Energy Injection into Two-dimensional Turbulence: a Scaling Regime Controlled by Drag

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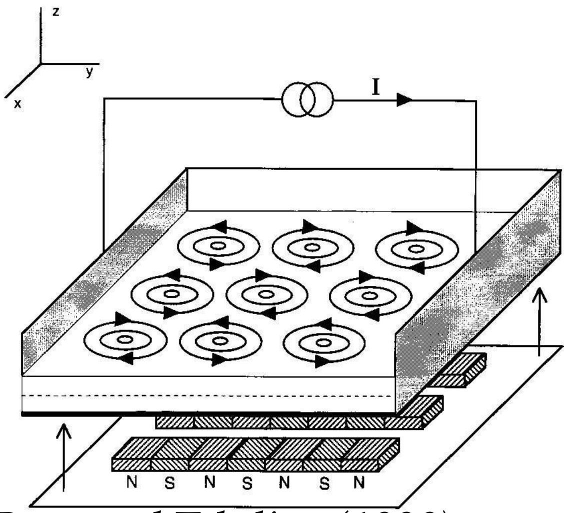
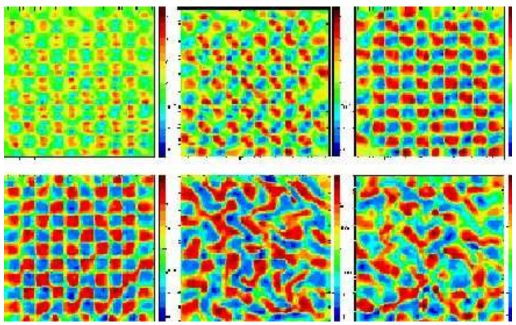
William R. Young

Forced-dissipative 2D Systems

Conducting fluid layer



Paret and Tabeling (1998)



http://www.fluid.tue.nl

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bottom wall friction

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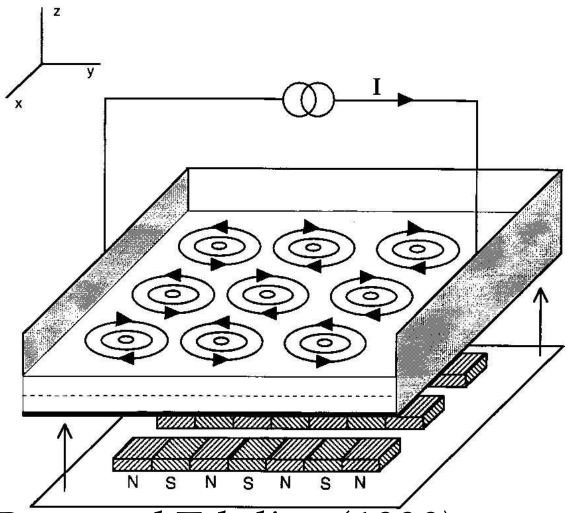

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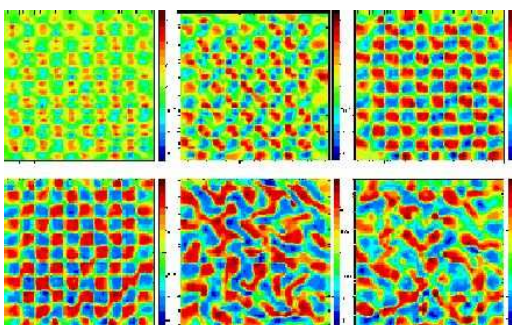

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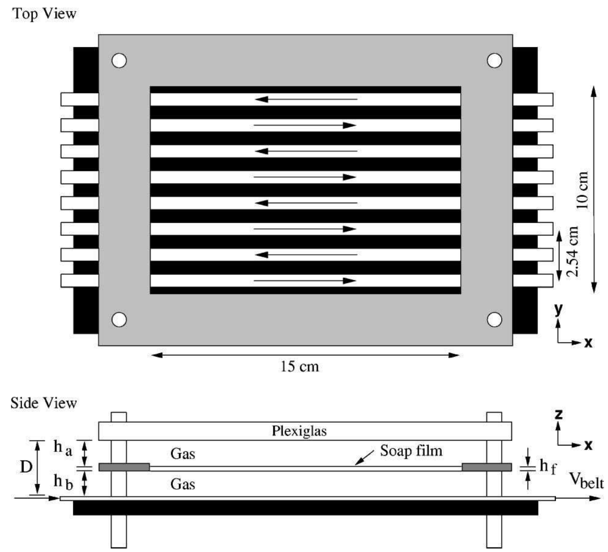
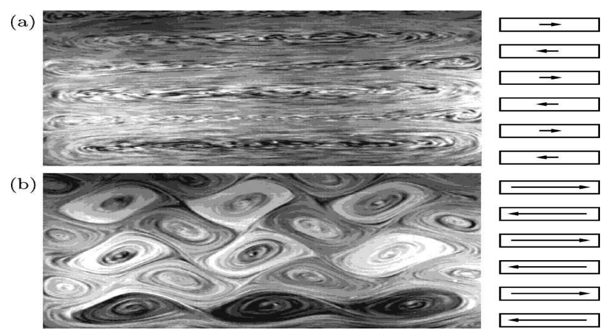
Soap film





