Ellipsoidal vortices
beyond the quasi-geostrophic approximation

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Rotating, Continuously Stratified Flows

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} + f_0\,\hat{k} \times \vec{u} = -\frac{1}{\rho_0}\nabla\Phi + b\hat{k}$$

$$\rho(\vec{x}) = \rho_0 + \bar{\rho}(z) + \rho'(\vec{x})$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} + N^2 w = 0$$

$$b = -\rho_0^{-1} g\,\rho'$$

$$\vec{\omega} = \nabla \times \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

- rotating: f-plane approximation, $\vec{\Omega} = \frac{1}{2} f_0\,\hat{k}$
- stratified: constant buoyancy frequency, $N^2 = -\rho_0^{-1} g\,\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}$

$$\sigma = \frac{f_0}{N} = 0.1$$

- potential vorticity anomaly:

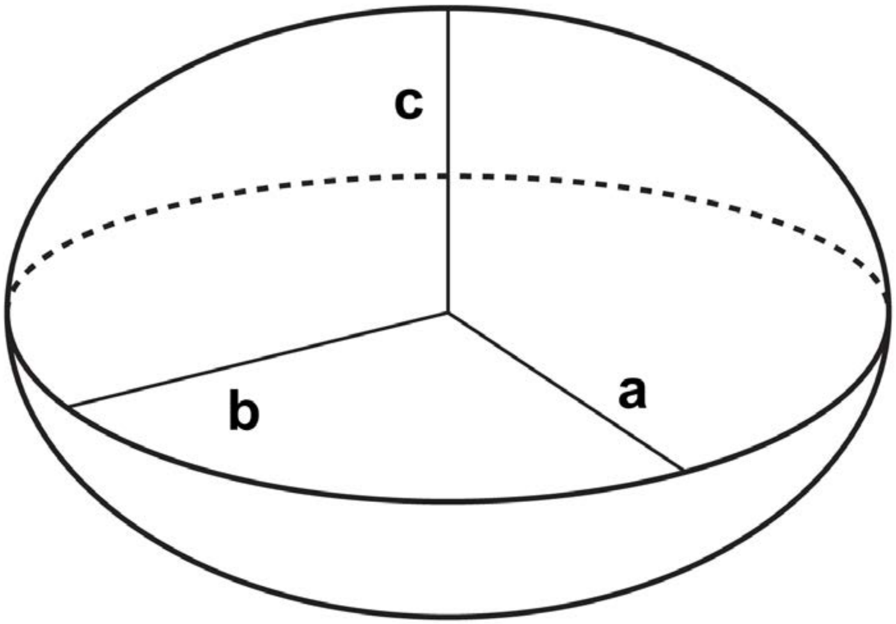
$$Q \equiv \frac{\vec{\omega} + f_0\hat{k}}{f_0} \cdot \frac{\nabla b + N^2\hat{k}}{N^2} = 1 + q \sim (\vec{\omega} + 2\vec{\Omega}) \cdot \nabla\rho$$

Computational Domain and Initial Conditions

- periodic three-dimensional domain: $L \times L \times H$
- small aspect ratio: $H = \sigma L$
- initial conditions: an ellipsoidal volume of constant PV anomaly q_0 in a *near balanced state*
- aspect ratios λ and μ:

$$\lambda = \frac{a}{b} \quad (a < b)$$

$$\mu = \frac{c}{\sigma \sqrt{ab}}$$



Numerical methods

- change of variables: $\{b, \vec{u}_h\} \rightarrow \{q, \vec{A}_h\}$

$$\vec{A}_h = \vec{\omega}_h / f_0 \; + \; \nabla_h b / f_0^2$$

- equations of motion:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0$$

$$\frac{\mathrm{D}\vec{A}_h}{\mathrm{D}t} + f_0 \hat{k} \times \vec{A}_h = \mathcal{N}(\vec{A}_h, q)$$

- q is materially conserved $\Rightarrow$ equations can be solved efficiently by the Contour-Advective Semi-Lagrangian (CASL) algorithm (Dritschel & Viúdez 2003 , JFM **488**, 123-150)

