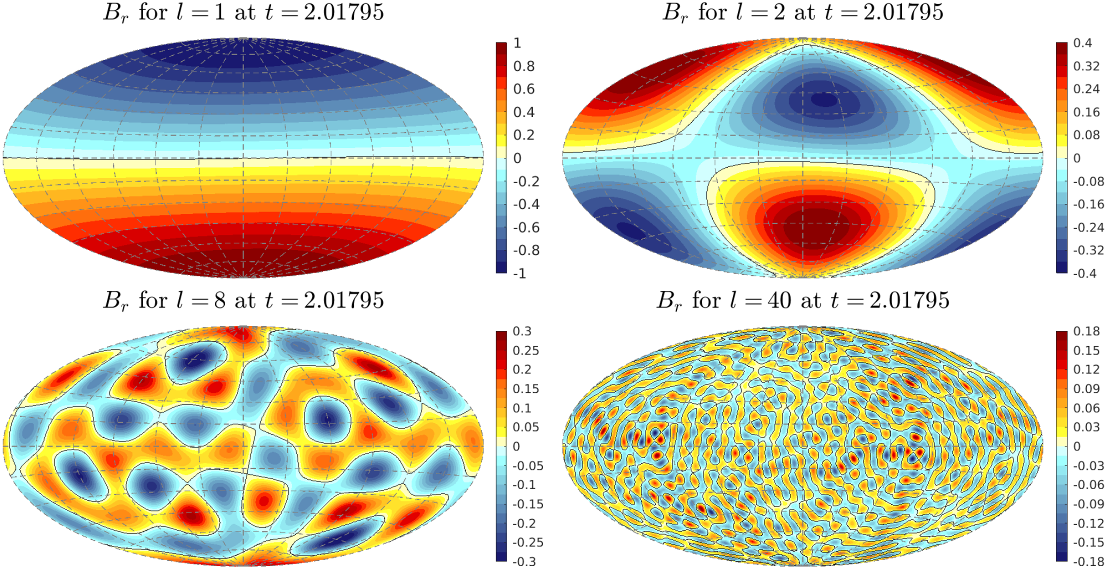
Scaling of the geomagnetic secular variation time scales

Yue-Kin Tsang

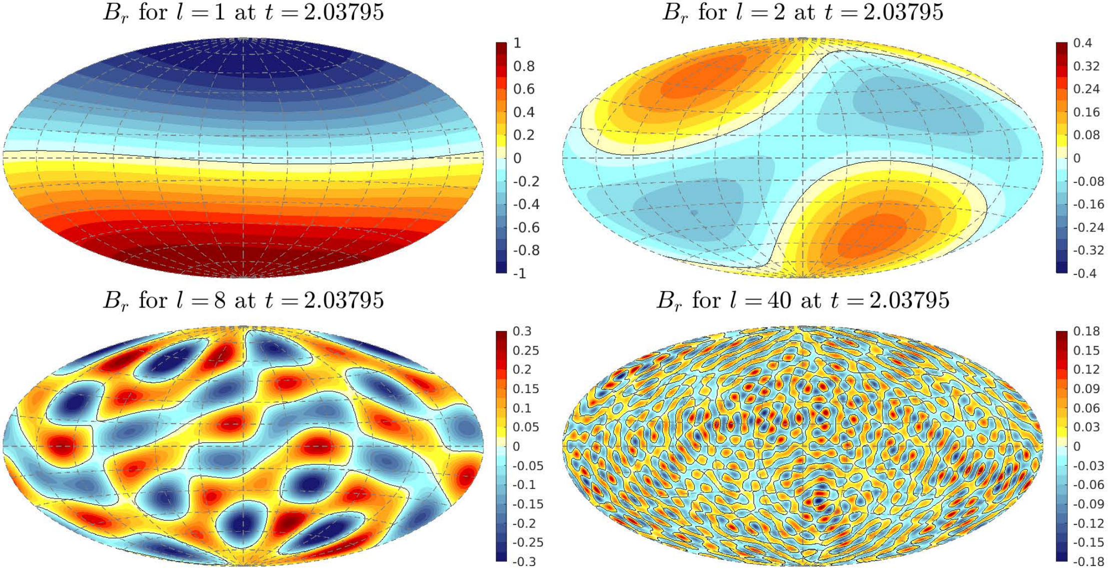
School of Mathematics, Statistics and Physics
Newcastle University