Burgess et al. (1999)

driving belt
induced motion

drag from
surrounding gas

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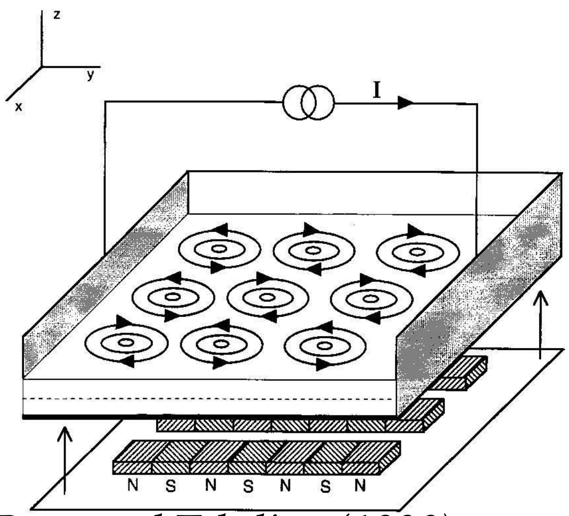

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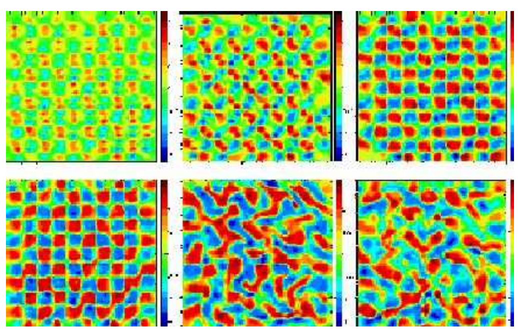

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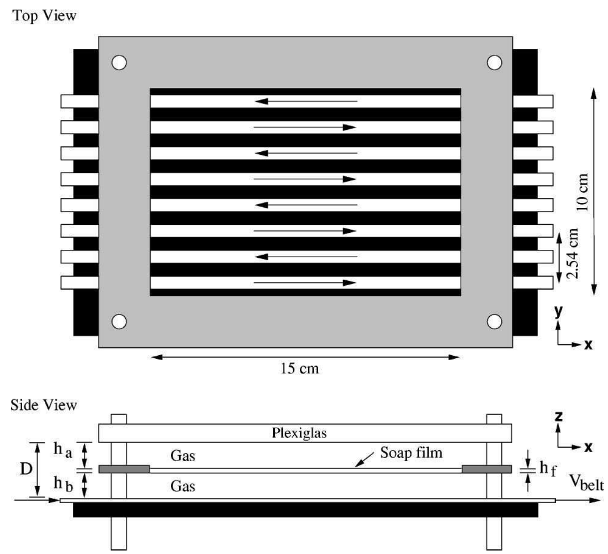

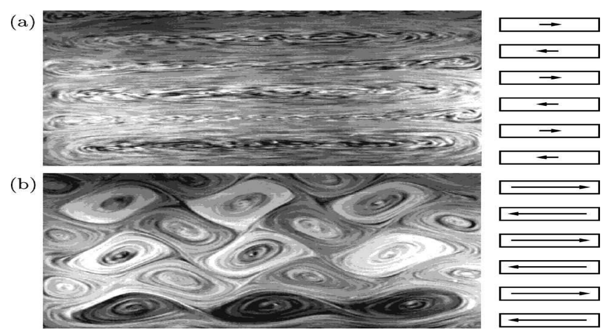

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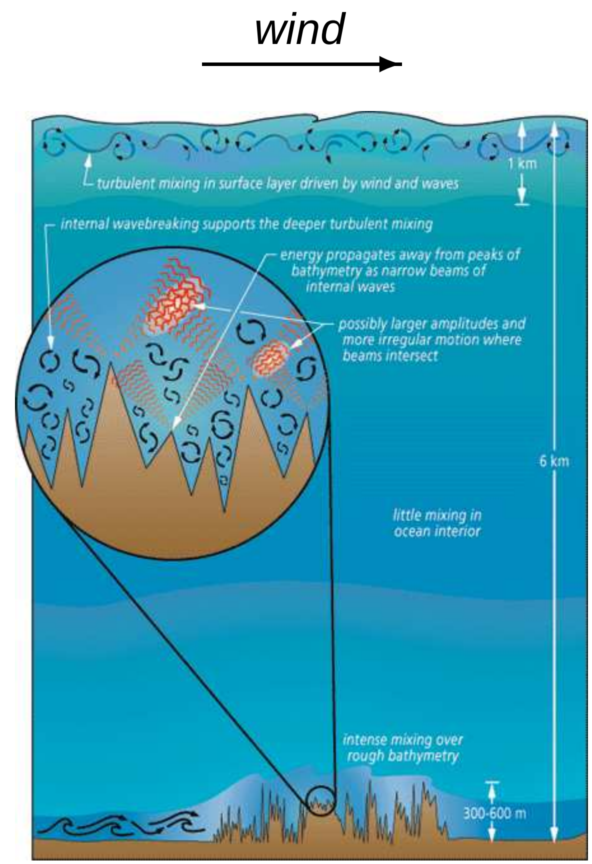
The Ocean



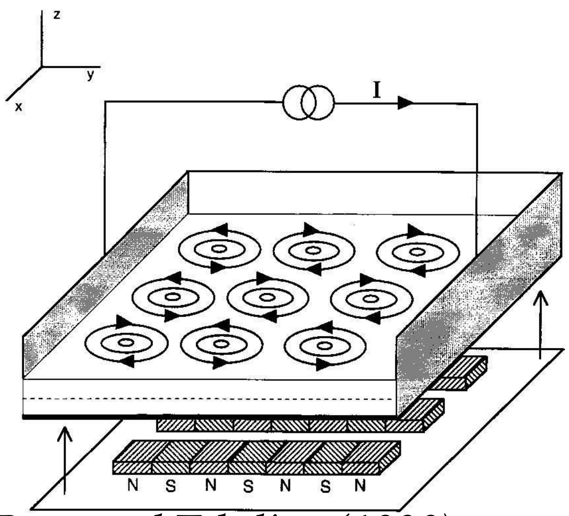
J. M. Toole (1996)

