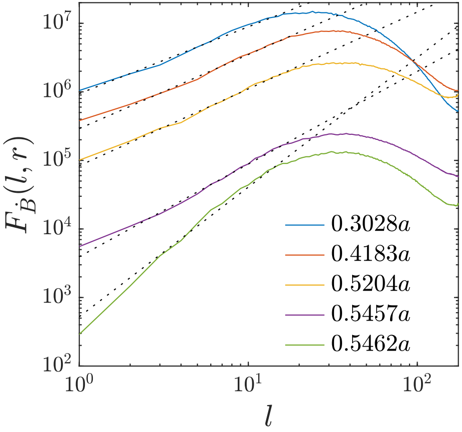
$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}}$$

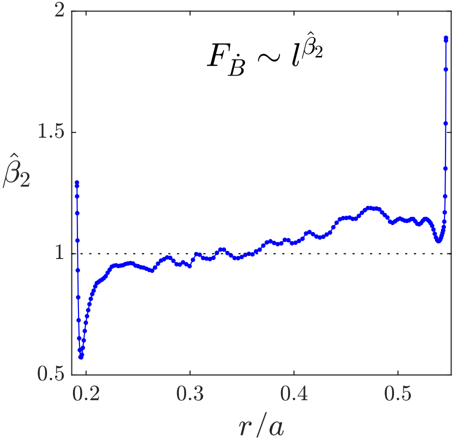
Change in the scaling of τ: who causes it?



$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{F_{\dot{B}}}} \sim {F_{\dot{B}}}^{-\frac{1}{2}}$$

Change in the scaling of τ: who causes it?



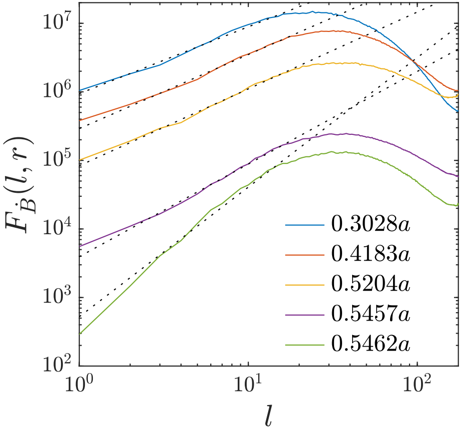


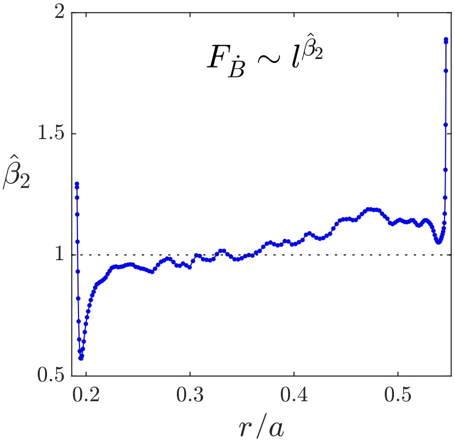
$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{F_{\dot{B}}}} \sim F_{\dot{B}}^{-\frac{1}{2}}$$

$$F_{\dot{B}} \sim l \implies \tau \sim l^{-0.5} \quad \text{(interior)}$$

$$F_{\dot{B}} \sim l^2 \implies \tau \sim l^{-1} \quad \text{(surface)}$$

Change in the scaling of τ: who causes it?



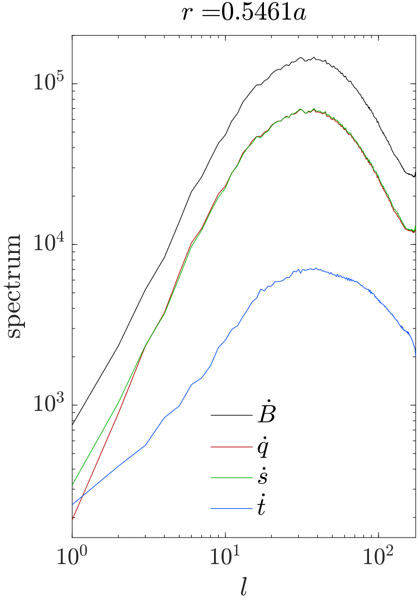


$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{F_{\dot{B}}}} \sim F_{\dot{B}}^{-\frac{1}{2}}$$

$F_{\dot{B}} \sim l \implies \tau \sim l^{-0.5}$ (interior)

$F_{\dot{B}} \sim l^2 \implies \tau \sim l^{-1}$ (surface) frozen flux?