Chris Jones
University of Leeds

Time variation of the geomagnetic field at different spatial scales



Time variation of the geomagnetic field at different spatial scales



Spectra: to study properties at different spatial scales

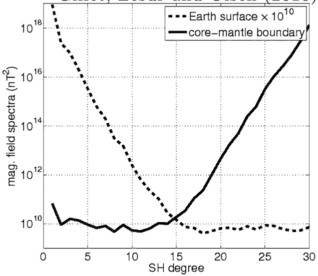
(1) Lowes spectrum $(r \geqslant r_{\rm cmb})$

$$R(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[\, g_{lm}^2(t) + h_{lm}^2(t) \,\right],$$

$$\sum_{l=1}^{\infty} R(l,r,t) = \frac{1}{4\pi} \oint |\boldsymbol{B}(r,\theta,\phi,t)|^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

(a = Earth's radius)

Gillet, Lesur and Olsen (2010)

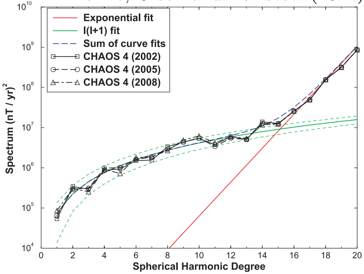


(2) Secular variation spectrum $(r \geqslant r_{\rm cmb})$

$$R_{\rm sv}(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[\, \dot{g}_{lm}^2(t) + \dot{h}_{lm}^2(t) \,\right]$$

$$\sum_{l=1}^{\infty} R_{\rm sv}(l,r,t) = \frac{1}{4\pi} \oint |\dot{\boldsymbol{B}}(r,\theta,\phi,t)|^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi\,, \quad \dot{\boldsymbol{B}} = \frac{\partial \boldsymbol{B}}{\partial t}$$

Holme, Olsen and Bairstow (2011)



Secular variation time-scale spectrum

$$R(l) \sim \text{“amount” of } B^2 \text{ in spatial scale } l$$

$$R_{\rm sv}(l) \sim \text{“amount” of } \dot{B}^2 \text{ in spatial scale } l$$

$$\tau_{\rm sv}(l,t) = \sqrt{\frac{R}{R_{\rm sv}}} = \sqrt{\frac{\sum_{m=0}^{l}\left(g_{lm}^2 + h_{lm}^2\right)}{\sum_{m=0}^{l}\left(\dot{g}_{lm}^2 + \dot{h}_{lm}^2\right)}} \quad (r \geqslant r_{\rm cmb})$$

- characteristic time scale of magnetic field structures with spatial scale characterised by l
- numerical simulations and *some* satellite data support the simple power-law: $\tau_{\rm sv}(l) \sim l^{-1}$ (there are still some debates about this)
- theoretically, a common argument based on the frozen-flux hypothesis:

$$\dot{B}_r = -\nabla_{\rm h} \cdot (\boldsymbol{u}_{\rm h} B_r)$$

$$\nabla_{\rm h} \sim \sqrt{l(l+1)} \sim l \quad \text{and} \quad \boldsymbol{u}_{\rm h} \sim U$$

$$\tau_{\rm sv} \sim B_r/\dot{B}_r \sim l^{-1}$$

Questions

1. τ_{sv} is defined using the Gauss coefficients obtained from $\boldsymbol{B}$ outside the outer core.

   Do τ_{sv} and the scaling law $\tau_{\text{sv}} \sim l^{-1}$ describe the time variation of $\boldsymbol{B}$ *inside* the outer core?

   [No. Inside the outer core, $\boldsymbol{B}$ is not potential. $\dot{B}_r$ $\dot{B}_\theta$ and $\dot{B}_\phi$ may all be important.]

2. Does the *frozen-flux argument* explain the scaling $\tau_{\text{sv}} \sim l^{-1}$ observed at the surface?

   [No. Magnetic diffusion is important near the CMB.]