wind forcing,
lunisolar tide

sea floor drag,
instabilities

Forced-dissipative 2D Systems

Conducting fluid layer	Soap film	The Ocean



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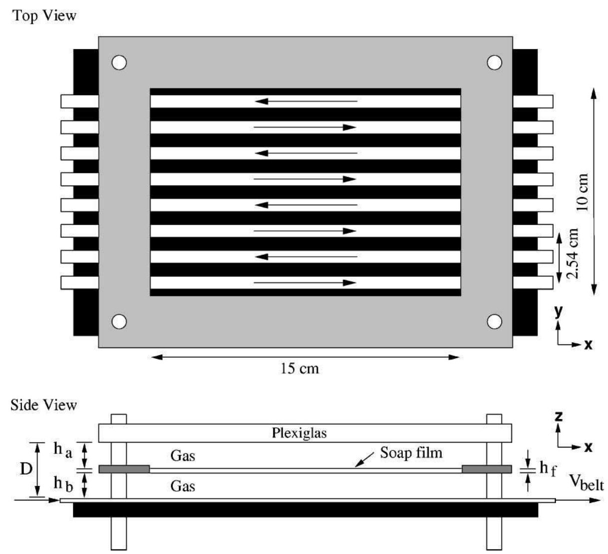
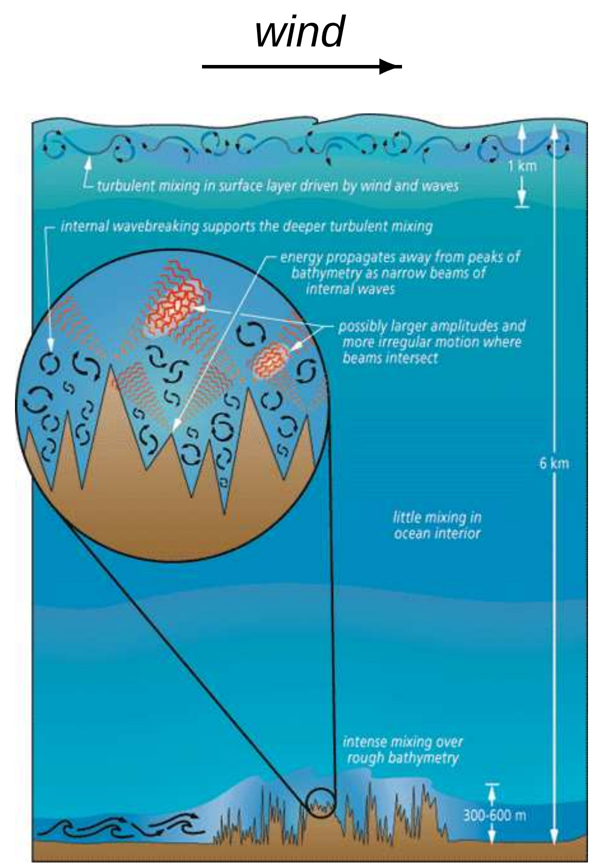




J. M. Toole (1996)

Forcing (energy input)

Nonlinear interaction (energy redistributed)

Dissipation (energy removed)

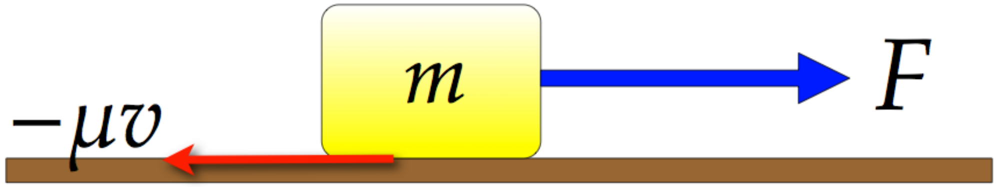
How much power is needed to drive these systems??

