Advection-condensation of water vapor with coherent stirring: a stochastic approach

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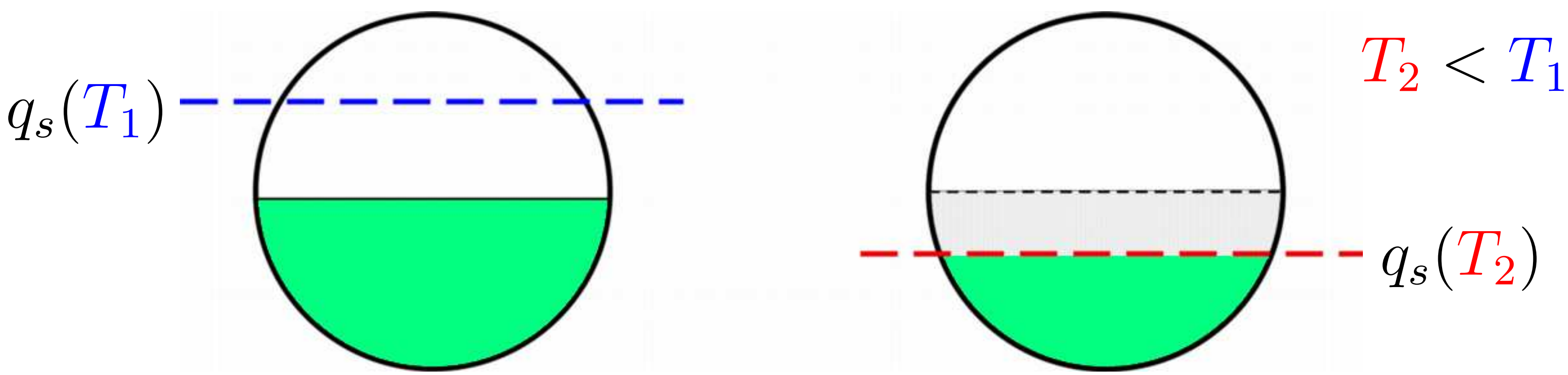
Condensation of water vapour

- specific humidity of an air parcel:

$$q = \frac{\text{mass of water vapor}}{\text{total air mass}}$$

- saturation specific humidity, $q_s(T)$
  - when $q > q_s$, condensation occurs
  - excessive moisture precipitates out, $q \to q_s$
  - $q_s(T)$ decreases with temperature T
  - T decreases with latitude $\Rightarrow q_s$ position dependent



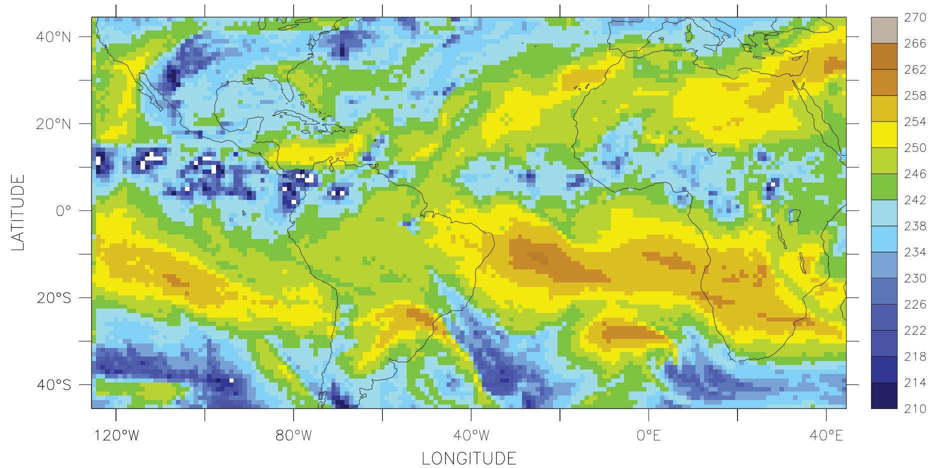
Atmospheric moisture and climate

- Earth's radiation budget:
  - absorption of incoming shortwave radiation generates heat
  - heat carried away by outgoing longwave radiation (OLR)
- water vapor is a greenhouse gas that traps OLR
- $\mathrm{OLR} \sim -\langle \log q \rangle$
- $\mathrm{OLR} \sim -\langle \log[\langle q \rangle + q'] \rangle \approx -\log \langle q \rangle + \dfrac{1}{2\,\langle q \rangle^2} \langle q'^2 \rangle$
- how fluctuation q' is generated?
- what is the probability distribution of water vapor in the atmosphere?