3. What *mechanisms* lead to the observed scaling $\tau_{\text{sv}} \sim l^{-1}$?

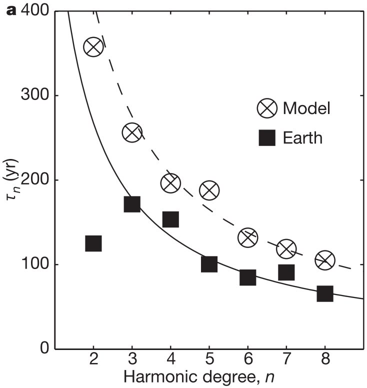
   [Briefly, balance between $\nabla \times (\boldsymbol{u} \times \boldsymbol{B})$ and $\nabla^2 \boldsymbol{B}$ at the CMB.

   Details depend on the boundary conditions.]

Scaling of $\tau_{\rm sv}(l)$: observations and numerical models

Christensen and Tilgner (2004)

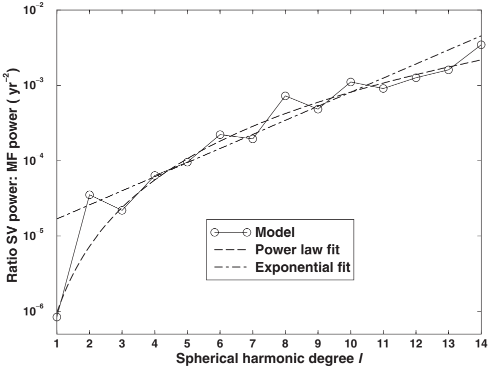
observation data 1840 – 1990
and numerical dynamo models



$$\tau_{\rm sv} \sim l^{-1}$$

Holme and Olsen (2006)

satellite data 1999 – 2003



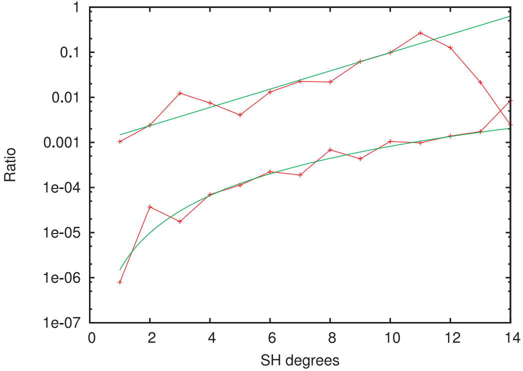
$$\frac{R_{\rm sv}}{R} \sim l^{2.9}$$

$$\Longrightarrow \quad \tau_{\rm sv} \sim l^{-1.45}$$

Scaling of $\tau_{sv}(l)$: observations and numerical models

Lesur et al. (2008)

6yr CHAMP + 5yr observatory data

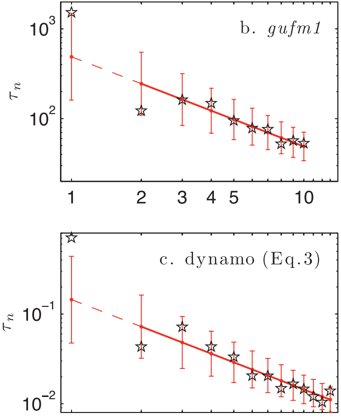


$$\frac{R_{sv}}{R} \sim l^{2.75}$$

$$\Longrightarrow \tau_{sv} \sim l^{-1.38}$$

Lhuillier et al. (2011)

'historical data' 1840 − 1990, satellite data (2005) and numerical dynamo models



$$\tau_{sv} \sim l^{-1}$$

The scaling exponent γ

$$\tau_{\text{sv}}(l) \sim l^{-\gamma} \qquad \text{(excluding } l=1\text{)}$$

- numerical models: $\gamma = 1$
- observations: mixed results, $1.32 < \gamma < 1.45$ and $\gamma = 1$
- time average vs. snapshot
- why study τ_{sv} : infer properties of the magnetohydrodynamics inside the outer core from observations at the surface

We should first ask :

Is τ_{sv} relevant to the time scale of $\dot{\boldsymbol{B}}$ *inside* the outer core?