||

ε

Energy Injection Rate

Pulling a block on a rough surface by a constant force



Newton's second law,

$$F - \mu v = m\frac{dv}{dt}$$

Steady state velocity,

$$v = \frac{F}{\mu}$$

Energy injection rate (Power input),

$$\varepsilon = Fv = F\left(\frac{F}{\mu}\right) \sim \mu^{-1}$$

Dependence of ε on μ

Forced harmonic oscillator

$$\ddot{x} + \omega_0^2 x = -\mu \dot{x} + A \cos \omega t$$

instantaneous : $$\varepsilon_{int}(t) = \dot{x}(t)\, A \cos \omega t$$

averaged : $$\varepsilon = \frac{1}{T} \int_0^T \varepsilon_{int}(t')\, dt'$$

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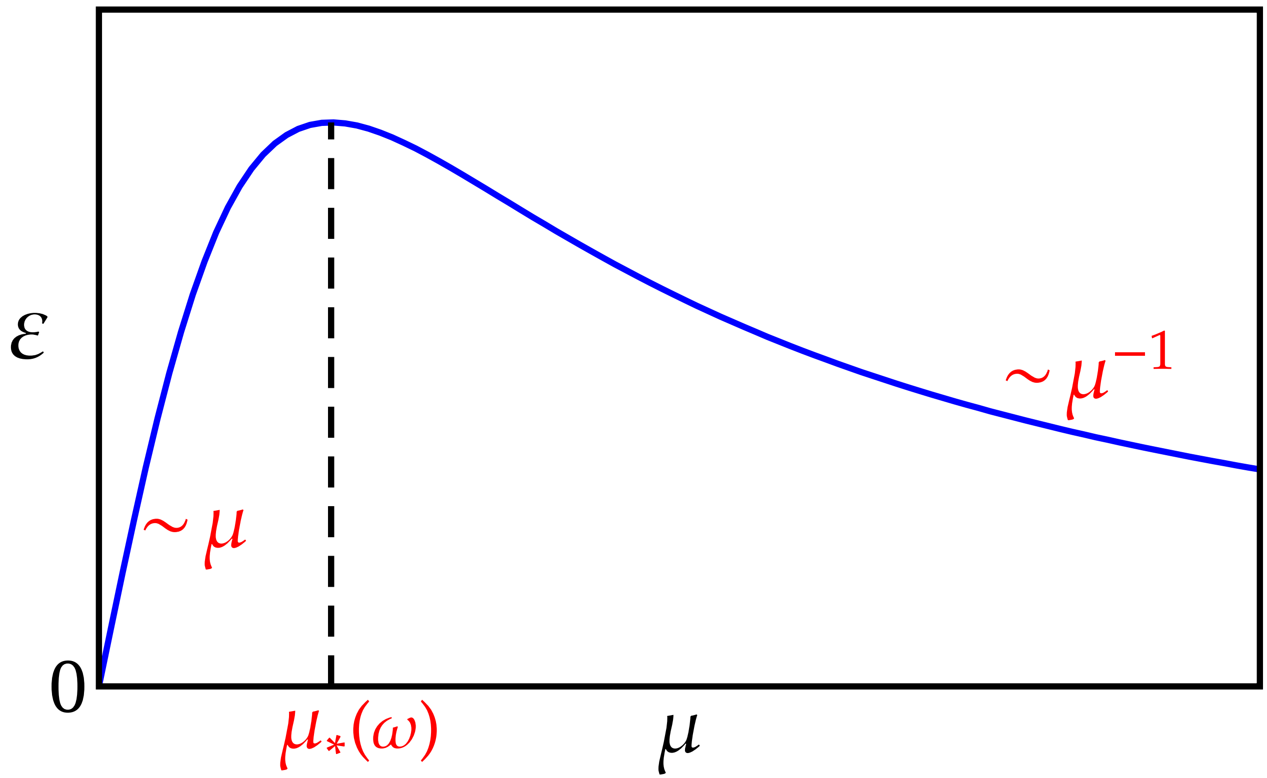


Two-dimensional turbulence

$$\zeta_t + \boldsymbol{u} \cdot \nabla\zeta = f(\boldsymbol{x}, t) - \mu\zeta + \nu\nabla^2\zeta \qquad \boldsymbol{u} = (u, v)$$

$$\zeta \equiv v_x - u_y = \nabla^2\psi$$

Energy injection rate

$$\varepsilon = -\langle \psi f \rangle$$

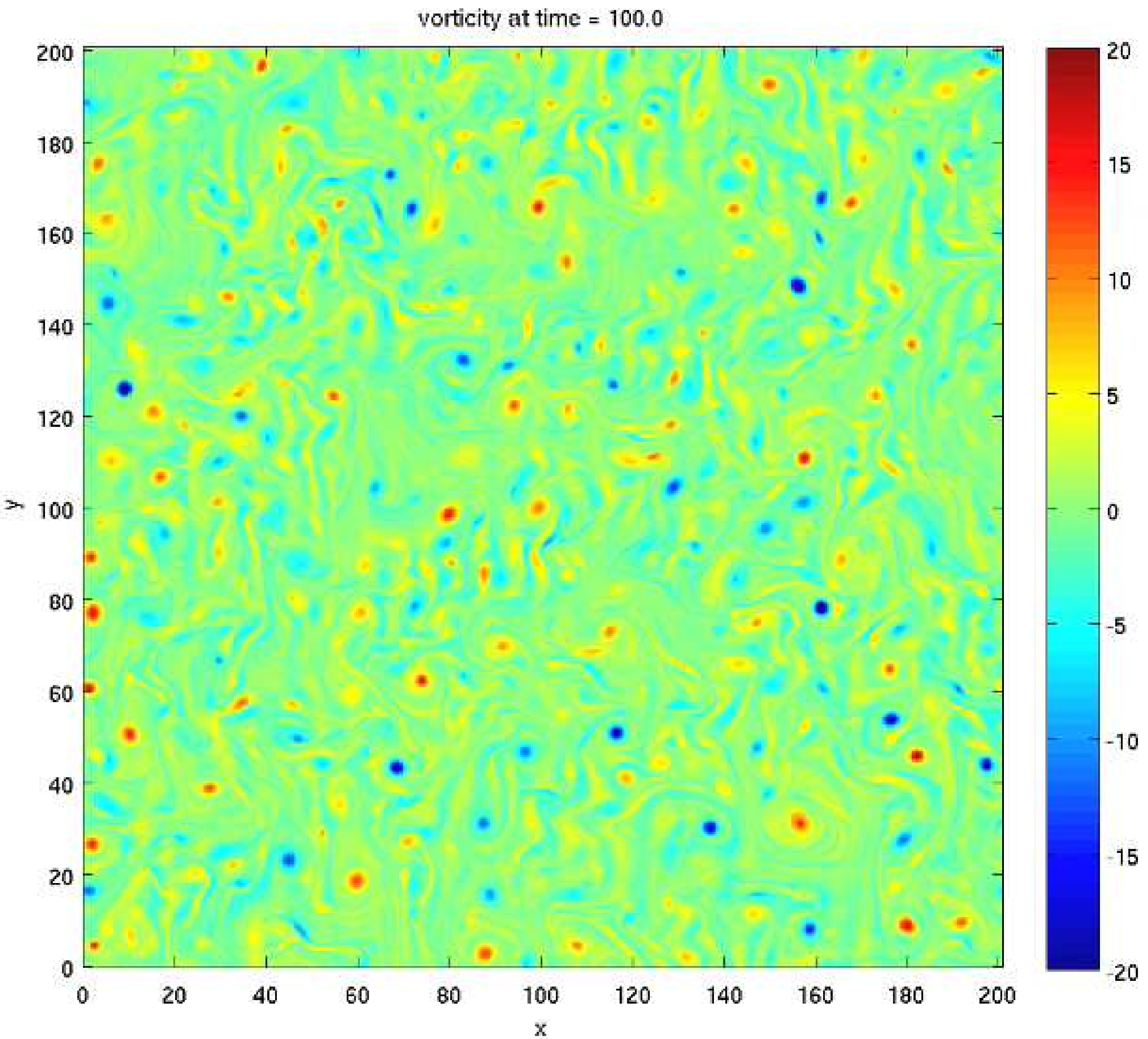
Power Integral (conservation of energy)