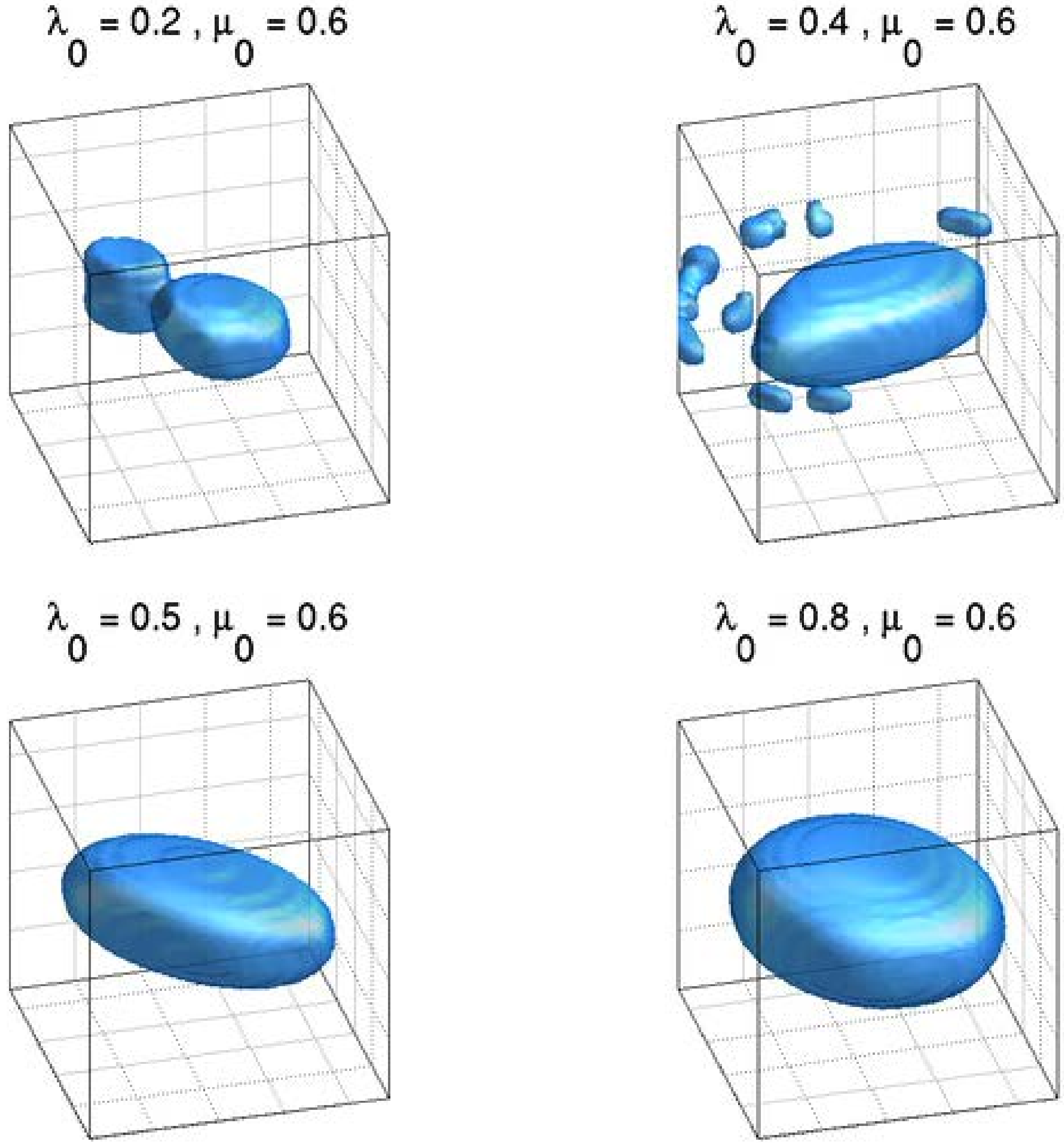
Beyond the QG approximation

- In the limit of asymptotically strong rotation and stratification, *geostrophic-hydrostatic balance* leads to the quasi-geostrophic (QG) approximation

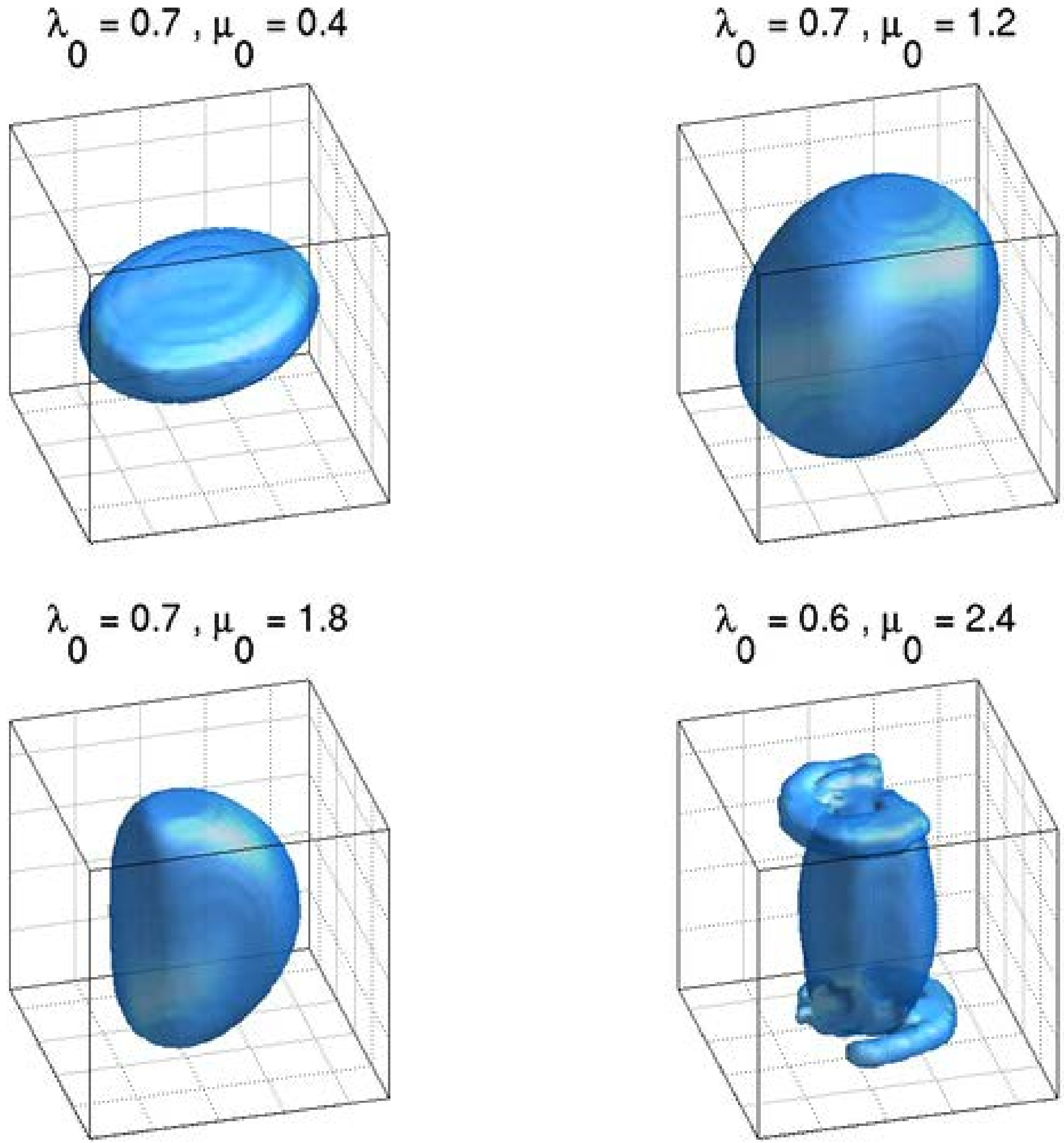
$$\frac{\mathrm{D}q_{\mathrm{QG}}}{\mathrm{D}t} = 0$$