Advection-condensation paradigm

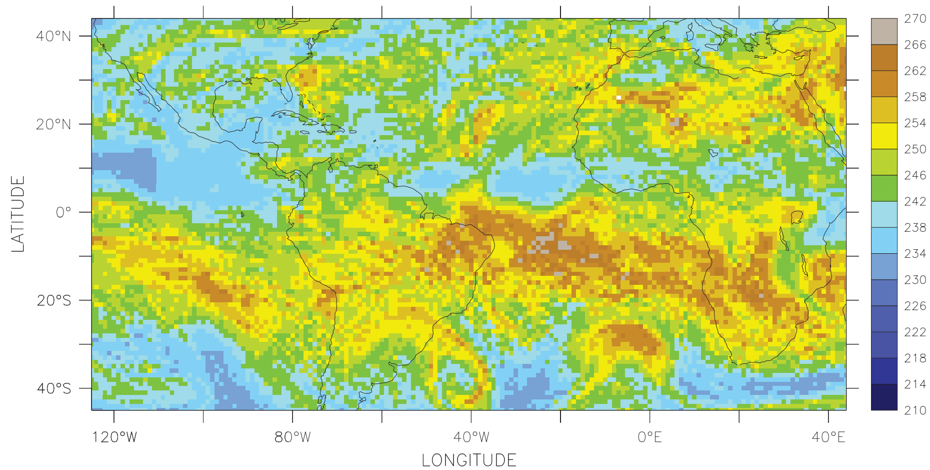
Large-scale advection + condensation

$\rightarrow$ reproduce (leading-order) observed humidity distribution



Observed WVEBBT

Observation



Simulated WVEBBT

Simulation

– velocity and q_s field from observation
– trace parcel trajectories backward to the lower boundary layer (source)
– track condensation along the way

ignore: cloud-scale microphysics, molecular diffusion, …

(Pierrehumbert & Roca, GRL, 1998)

Advection-condensation model

- PDE formualtion:

$$\frac{\partial q}{\partial t} + \vec{u} \cdot \nabla q = S - C$$

q is treated as a passive scalar advected by a prescribed $\vec{u}$

- Particle formulation:

$$\mathrm{d}\vec{X}(t) = \vec{u}\,\mathrm{d}t\,, \quad \mathrm{d}Q(t) = (S - C)\mathrm{d}t$$

air parcel at location $\vec{X}$ carrying specific humidity Q

  - $S =$ moisture source (evaporation)
  - $C =$ condensation sink, in the rapid condensation limit

$$C : q(\vec{x}, t) \mapsto \min\left[\, q(\vec{x}, t)\,,\; q_s(\vec{x})\,\right]$$

  - saturation profile: $q_s(y) = q_0 \exp(-\alpha y)$
  - $y =$ latitude (advection on a midlatitude isentropic surface) or altitude (vertical convection in troposphere)

Previous analytical results

1D stochastic models: $u \sim$ spatially uncorrelated random process

- **Pierrehumbert, Brogniez & Roca 2007**: white noise, $S = 0$
- **O'Gorman & Schneider 2006**: Ornstein-Uhlenbeck process, $S = 0$

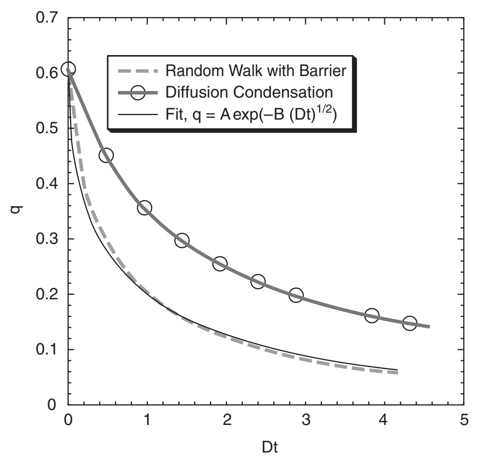


FIGURE 6.8. Decay of ensemble mean specific humidity at $y = 0.5$ for the bounded random walk with a barrier at $y = 0$. The thin

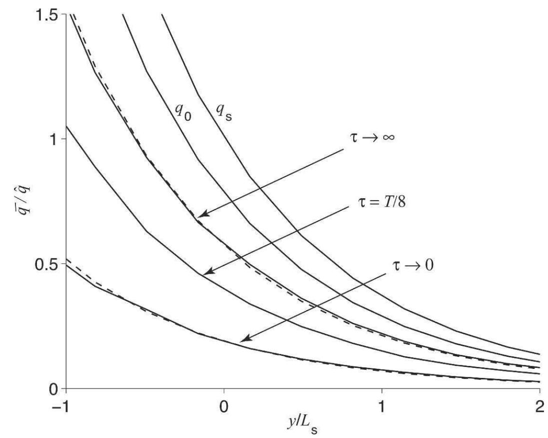
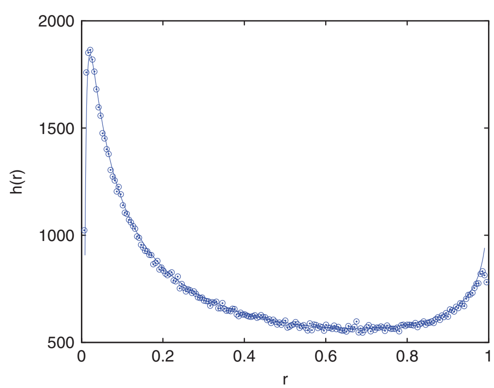
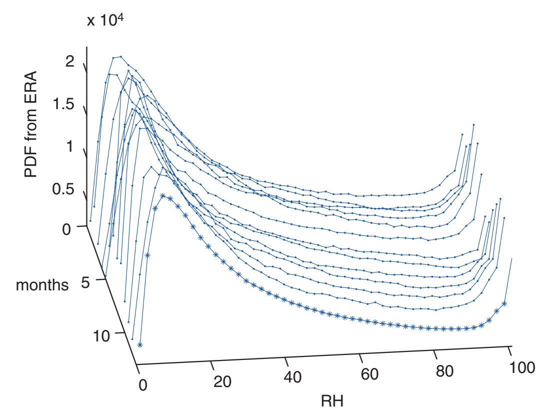