$$\varepsilon = \underbrace{\mu \langle u^2 + v^2 \rangle}_{\varepsilon_\mu} + \underbrace{\nu \langle \zeta^2 \rangle}_{\varepsilon_\nu}$$

- small-scale forcing: $f = \tau_f^{-2}\cos(k_f x)$, $k_f^{-1} \ll$ box size
- drag is the main dissipative mechanism: $\varepsilon_\mu \gg \varepsilon_\nu$

Numerical Model

$$\zeta_t + \boldsymbol{u} \cdot \nabla\zeta = \cos x - \mu\zeta - \nu\nabla^{8}\zeta$$



$$L = 32(2\pi)$$

$$N = 1024^2$$

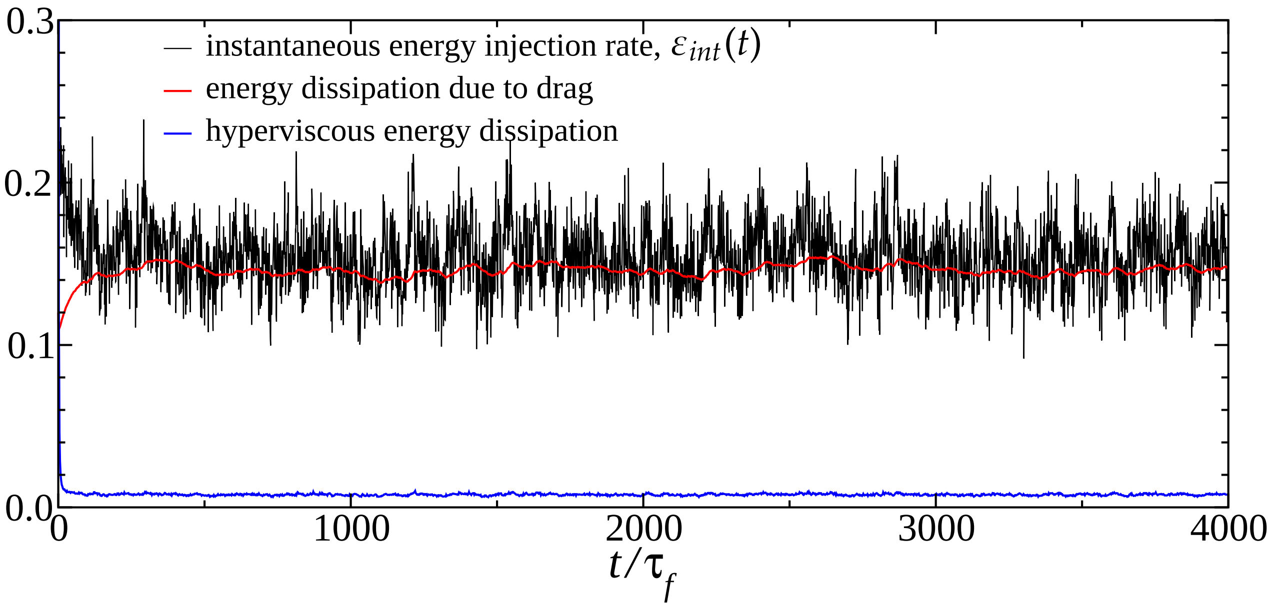
$$\mu = 0.007$$

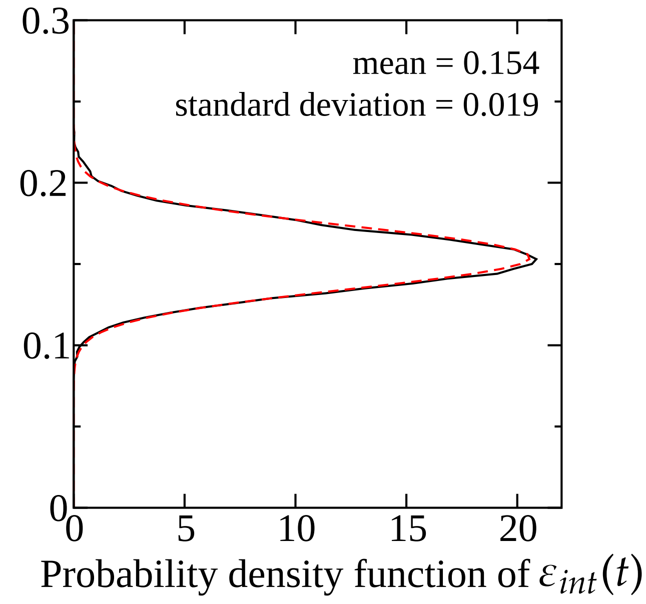
$$\nu = 10^{-5}$$

$$\varepsilon = \langle \zeta \cos x \rangle$$

$$\varepsilon_{int}(t) = \overline{\zeta \cos x}^{x,y}$$

Instantaneous energy injection rate

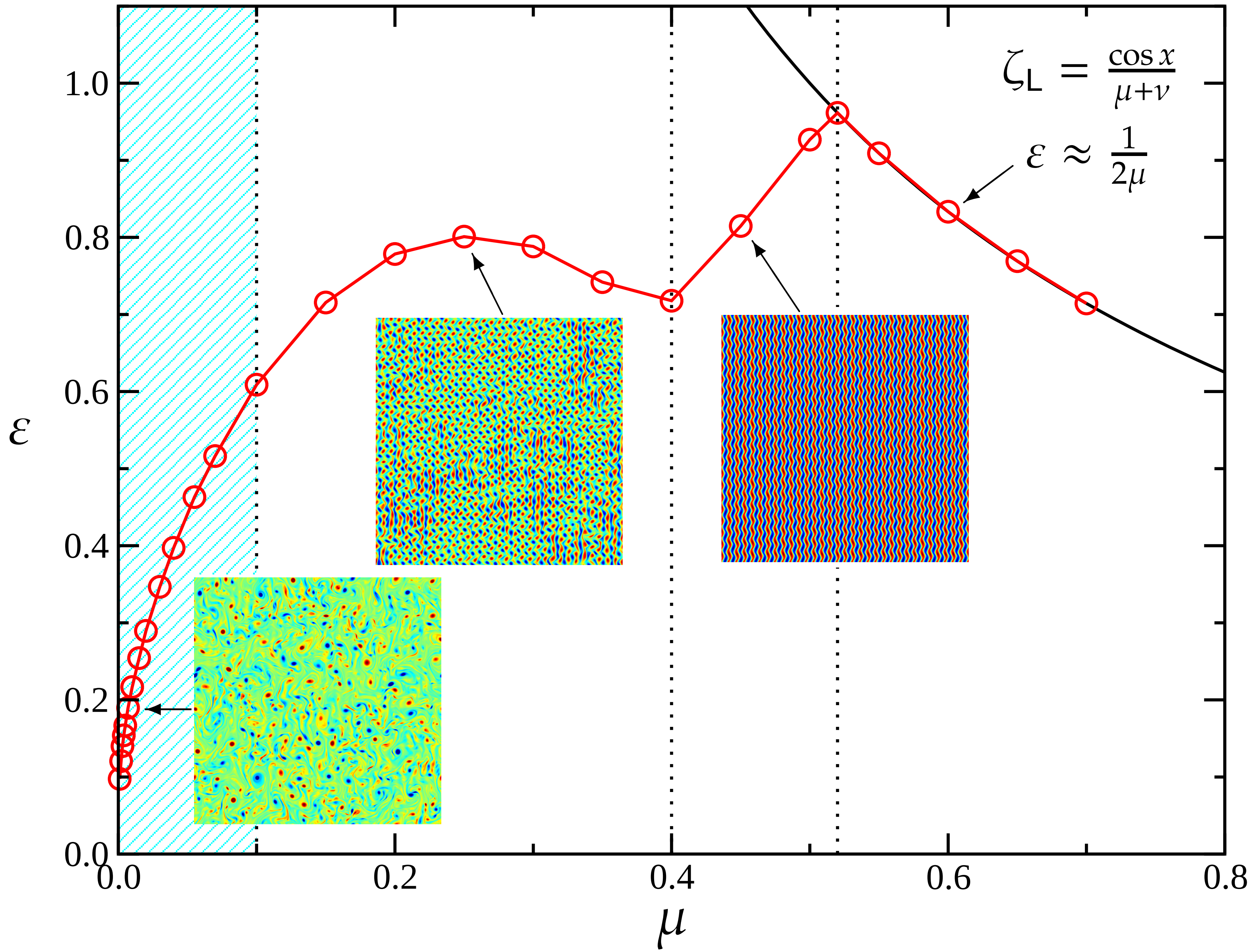




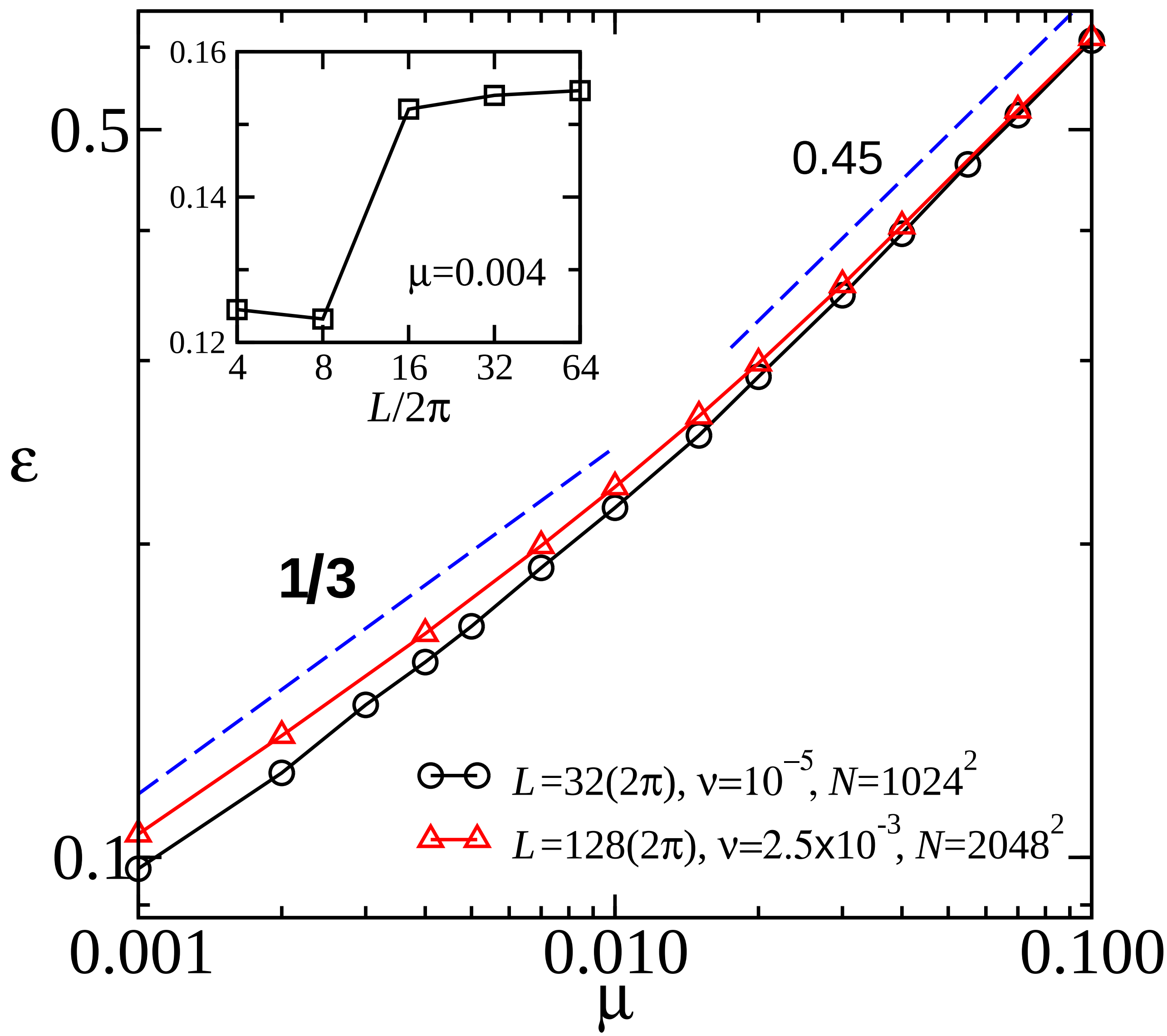