- $\vec{A}_h$: a measure of the leading order imbalance
- in contrast to the QG system, the ellipsoid is not an exact solution to the full non-hydrostatic system
- extend previous works on ellipsoidal vortices in the QG approximation (much less computational intensive)
- problem parameters: q_0, λ and μ

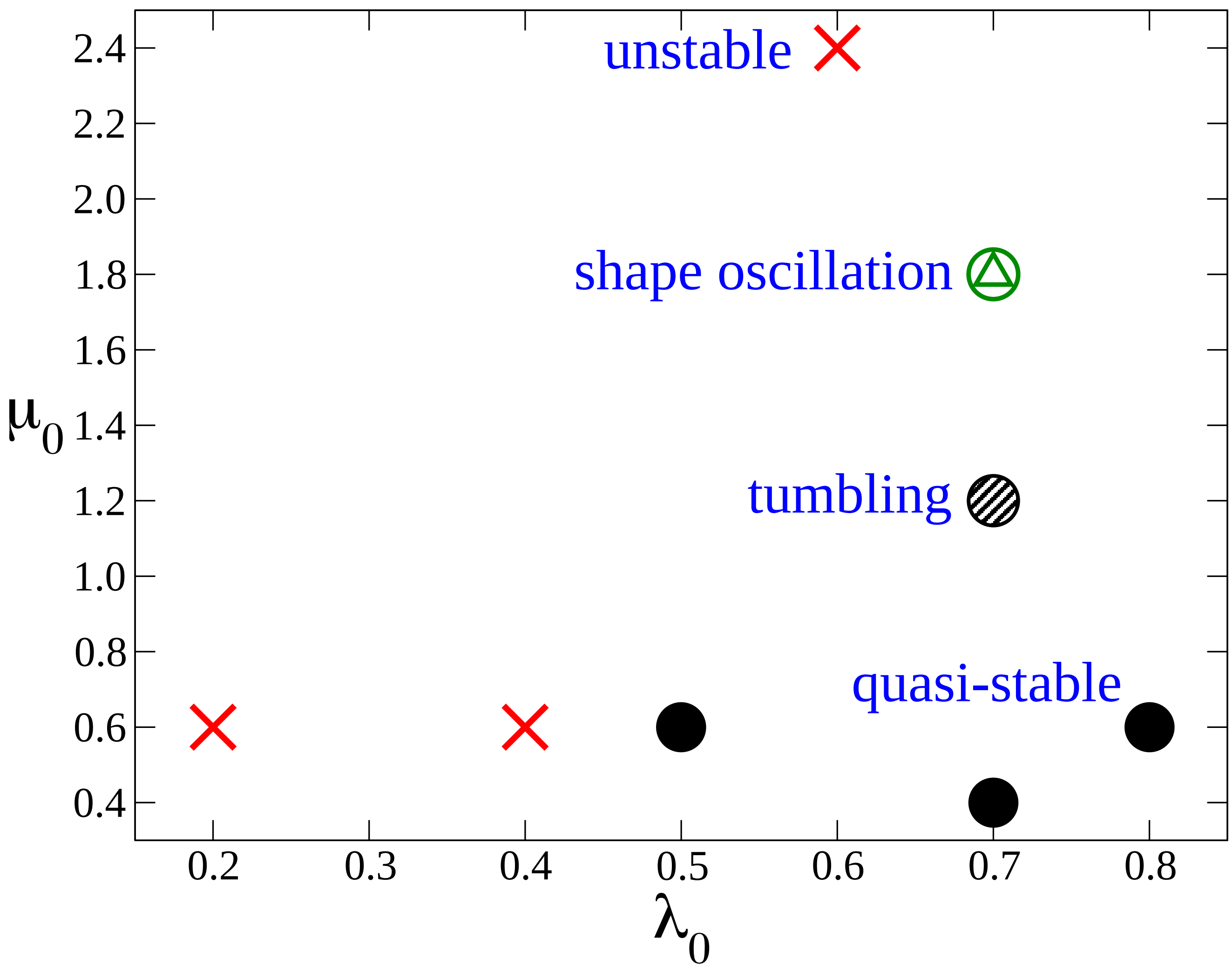
Effects of horizontal aspect ratio λ: q_0 = 0.5