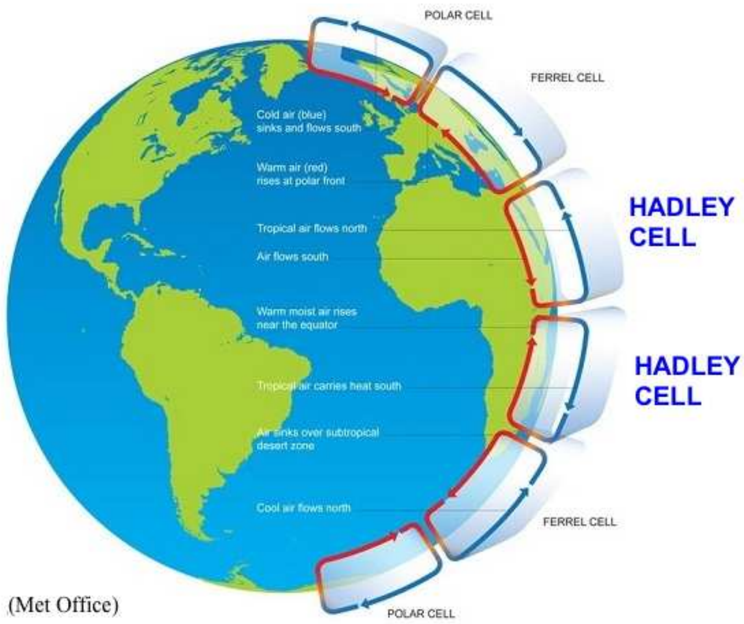
FIG. 2. Mean specific humidity vs meridional distance for initial value problem. Moisture distributions are shown after the evolution times T at which $L(T) = 4L_s$ in each case. Solid lines are

- **Sukhatme & Young 2011**: white noise with a boundary source





Coherent circulation in the atmosphere



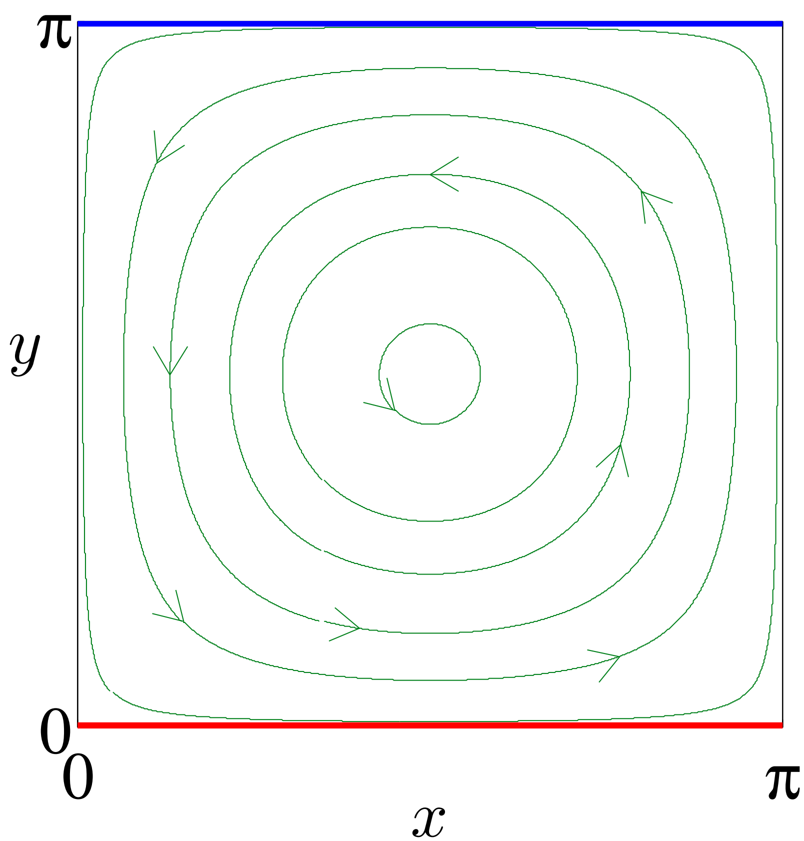
(Met Office)

- moist, warm air rises near the equator
- poleward transport in the upper troposphere
- subsidence in the subtropics ($\sim$ 30°N and 30°S)
- transport towards the equator in the lower troposphere

Q: response of rainfall patterns to changes in the Hadley cells?