Balance of terms for the large scales in the CMB boundary layer



$$\dot{\boldsymbol{B}} = \nabla \times \boldsymbol{\mathcal{G}} + \eta \nabla^2 \boldsymbol{B}, \quad \boldsymbol{\mathcal{G}} \equiv \boldsymbol{u} \times \boldsymbol{B}$$

$$\boldsymbol{\mathcal{G}} = \sum_{lm} \left[q_{\mathcal{G}}\, \hat{\boldsymbol{Y}} + s_{\mathcal{G}}\, \hat{\boldsymbol{\Psi}} + t_{\mathcal{G}}\, \hat{\boldsymbol{\Phi}} \right], \quad (l, m \text{ omitted})$$

$$\boldsymbol{B} = \sum_{lm} \left[q\, \hat{\boldsymbol{Y}} + s\, \hat{\boldsymbol{\Psi}} + t\, \hat{\boldsymbol{\Phi}} \right], \quad \dot{\boldsymbol{B}} = \sum_{lm} \left[\dot{q}\, \hat{\boldsymbol{Y}} + \dot{s}\, \hat{\boldsymbol{\Psi}} + \dot{t}\, \hat{\boldsymbol{\Phi}} \right]$$

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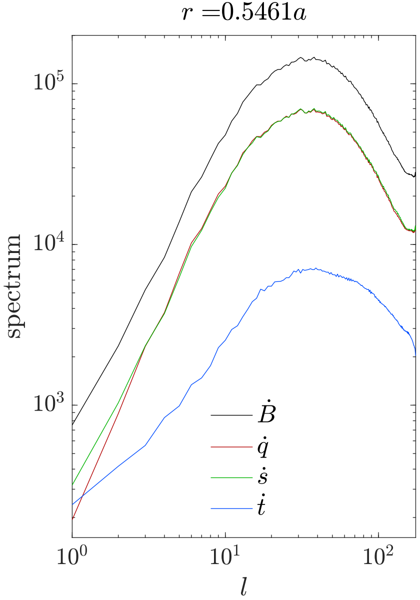

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$$\dot{q} = -\frac{\sqrt{l(l+1)}}{r} t_{\mathcal{G}} + \eta \left[-\frac{l(l+1)}{r^2} q + \frac{\sqrt{l(l+1)}}{r} s' + \frac{\sqrt{l(l+1)}}{r^2} s \right]$$

$$\dot{s} = t'_{\mathcal{G}} - \frac{t_{\mathcal{G}}}{r} + \eta \left[-\frac{\sqrt{l(l+1)}}{r} q' + s'' + \frac{2}{r} s' \right] \qquad ({}' : \partial_r)$$

$$\dot{t} = s'_{\mathcal{G}} + \frac{s_{\mathcal{G}}}{r} - \frac{\sqrt{l(l+1)}}{r} q_{\mathcal{G}} + \eta \left[t'' + \frac{2}{r} t' - \frac{l(l+1)}{r^2} t \right]$$

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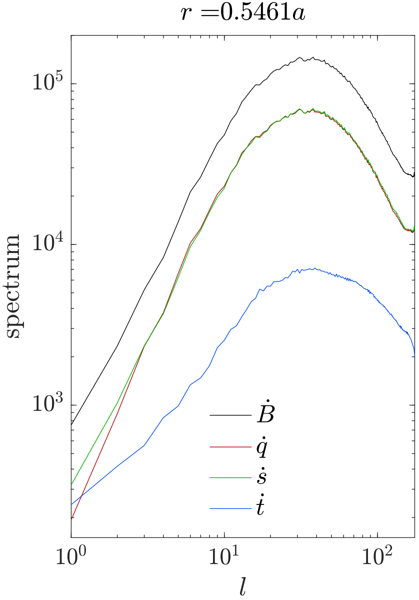

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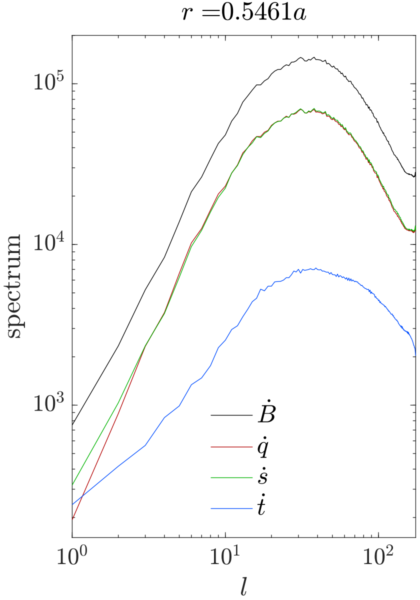
$$F_{\dot{B}} = \frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|\dot{q}|^2 + |\dot{s}|^2 + |\dot{t}|^2 \right) (4 - 3\delta_{m,0})$$

$$F_{\dot{B}} = F_{\dot{q}} + F_{\dot{s}} + F_{\dot{t}}$$

Balance of terms for the large scales in the CMB boundary layer



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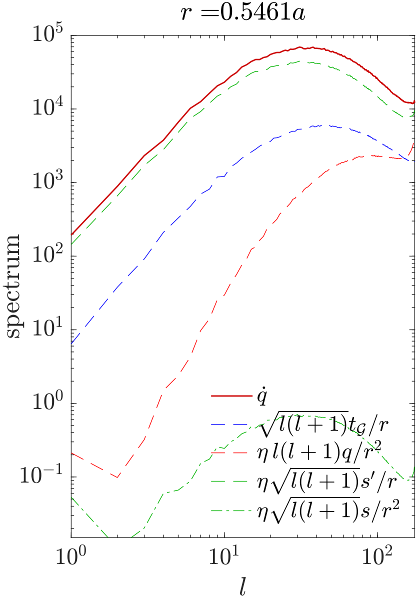
$$F_{\dot{B}} = F_{\dot{q}} + F_{\dot{s}} + F_{\dot{t}}$$