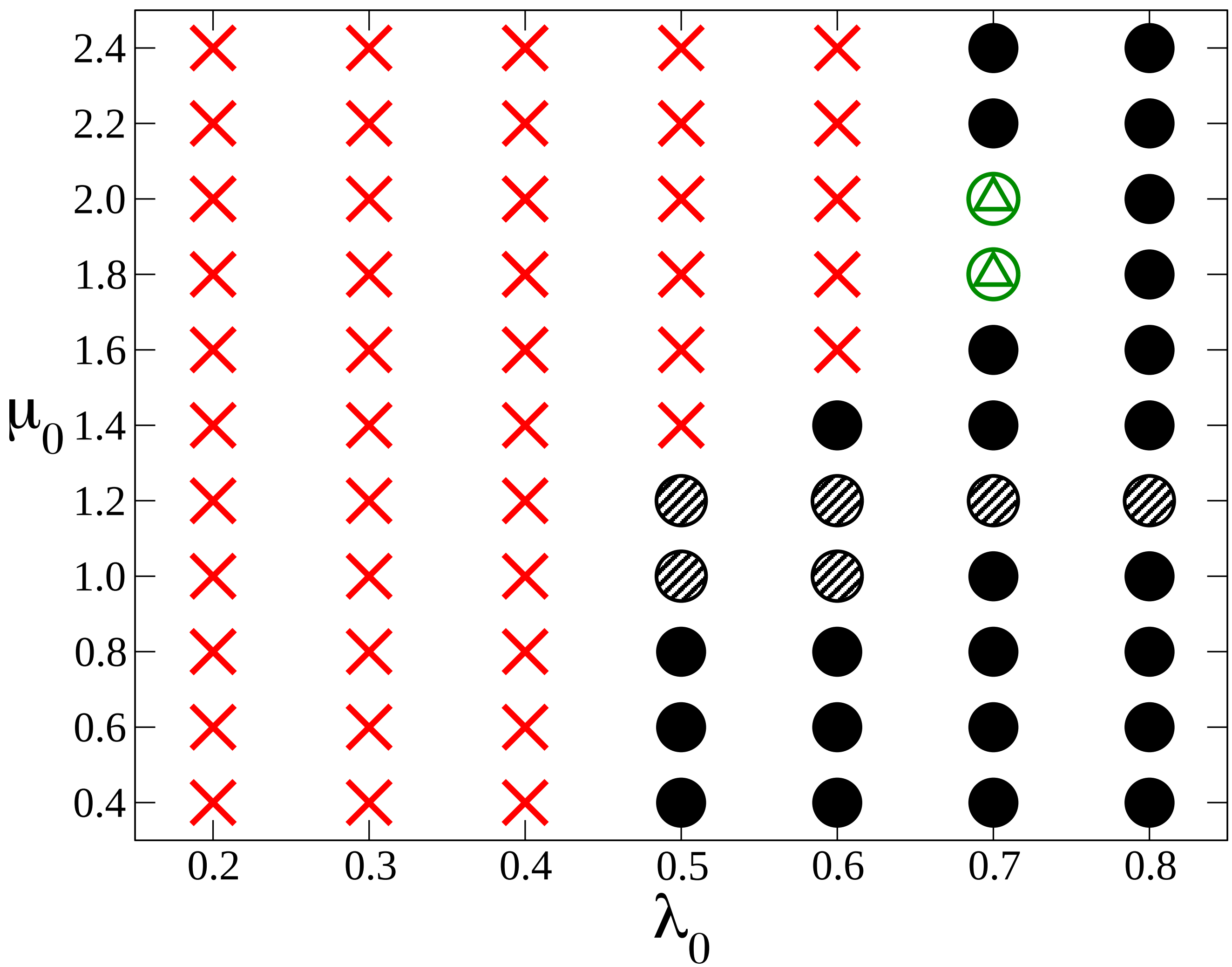
Effects of vertical aspect ratio μ: q_0 = 0.5