Steady-state problem

- bounded domain: $[0, \pi] \times [0, \pi]$, reflective B.C.
- $q_s(y) = q_{\max} \exp(-\alpha y)$: $q_s(0) = q_{\max}$ and $q_s(\pi) = q_{\min}$
- resetting source: $Q = q_{\max}$ if particle hits $y = 0$

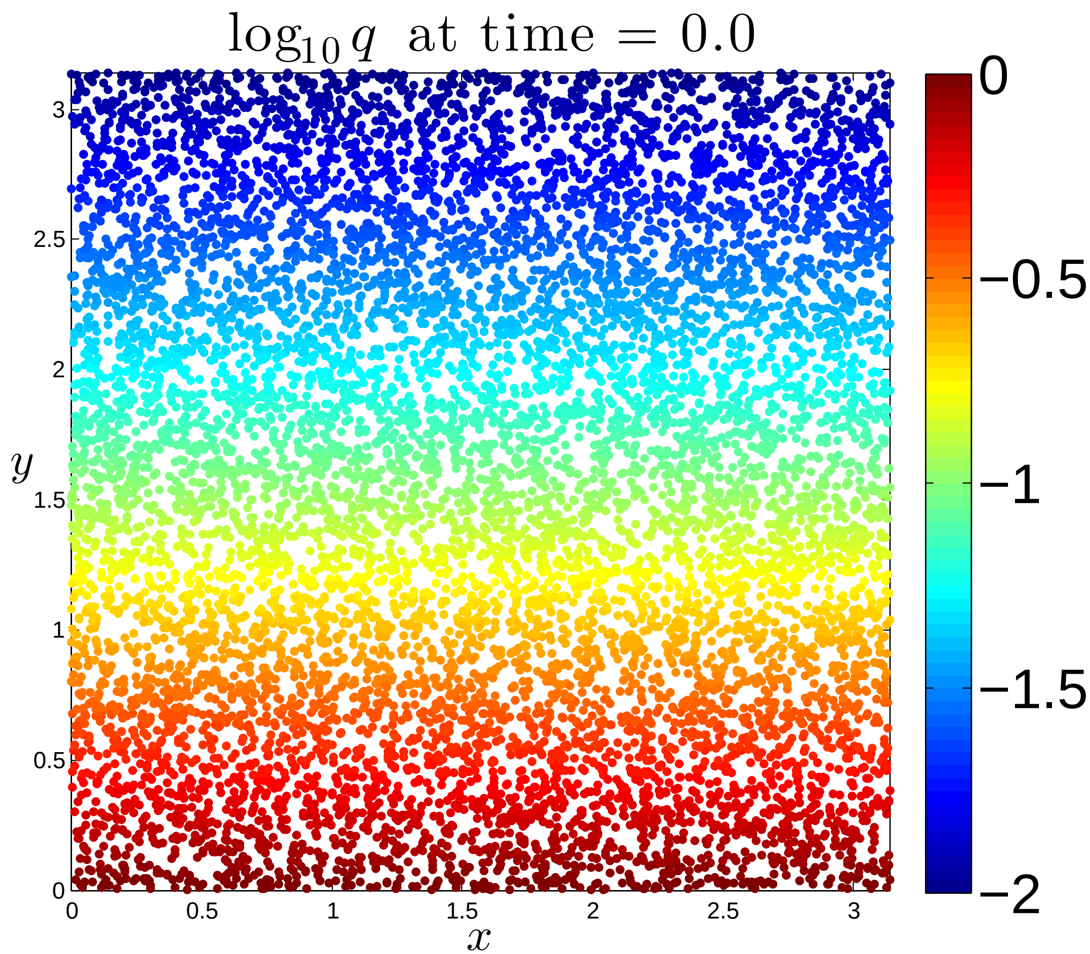


cellular flow: $\psi = -U \sin(x) \sin(y)$; $\quad (u, v) = (-\psi_y, \psi_x)$

Stochastic system with source

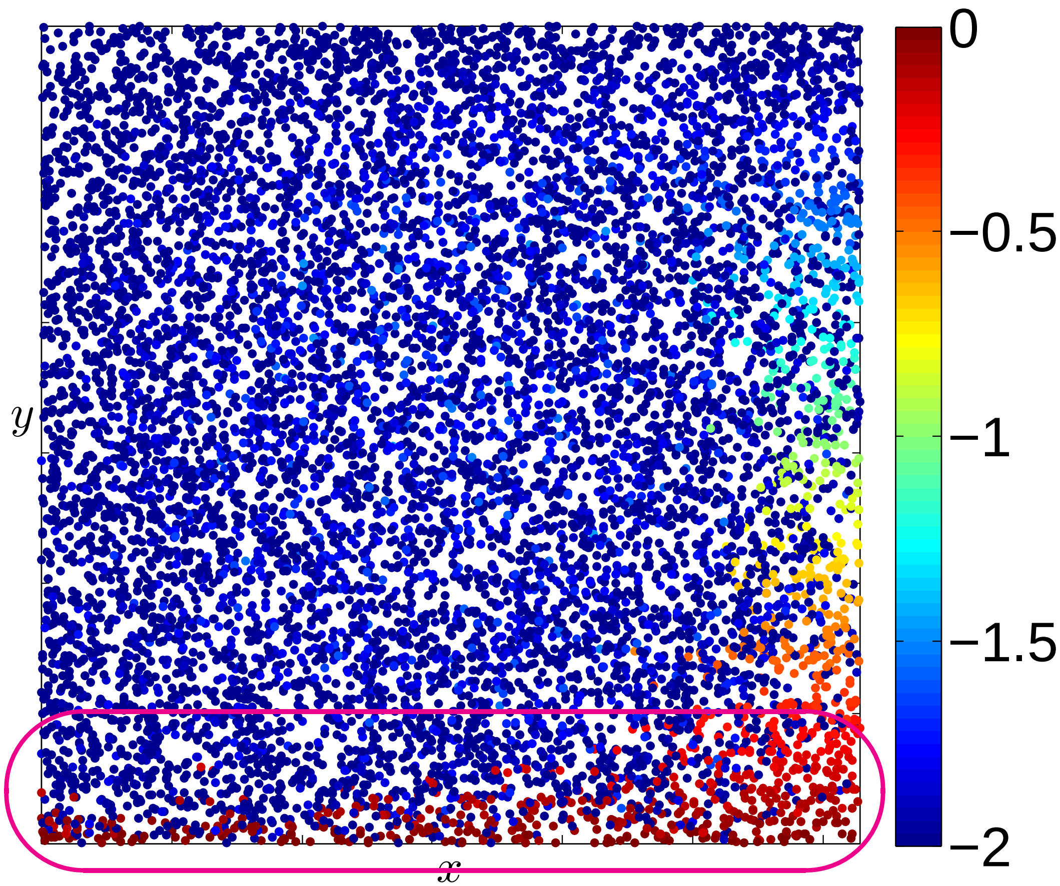
$$\mathrm{d}X(t) = u(X,Y)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_1(t)$$
$$\mathrm{d}Y(t) = v(X,Y)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_2(t)$$
$$\mathrm{d}Q(t) = [S(Y) - C(Q,Y)]\mathrm{d}t$$