Generalisation to inside the dynamo region (outer core)

Recall the definition of the Lowes spectrum $R(l,r,t)$ for $r \geqslant r_{\mathrm{cmb}}$,

$$\boldsymbol{B} = -\nabla\Psi\,, \quad \Psi(r,\theta,\phi,t) = a\sum_{l=1}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^{l+1}\hat{P}_{lm}(\cos\theta)\left[g_{lm}(t)\cos m\phi + h_{lm}(t)\sin m\phi\right]$$

$$\frac{1}{4\pi}\oint|\boldsymbol{B}(r,\theta,\phi,t)|^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \sum_{l=1}^{\infty}R(l,r,t)$$

$$R(l,r,t) = \left(\frac{a}{r}\right)^{2l+4}(l+1)\sum_{m=0}^{l}\left[\,g_{lm}^2(t) + h_{lm}^2(t)\,\right]$$

For any r, expand in vector spherical harmonics,

$$\boldsymbol{B}(r,\theta,\phi,t) = \sum_{lm}\left[q_{lm}(r,t)\hat{\boldsymbol{Y}}_{lm}(\theta,\phi) + s_{lm}(r,t)\hat{\boldsymbol{\Psi}}_{lm}(\theta,\phi) + t_{lm}(r,t)\hat{\boldsymbol{\Phi}}_{lm}(\theta,\phi)\right]$$

We define the magnetic energy spectrum $F(l,r,t)$ for all r:

$$\sum_{l=1}^{\infty}F(l,r,t) \equiv \frac{1}{4\pi}\oint|\boldsymbol{B}(r,\theta,\phi,t)|^2\,\mathrm{d}\Omega = \sum_{l=1}^{\infty}\left[\frac{1}{(2l+1)}\sum_{m=0}^{l}\left(|q_{lm}|^2 + |s_{lm}|^2 + |t_{lm}|^2\right)(4-3\delta_{m,0})\right]$$

Generalisation to inside the dynamo region (outer core)

$$F(l,r,t) = \frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|q_{lm}|^2 + |s_{lm}|^2 + |t_{lm}|^2\right)(4 - 3\delta_{m,0})$$

Similarly, define the time variation spectrum $F_{\dot{B}}(l,r,t)$:

$$\dot{\boldsymbol{B}}(r,\theta,\phi,t) = \sum_{lm} \left[\dot{q}_{lm}(r,t)\hat{\mathbf{Y}}_{lm}(\theta,\phi) + \dot{s}_{lm}(r,t)\hat{\mathbf{\Psi}}_{lm}(\theta,\phi) + \dot{t}_{lm}(r,t)\hat{\mathbf{\Phi}}_{lm}(\theta,\phi)\right]$$

$$\sum_{l=1}^{\infty} F_{\dot{B}}(l,r,t) \equiv \frac{1}{4\pi} \oint |\dot{\boldsymbol{B}}(r,\theta,\phi,t)|^2 \, \mathrm{d}\Omega = \sum_{l=1}^{\infty} \left[\frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|\dot{q}_{lm}|^2 + |\dot{s}_{lm}|^2 + |\dot{t}_{lm}|^2\right)(4 - 3\delta_{m,0})\right]$$

Then, the magnetic time-scale spectrum is defined as:

$$\tau(l,r) = \left\langle \sqrt{\frac{F(l,r,t)}{F_{\dot{B}}(l,r,t)}} \right\rangle_t$$

Outside the dynamo region: $F = R$, $F_{\dot{B}} = R_{\mathrm{sv}}$, $\tau = \tau_{\mathrm{sv}}$

A numerical model of geodynamo

Boussinesq, compositional driven, rotating convection of a electrically conducting fluid:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\frac{Pm}{Ek}\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\frac{Pm}{Ek}\nabla\Pi' + \left(\frac{RaPm^2}{Pr}\right)C'\boldsymbol{r} + \frac{Pm}{Ek}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + Pm\nabla^2\boldsymbol{u},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla^2\boldsymbol{B}$$