Various phases for $q_0 = 0.5$

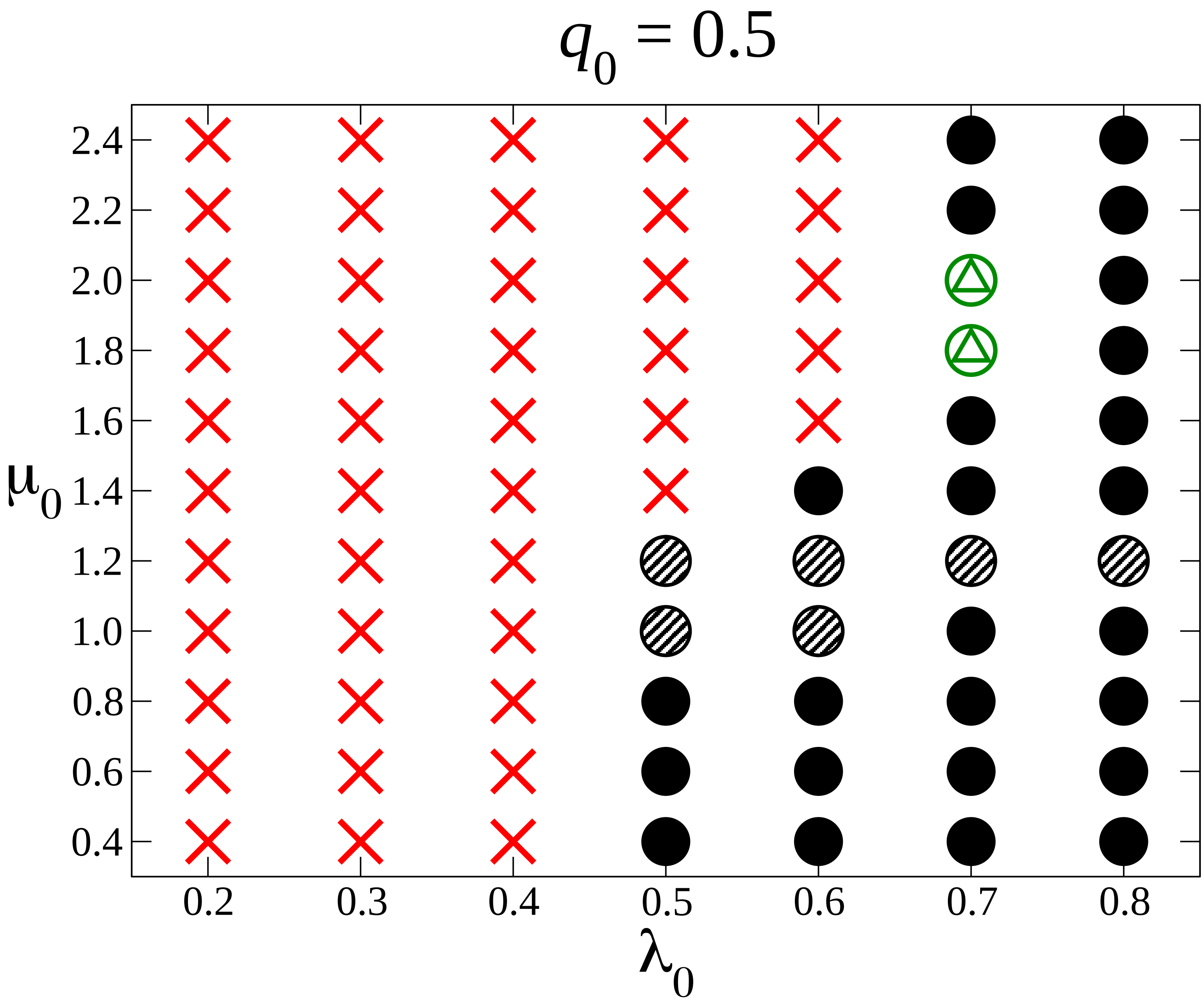


Stability and vortex geometry: $q_0 = 0.5$

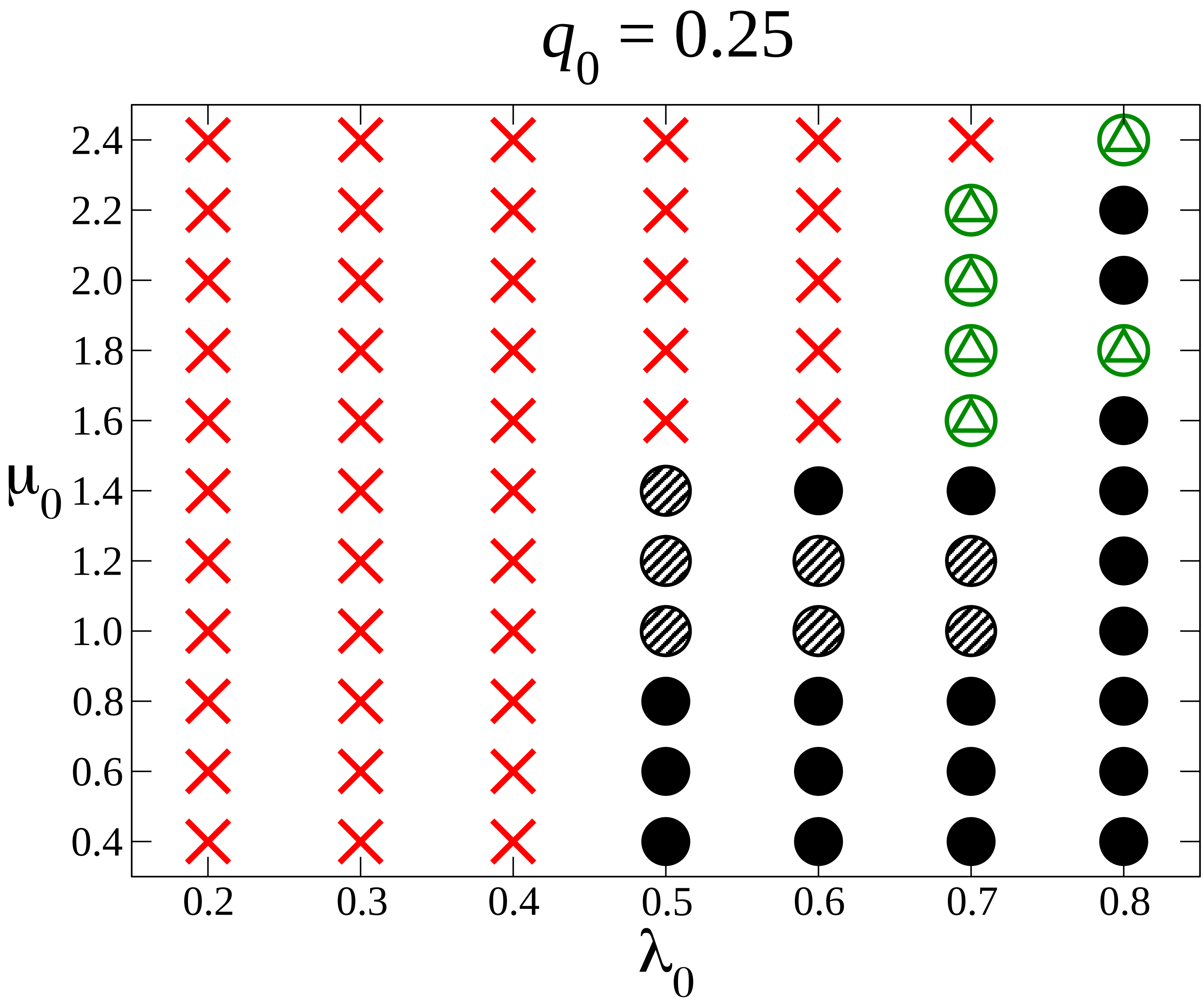


- oblate (small μ) vortices are more stable
- vortices with close to circular cross-section (large λ) are more stable