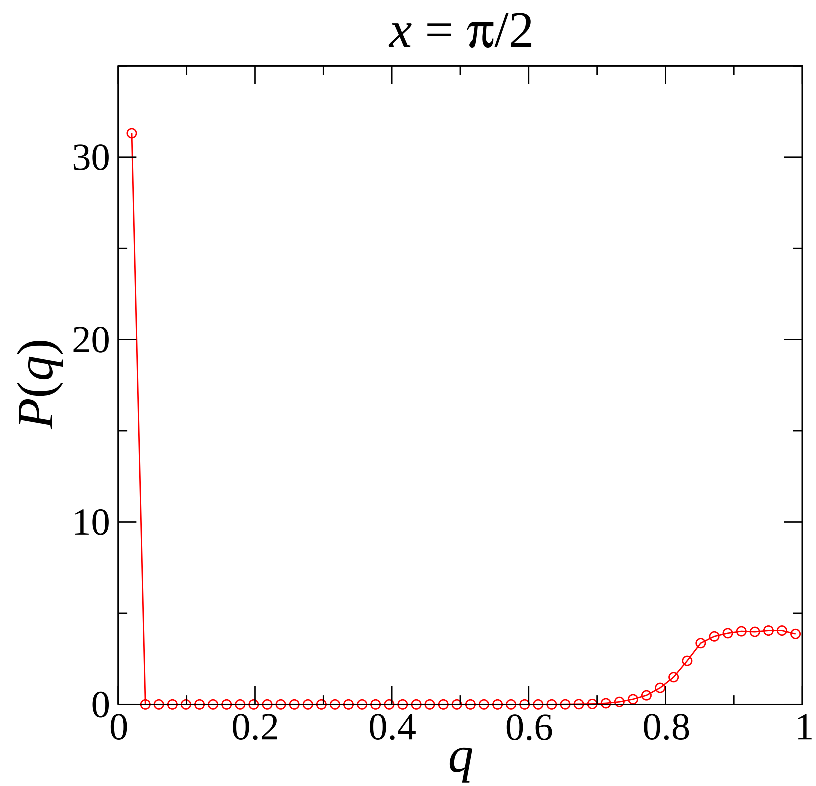
$$\psi = -U \sin x \sin y$$
$$u = -\psi_y$$
$$v = \psi_x$$