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{Pm}{Pr}\nabla^2 C - 1$$

$$\nabla \cdot \boldsymbol{u} = 0$$

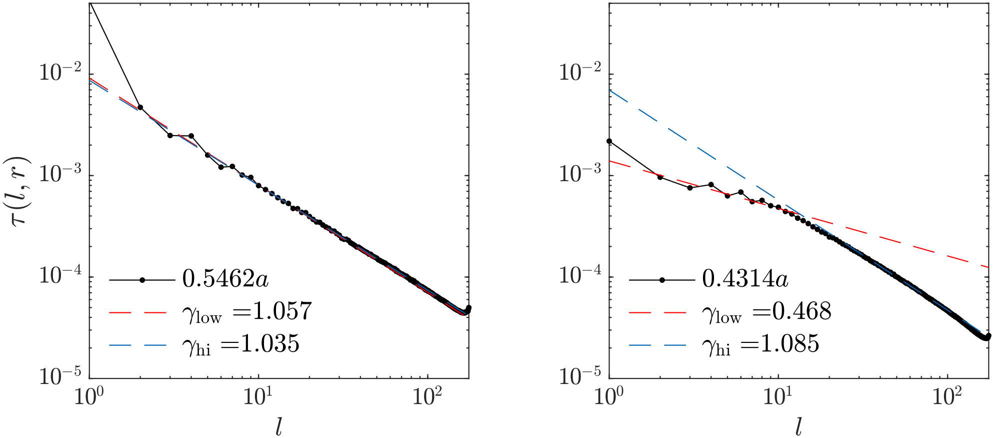
$$\nabla \cdot \boldsymbol{B} = 0$$

Boundary conditions: no-slip for $\boldsymbol{u}$, Neumann for C

Domain: a spherical shell $0.1912\,a \leqslant r \leqslant 0.5462\,a$

$Ra = 2.7 \times 10^8\,, Ek = 2.5 \times 10^{-5}\,, Pm = 2.5\,, Pr = 1$

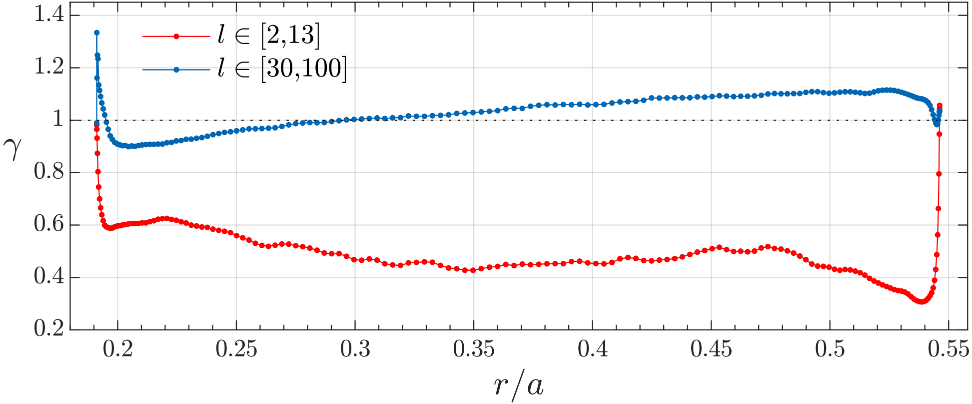
Magnetic time-scale spectrum $\tau(l, r)$ at different depth



For the large-scale modes (small l),

- at the surface: $\tau \sim l^{-1}$
- in the interior: $\tau \sim l^{-0.5}$, *the large-scale modes speeds up in the interior!*

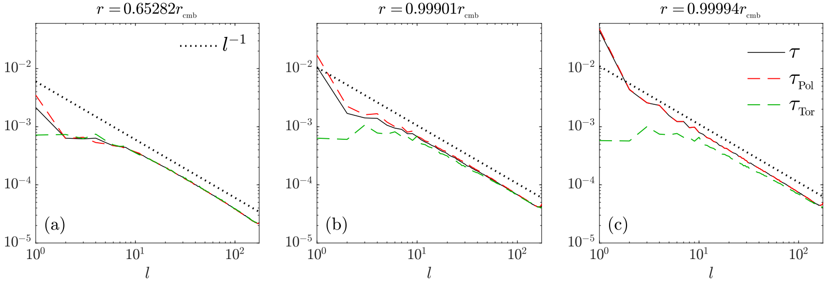
Change in the scaling of τ: where does it occur?