Stability and the Rossby number q_0



Stability and the Rossby number q_0



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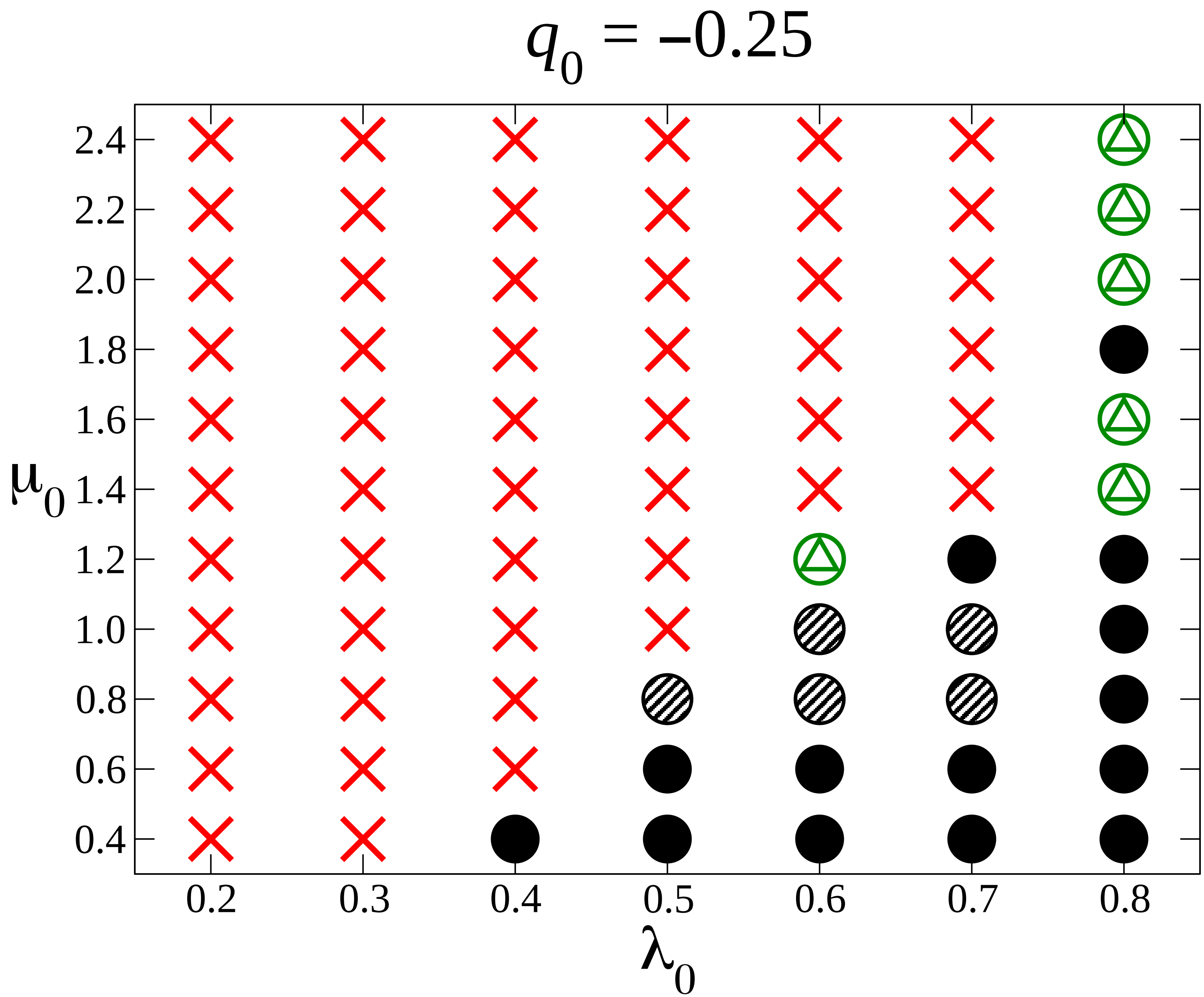

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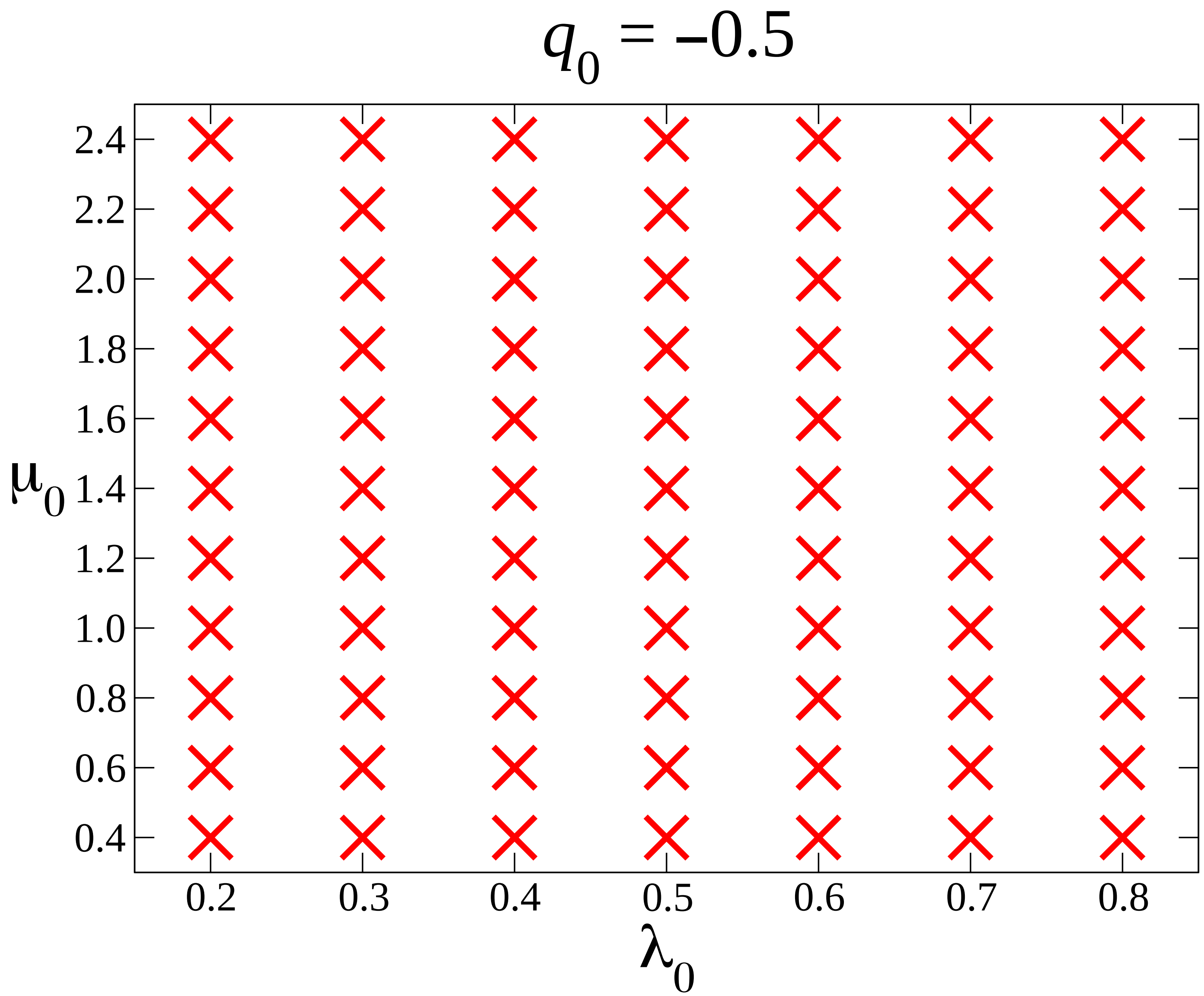