$$F_{\dot{B}} \sim F_{\dot{q}} \sim F_{\dot{s}} \sim l^2$$

Balance of terms for the large scales in the CMB boundary layer



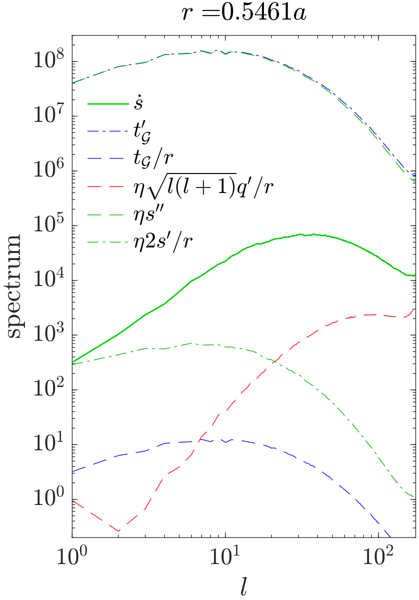
$$F_{\dot{B}} \sim F_{\dot{q}} \sim F_{\dot{s}} \sim l^2$$

$$F_{\dot{B}} = F_{\dot{q}} + F_{\dot{s}} + F_{\dot{t}}$$

$$\dot{q} = -\frac{\sqrt{l(l+1)}}{r}t_{\mathcal{G}} - \eta\frac{l(l+1)}{r^2}q + \eta\frac{\sqrt{l(l+1)}}{r}s' + \eta\frac{\sqrt{l(l+1)}}{r^2}s$$

$$\dot{s} = t'_{\mathcal{G}} - \frac{t_{\mathcal{G}}}{r} - \eta\frac{\sqrt{l(l+1)}}{r}q' + \eta s'' + \eta\frac{2}{r}s'$$

Balance of terms for the large scales in the CMB boundary layer



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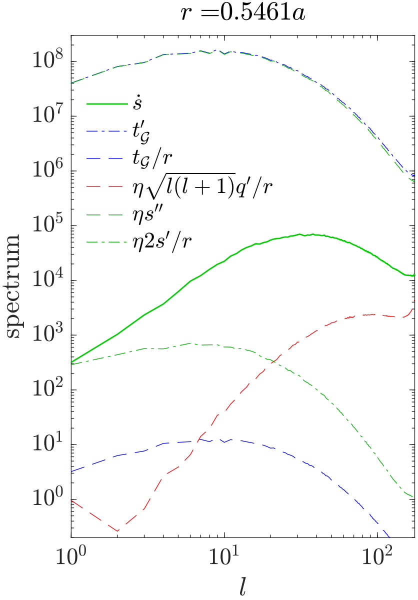
$$F_{\dot{B}} = F_{\dot{q}} + F_{\dot{s}} + F_{\dot{t}}$$

$$\dot{q} = -\frac{\sqrt{l(l+1)}}{r} t_{\mathcal{G}} - \eta \frac{l(l+1)}{r^2} q + \eta \frac{\sqrt{l(l+1)}}{r} s' + \eta \frac{\sqrt{l(l+1)}}{r^2} s$$

$$\dot{s} = t'_{\mathcal{G}} - \frac{t_{\mathcal{G}}}{r} - \eta \frac{\sqrt{l(l+1)}}{r} q' + \eta s'' + \eta \frac{2}{r} s'$$

- $t'_{\mathcal{G}}$ and $\eta s''$ dominate in magnitude
- spectra of $t'_{\mathcal{G}}$ and $\eta s''$ are shallow
- $t'_{\mathcal{G}}$ and $\eta s''$ cancel each other to leading order
- the steeper $F_{\dot{s}} \sim l^2$ scaling: a higher-order effect

Balance of terms for the large scales in the CMB boundary layer



$$F_{\dot{B}} \approx F_{\dot{q}} + F_{\dot{s}}$$

$$\dot{q} \approx \eta \frac{\sqrt{l(l+1)}}{r} s'$$

$$\dot{s} \approx t'_{\mathcal{G}} + \eta s''$$

$$F_{\dot{B}} \sim F_{\dot{q}} \sim F_{\dot{s}} \sim l^2$$

$$\implies \tau \sim l^{-1}$$

In contrast to the argument based on the frozen flux hypothesis:

- the diffusion term plays a role
- radial derivative is important

Summary

- at the small scales, scaling of τ only varies weakly with r
- at the large scales, $\tau \sim l^{-0.5} \rightarrow \tau \sim l^{-1}$ as r increases
- transition occurs within a boundary layer under the CMB
- boundary conditions, specifically diffusion, plays a role in the $\tau \sim l^{-1}$ observed at the surface