- γ for the large-scale modes increases sharply within a boundary layer under CMB

Focus on the large scales in following discussion ...

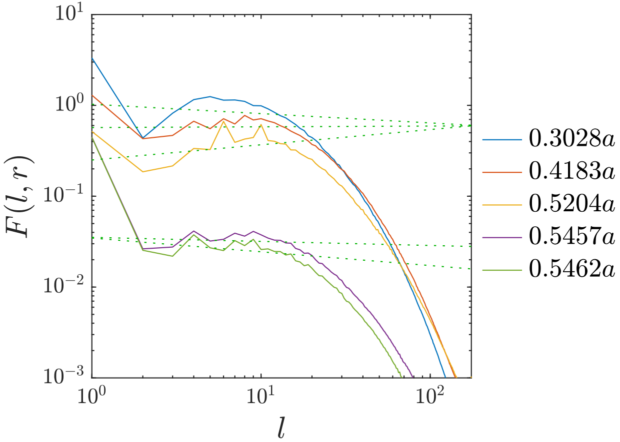
Poloidal and toroidal time scales



$$\boldsymbol{B} = \boldsymbol{B}_{\rm Pol} + \boldsymbol{B}_{\rm Tor}\ , \qquad \tau_{\rm Pol} = \sqrt{\frac{\text{spectrum of } \boldsymbol{B}_{\rm Pol}}{\text{spectrum of } \dot{\boldsymbol{B}}_{\rm Pol}}}\ , \qquad \tau_{\rm Tor} = \sqrt{\frac{\text{spectrum of } \boldsymbol{B}_{\rm Tor}}{\text{spectrum of } \dot{\boldsymbol{B}}_{\rm Tor}}}$$

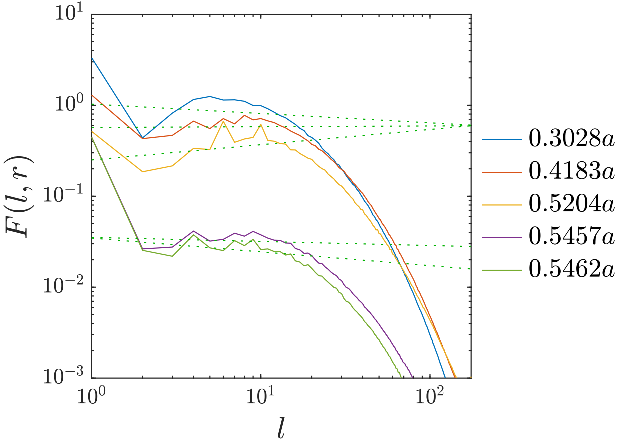
- interior: $\boldsymbol{B}_{\rm Pol}$ and $\boldsymbol{B}_{\rm Tor}$ are equally important, $\tau = \tau_{\rm Pol} = \tau_{\rm Tor}$ all have the same shape
- CMB: $\boldsymbol{B}_{\rm Tor} \to \boldsymbol{0}$ due to the magnetic boundary condition, so $\boldsymbol{B} \approx \boldsymbol{B}_{\rm Pol}$
  - $\tau_{\rm Tor}$ has the same shape as in the interior but it is irrelevant
  - $\tau_{\rm Pol}$ changes shape as $r \to r_{\rm cmb}$, $\tau = \tau_{\rm Pol} \sim l^{-1}$
- contribution of $\dot{B}_\theta$ and $\dot{B}_\phi$ to $\dot{\boldsymbol{B}}$ in the interior masked by the boundary conditions

Change in the scaling of τ: who causes it?



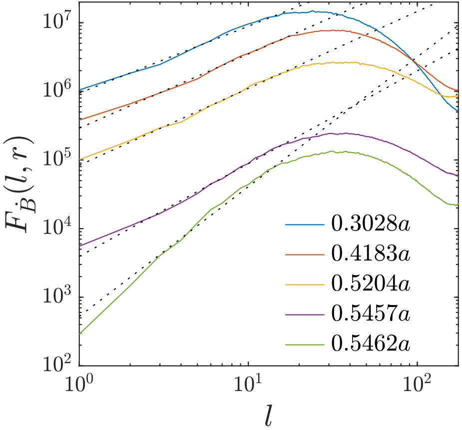
$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}}$$