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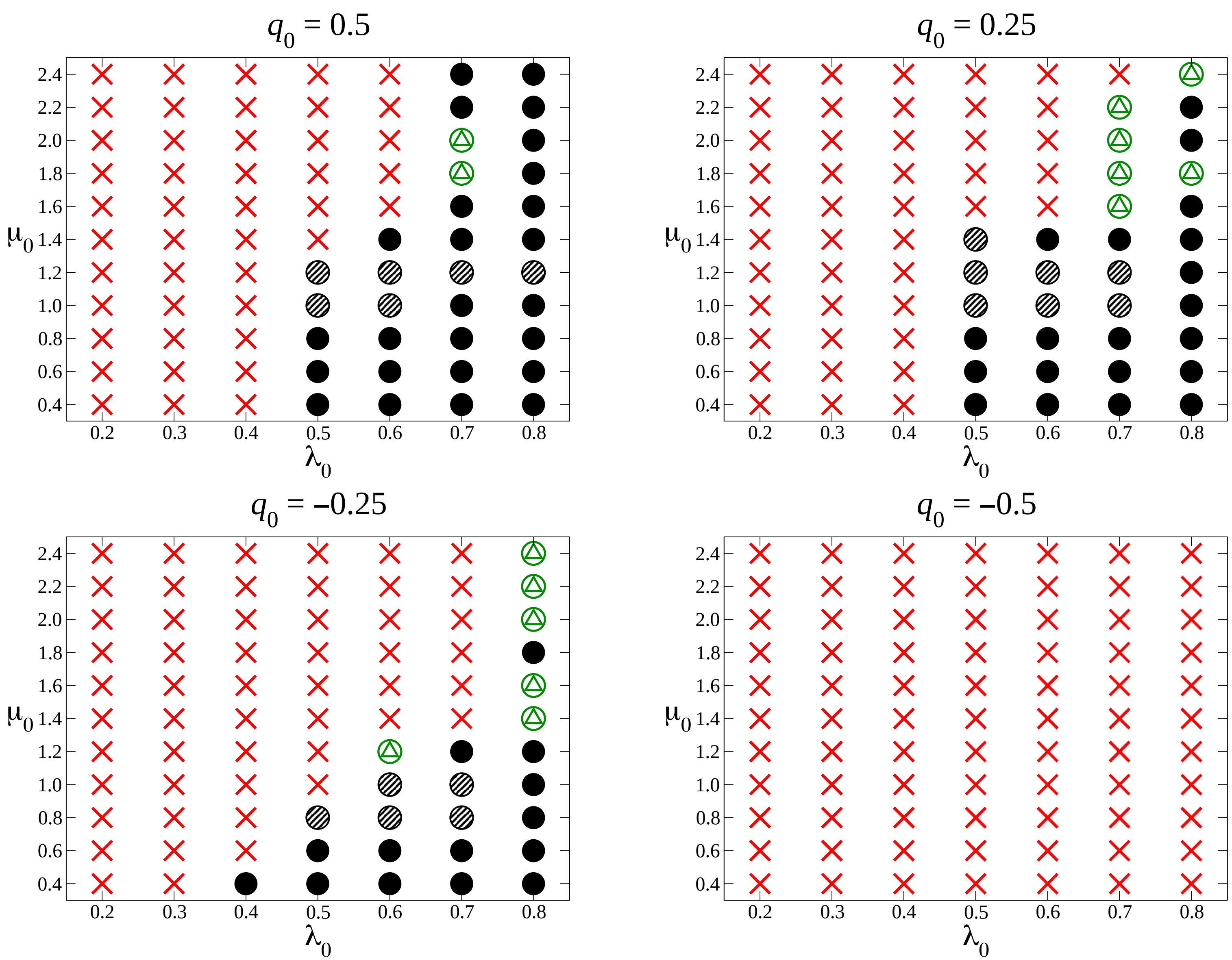
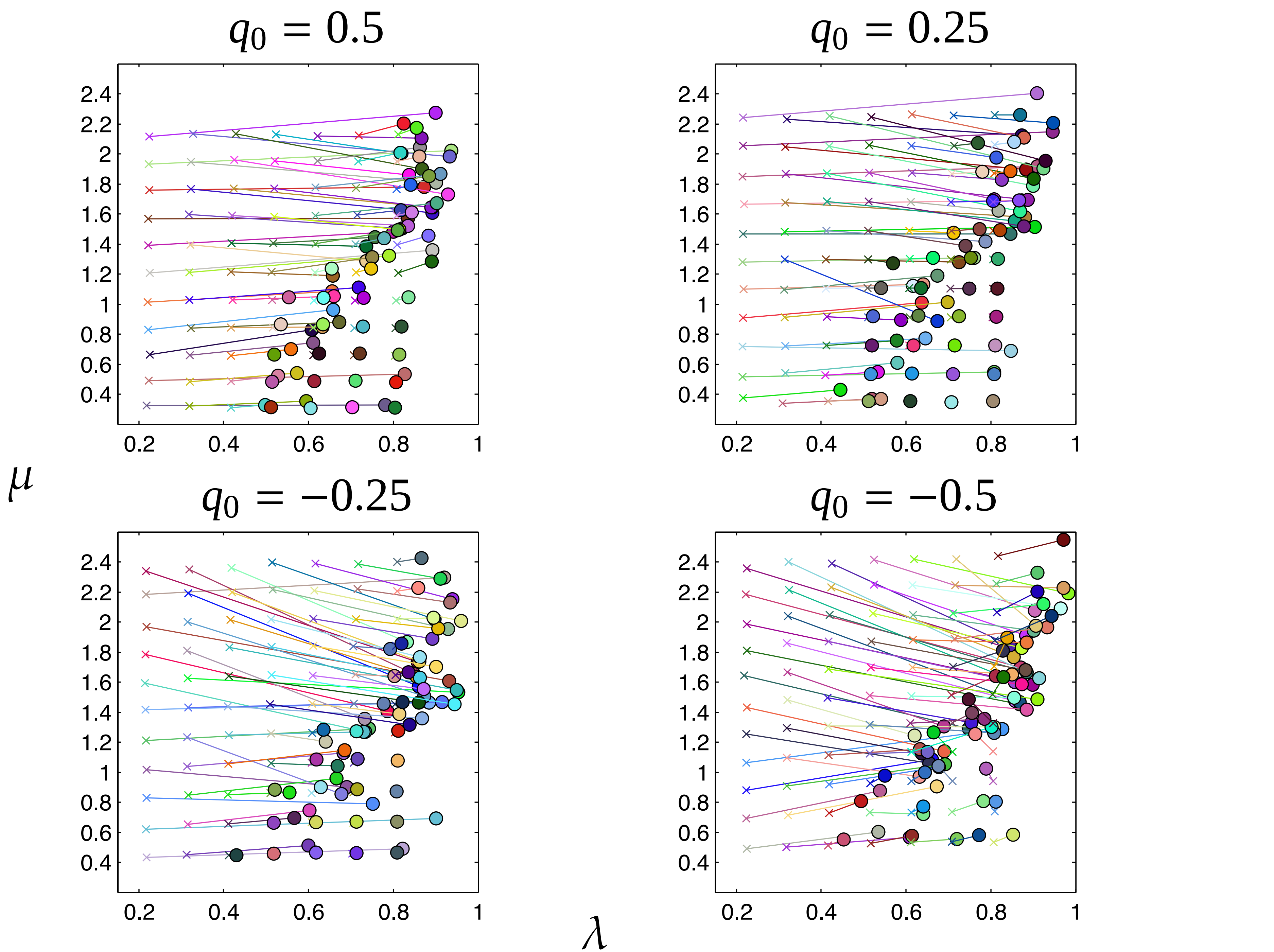
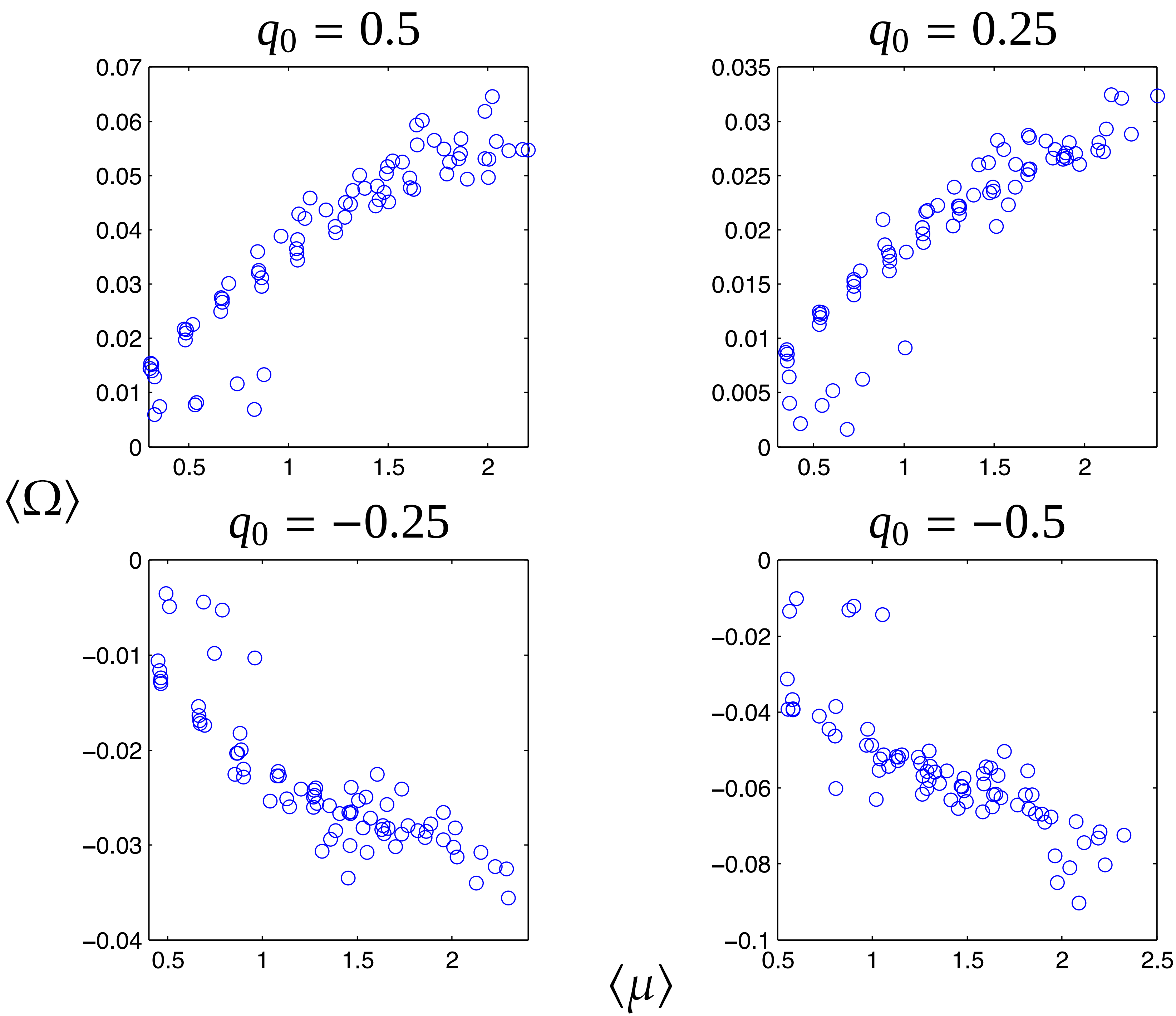


- For a given (λ_0, μ_0), stability generally decreases with q_0
- cyclones are more stable than anti-cyclones

Where do the unstable vortices go?



Vortex rotation rate Ω



Summary

- study the stability and evolution of an ellipsoidal vortex in a rotating stratified fluid using the full Boussinesq's equations
- oblate vortices with almost circular cross-section are the most stable
- increase in Rossby number q_0 enhances stability
- cyclones are more stable than anti-cyclones
- the vortex rotation rate scales with the vertical aspect ratio