$$\varepsilon_{int}(t) = \frac{1}{L^2} \iint \zeta(x,y,t) \cos x \ dxdy$$

$$\varepsilon = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \varepsilon_{int}(t')\, dt'$$

Energy Injection Rate vs. Drag



Scaling Law for $\varepsilon(\mu)$



Results are insensitive to ν and (large enough) L

Theory: The Model

$$\zeta(x,y,t) = \underbrace{A(t)\cos(k_f x) + B(t)\sin(k_f x)}_{\text{forced mode, } \hat{\zeta}(x,t)} + \tilde{\zeta}(x,y,t)$$

$$\varepsilon = (k_f/\tau_f)^2 \left\langle \hat{\zeta}\cos(k_f x) \right\rangle$$

Random Sweeping Model

$$\hat{\zeta}_t + U\hat{\zeta}_x + V\hat{\zeta}_y = \tau_f^{-2}\cos(k_f x) - \mu\hat{\zeta} - \eta\hat{\zeta} \tag{1}$$

$$\varepsilon \approx \mu\langle U^2 + V^2\rangle \approx 2\mu\langle U^2\rangle \tag{2}$$

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- advection by large-scale eddies (U, V)
  - isotropic: $\langle U^2\rangle = \langle V^2\rangle = U_{rms}^2$
  - vary on scales $\gg k_f^{-1}$
  - large-eddy turnover time $\sim \mu^{-1} \gg U_{rms}k_f$

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  - vary on scales $\gg k_f^{-1}$
  - large-eddy turnover time $\sim \mu^{-1} \gg U_{rms} k_f$
- nonlinear energy transfer out of the forced mode
  - $\eta \gg \mu \gg \nu$

Theory: The Solution

$$\hat{\zeta}_t + U\hat{\zeta}_x + V\hat{\zeta}_y = \tau_f^{-2}\cos(k_f x) - \mu\hat{\zeta} - \eta\hat{\zeta}$$

Neglect μ ($\eta \gg \mu$) and seek steady-state solution,

$$U\hat{\zeta}_x = \tau_f^{-2}\cos(k_f x) - \eta\hat{\zeta}$$

Since $U(x, y)$ varies on the large scales,

$$\hat{\zeta} \approx \frac{\cos(k_f x - \phi)}{\tau_f^2\sqrt{\eta^2 + (Uk_f)^2}} \quad , \qquad \tan\phi = \frac{Uk_f}{\eta}$$