$$U = 1$$
$$\kappa = 10^{-2}$$

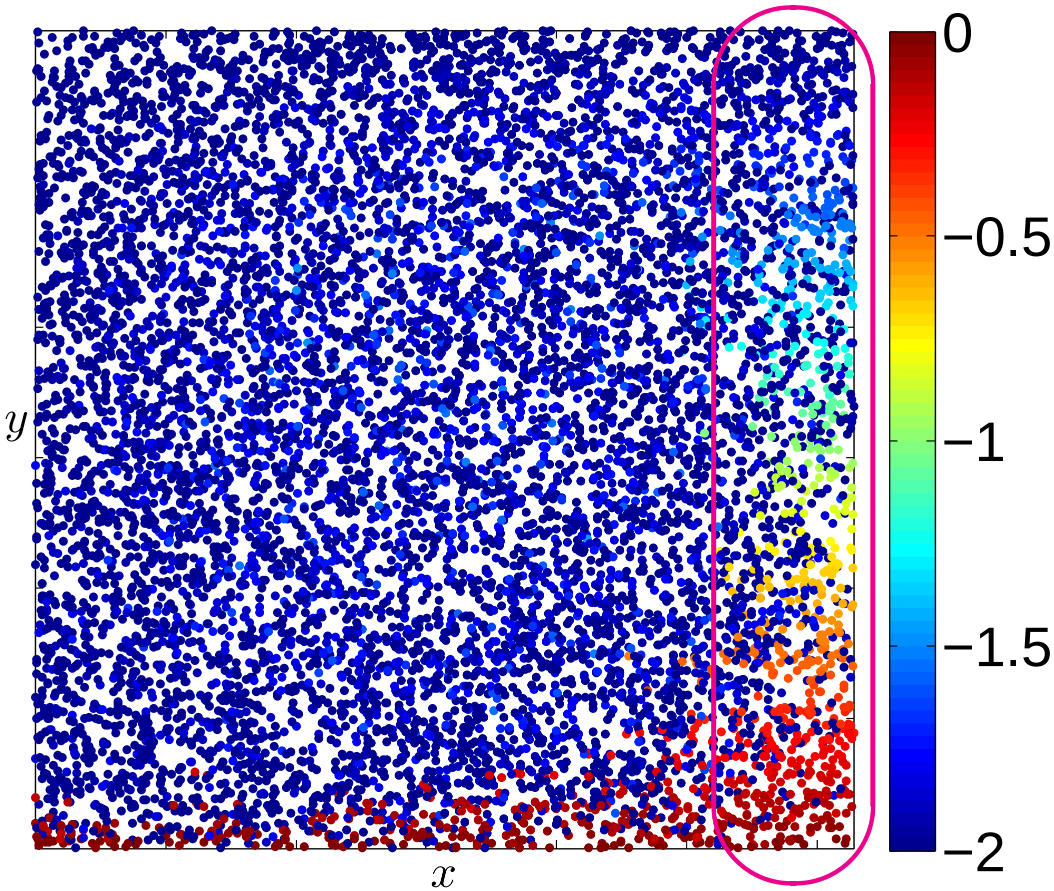
Source boundary layer

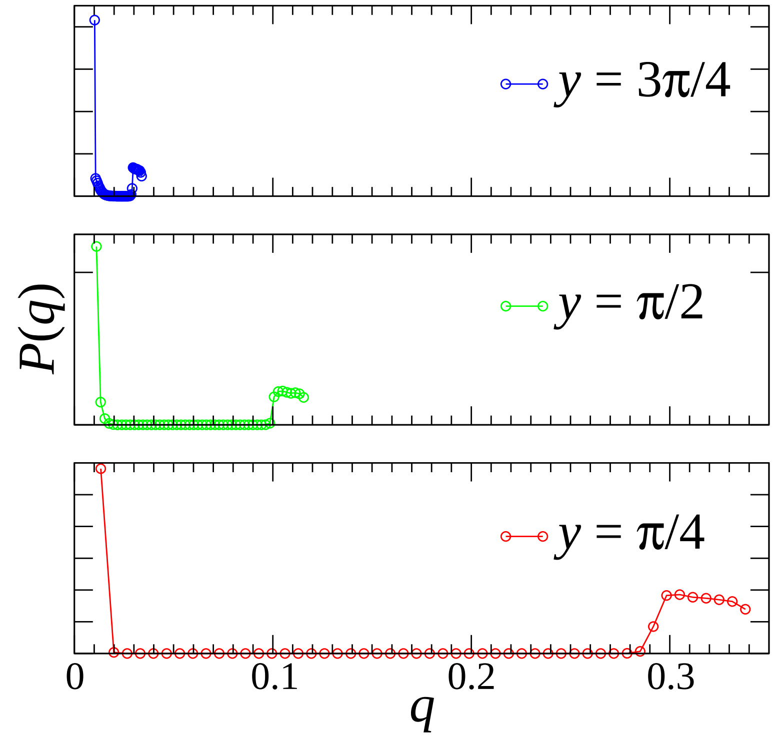




- **Bimodal distribution**: layer consists mainly of either:
  - $Q = q_{\rm min}$ from upstream of the flow and diffuse in from the domain interior
  - $Q \approx q_{\rm max}$ from the resetting source
- particles with $Q \approx q_{\rm max}$ spreading into the domain as x increases