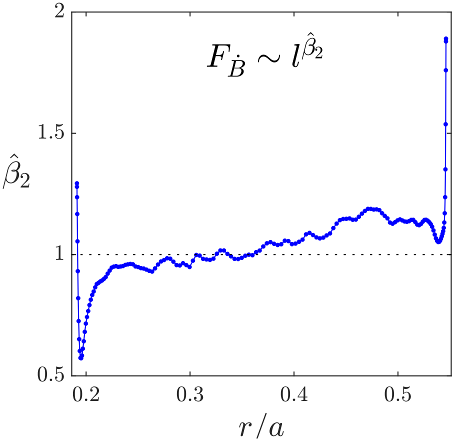
Change in the scaling of τ: who causes it?



$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{F_{\dot{B}}}} \sim F_{\dot{B}}{}^{-\frac{1}{2}}$$

Change in the scaling of τ: who causes it?



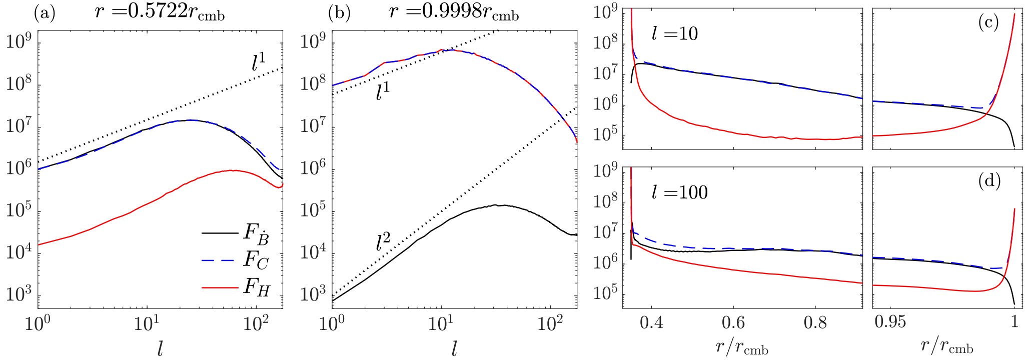


$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{F_{\dot{B}}}} \sim F_{\dot{B}}^{-\frac{1}{2}}$$

$$F_{\dot{B}} \sim l \implies \tau \sim l^{-0.5} \quad \text{(interior)}$$

$$F_{\dot{B}} \sim l^2 \implies \tau \sim l^{-1} \quad \text{(surface)}$$

Balance of terms (large scales) in the induction equation



$$\dot{\boldsymbol{B}} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} = \boldsymbol{C} + \boldsymbol{H}$$

- interior: $F_C \sim F_H \sim l$, $F_{\dot B} \approx F_C$ (magnetic diffusion negligible) , $F_{\dot B} \sim l$
- CMB: $F_C \sim F_H \sim l$, $F_C \approx F_H$ ($\boldsymbol{C}$ and $\boldsymbol{H}$ cancel to leading order) , $F_{\dot B} \sim l^2$
- $\boldsymbol{H}$ is important $\Rightarrow$ frozen-flux argument is not applicable in explaining $\tau \sim l^{-1}$ at CMB

Summary

- scaling of $\tau(l, r)$ with l observed outside the outer core is different from that in the interior
- for the large scales:

$$\begin{aligned} \tau &\sim l^{-0.5}, \quad && \text{in the interior} \\ \tau &\sim l^{-1}, && \text{at the CMB} \end{aligned}$$

  the transition occurs within a boundary layer under the CMB
- time variation of $\boldsymbol{B}_{\mathrm{Tor}}$ in the interior is hidden from surface observation
- for the large scales, $F_{\dot{B}}$ is responsible for the transition ($\tau = \sqrt{F/F_{\dot{B}}}$)
  - in the interior, induction term $\boldsymbol{C}$ dominates, $\dot{\boldsymbol{B}} \approx \boldsymbol{C}$ and $F_{\dot{B}} \sim l$
  - at the CMB (no-slip), balance between the induction term and magnetic diffusion leads to $F_{\dot{B}} \sim l^2$, meaning frozen-flux argument not applicable