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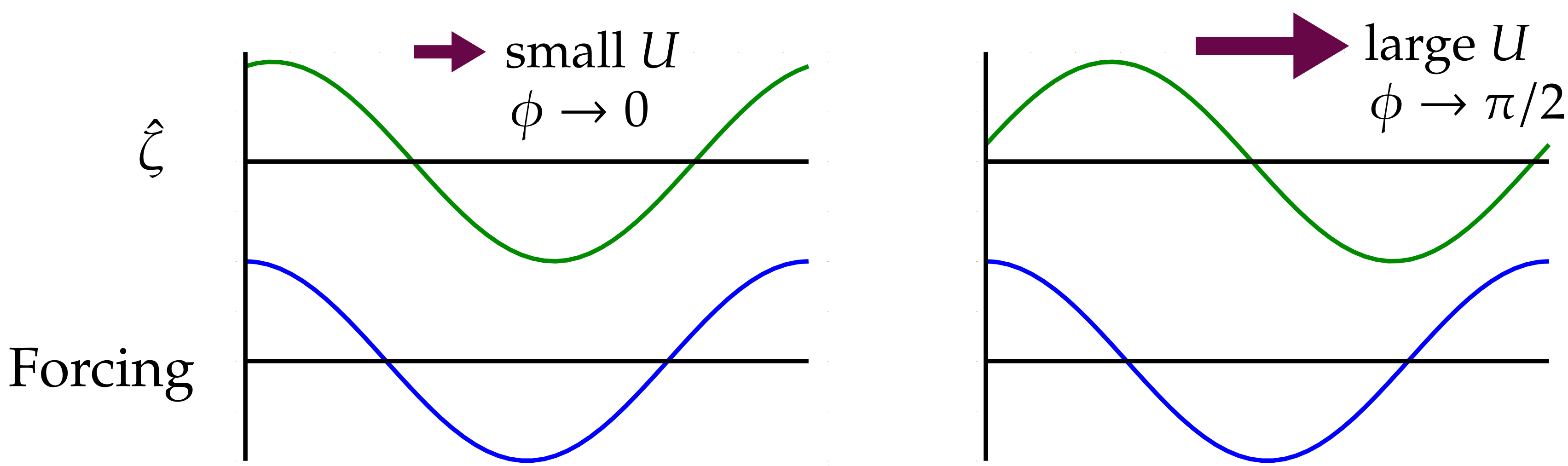

Theory: The Scaling Law

$$\text{typical } \frac{\eta}{Uk_f} \sim \frac{\langle\eta\rangle}{U_{rms}k_f} = \frac{\text{transfer rate}}{\text{sweeping rate}} \equiv \alpha$$

$$\left.\begin{array}{ll} \langle\eta\rangle \sim \text{shear at } k_f \sim \varepsilon^{\frac{1}{3}} & (\text{shear} \sim k\,[kE(k)]^{\frac{1}{2}}) \\ U_{rms}k_f \sim \varepsilon^{\frac{1}{2}}\mu^{-\frac{1}{2}} & (\varepsilon \approx 2\mu U_{rms}^2) \end{array}\right\} \quad \alpha \sim \mu^{\frac{4}{9}} \quad (\text{anticipate } \varepsilon \sim \mu^{\frac{1}{3}})$$

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From definition of ε,

$$\varepsilon \sim \left\langle \hat{\zeta}\cos(k_f x) \right\rangle \sim \left\langle \frac{\eta}{\eta^2 + (Uk_f)^2} \right\rangle$$

Let $U' = U/U_{rms}$ and $\eta' = \eta/\langle\eta\rangle$,

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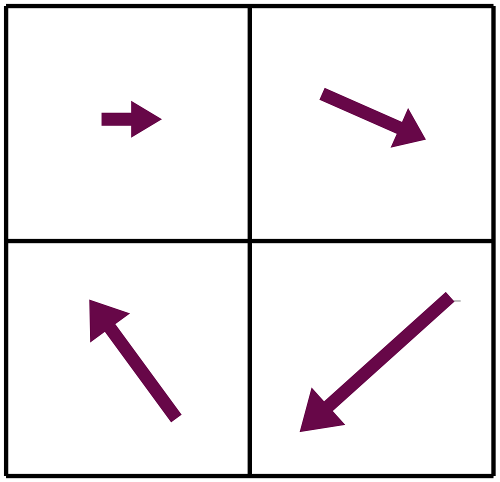

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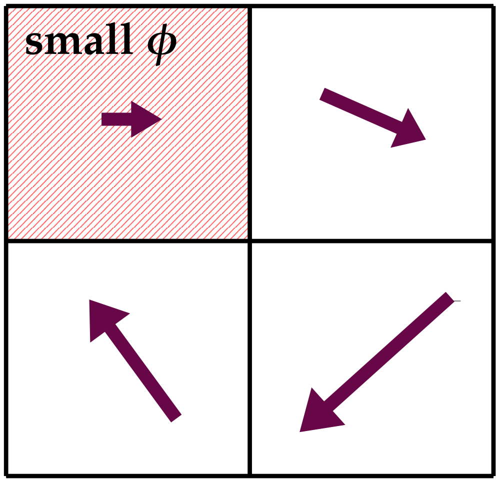

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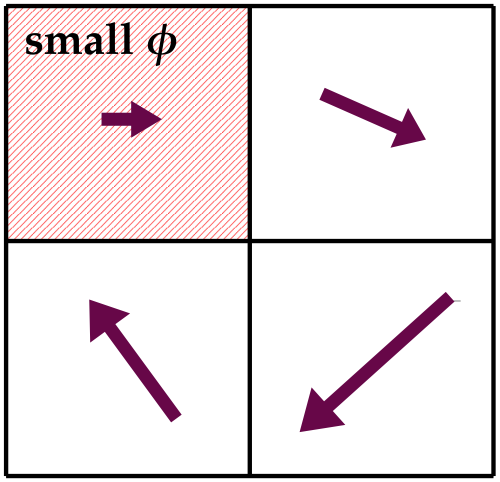

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So, $\varepsilon \sim U_{rms}^{-1}$ and $\varepsilon \approx 2\mu U_{rms}^2$ imply:

$$\boxed{\varepsilon \sim \mu^{\frac{1}{3}}}$$

Summary

- study energy injection rate ε in two-dimensional turbulence with drag μ and a prescribed small-scale body force
- discover a new scaling regime:

$$\varepsilon \sim \mu^{\frac{1}{3}} \quad \text{as} \quad \mu \to 0$$

- random sweeping model suggests energy input is mainly due to regions with small velocity