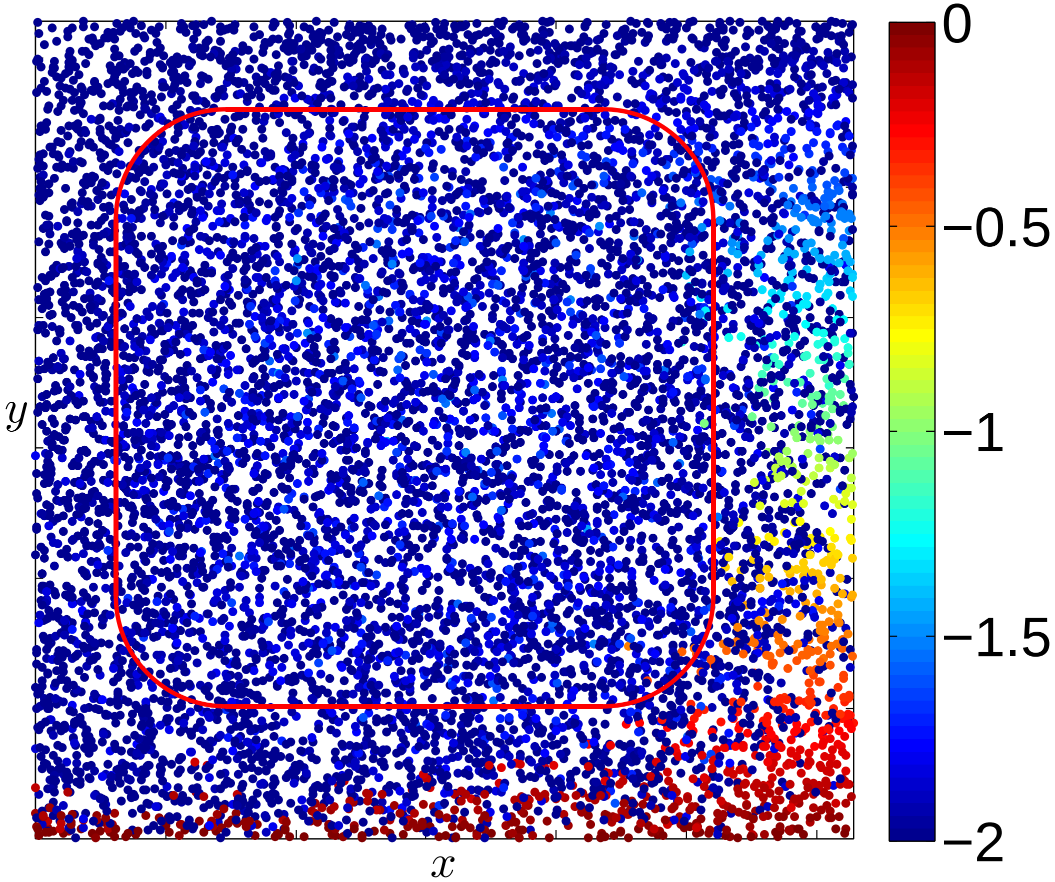
Condensation boundary layer

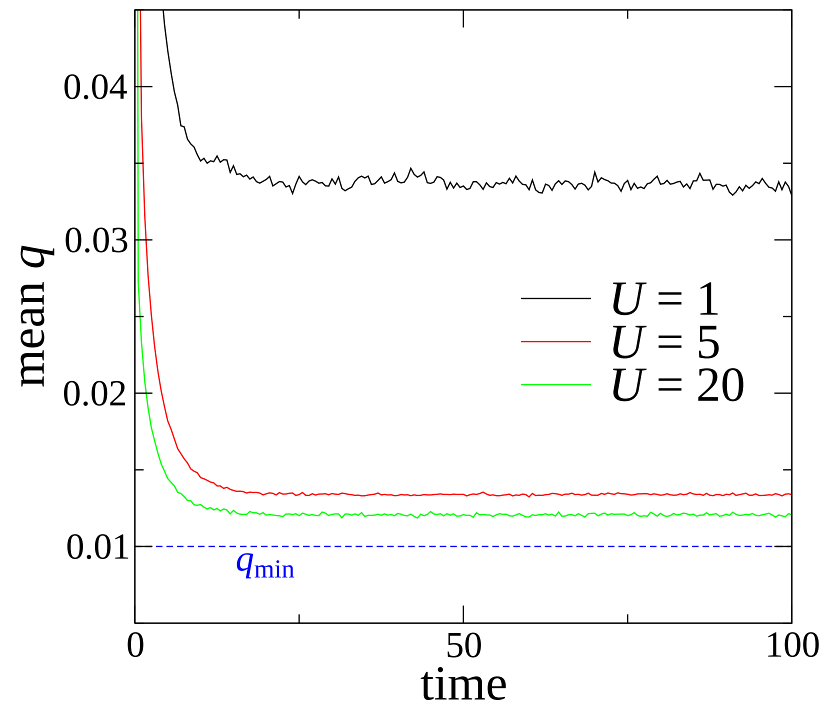




- moist particles move up into region of low $q_s(y)$
- at some fixed height y_1: mainly consists of $Q = q_{\min}$ (diffuse in from the interior) and $Q = q_s(y_1)$ — **Bimodal distribution**
- condensation $\Rightarrow$ localized rainfall over a narrow $O(\epsilon^{1/2})$ region

Interior region





- a homogeneous region of very dry air $Q \approx q_{\text{min}}$ is created in the domain interior
- the vortex "shields" the source from the interior
- interior effectively undergoing stochastic drying

Steady-state problem

Steady-state Fokker-Planck equation for $P(x, y, q)$:

$$\epsilon^{-1}\vec{u} \cdot \nabla P - \partial_q[(S - C)P] = \nabla^2 P\,, \qquad \epsilon = \kappa/(UL) \ll 1$$

Rapid condensation limit:

$$\left.\begin{aligned} P(x, y, q) \neq 0 \\ C = 0 \end{aligned}\right\} \text{ for } x, y \in [0, \pi] \text{ and } q \in [q_{\min}, q_s(y)]$$

Resetting source at bottom boundary:

$$P(x, y = 0, q) = \pi^{-1}\delta(q - q_{\max})$$

At the top boundary: $P(x, y = \pi, q) = \pi^{-1}\delta(q - q_{\min})$

Hence,

$$\epsilon^{-1}\vec{u} \cdot \nabla P = \nabla^2 P$$

which predicts a boundary layer of thickness $O(\epsilon^{1/2})$

Matched asymptotics

1. domain interior, to leading-order:

$$P_0 = \pi^{-2}\delta(q - q_{\min})$$

2. source boundary layer:

$$P_0 = G(x,y)\,\delta(q - q_{\min}) + H(q,x,y)$$

3. condensation boundary layer:

$$P_0 = G(x,y)\,\delta(q - q_{\min}) + [\pi^{-2} - G(x,y)]\,\delta(q - q_s(y))$$

In the $O(\epsilon^{1/2})$ boundary layers, introducing coordinates (Childress 1979):

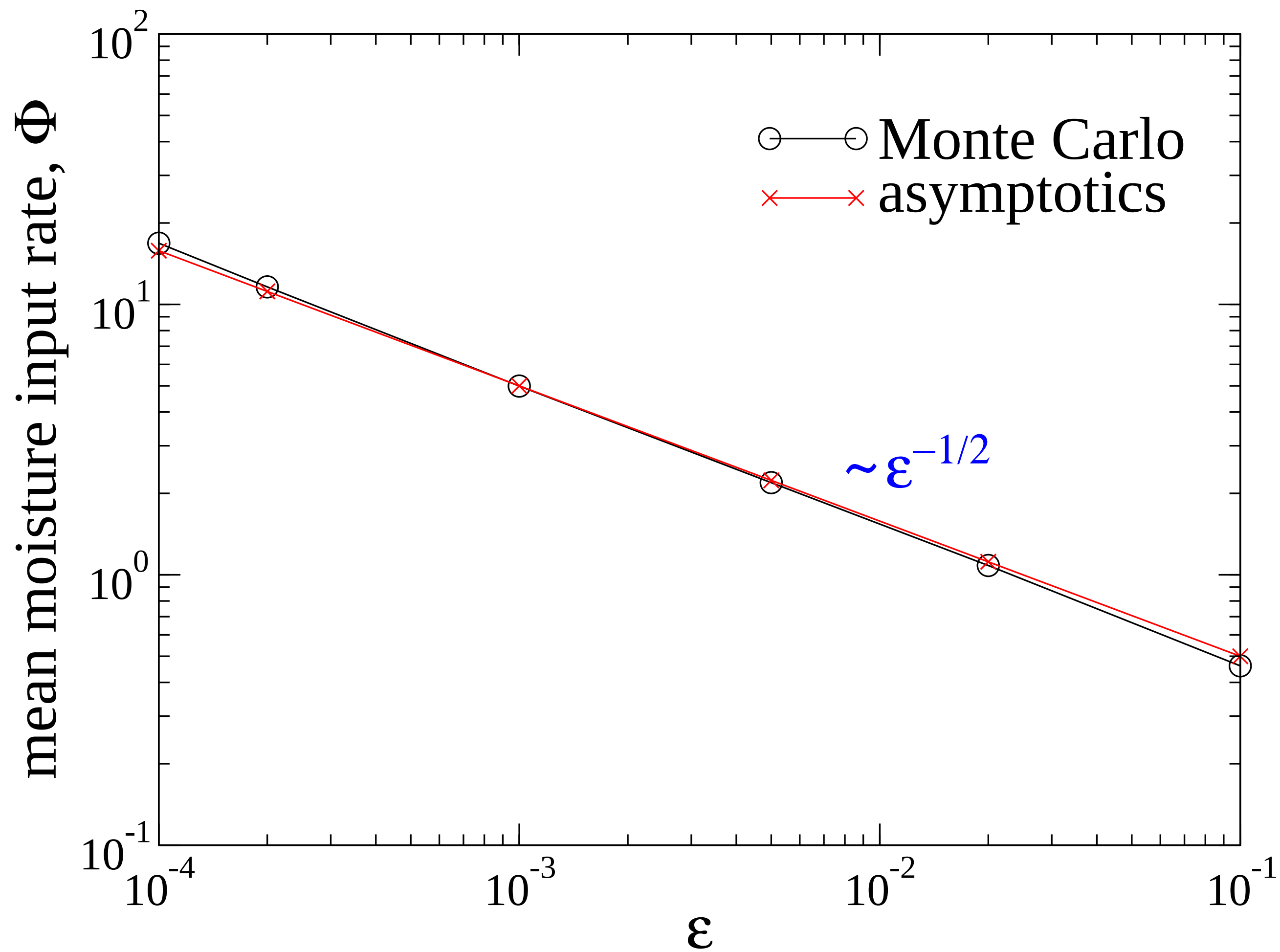
$$\zeta = \epsilon^{-1/2}\psi \quad \text{and} \quad \sigma = \int |\nabla\psi|\,\mathrm{d}l\,, \quad l = \text{arclength}$$

Equation for $G(\sigma,\zeta)$ reduces to:

$$\partial_\sigma G = \partial_{\zeta\zeta} G$$

Mean moisture input rate Φ

$$\Phi = \epsilon^{-1/2}\sqrt{\frac{8\kappa}{\pi}}\left(q_{\max} - q_{\min}\right), \quad \epsilon = \kappa/(UL)$$



Other diagnostics: horizontal rainfall profile, vertical moisture flux, …etc