How Fast Does a Passive Scalar Decay?

(Decay of Chaotically Advected Passive Scalars in the Zero Diffusivity Limit)

Yue-Kin Tsang

Courant Institute of Mathematical Sciences

New York University

Thomas M. Antonsen, Jr. and Edward Ott

University of Maryland, College Park

Decay of Variance

$$\frac{\partial \phi}{\partial t} + \vec{u} \cdot \nabla \phi = \kappa \nabla^2 \phi$$

$$\nabla \cdot \vec{u} = 0$$ *(incompressible)*

- $\phi(\vec{x}, 0) \sim \sin\left[\frac{2\pi}{L_D}(x+y)\right]$
- $\vec{u}(\vec{x}, t)$: doubly periodic with period L_f
- Mean is conserved: $\frac{d\langle\phi\rangle}{dt} = 0$
- Vairance = $\left\langle \phi^2 \right\rangle$ (*take* $\langle\phi\rangle = 0$)

Decay of Variance

$$\frac{d\left\langle \phi^2 \right\rangle}{dt} = -2\kappa \left\langle |\nabla \phi|^2 \right\rangle$$

- variance decay due to diffusion ($\kappa \neq 0$)
- decay rate *increases* with $|\nabla \phi|$

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stirring/stretching of fluid $\Rightarrow$ filaments $\Rightarrow$ large $|\nabla \phi|$

$\Rightarrow$ enhanced diffusion $\Rightarrow$ faster mixing/variance decay

Exponential Decay Rate γ_0

We are interested in long time behavior of ϕ as $\kappa \to 0$.

- numerical simulations and experiments show:

$$\langle \phi^2 \rangle \sim e^{-\gamma(\kappa)t}$$

- some numerical evidence support the prediction:

$$\lim_{\kappa \to 0^+} \gamma(\kappa) \equiv \gamma_0$$

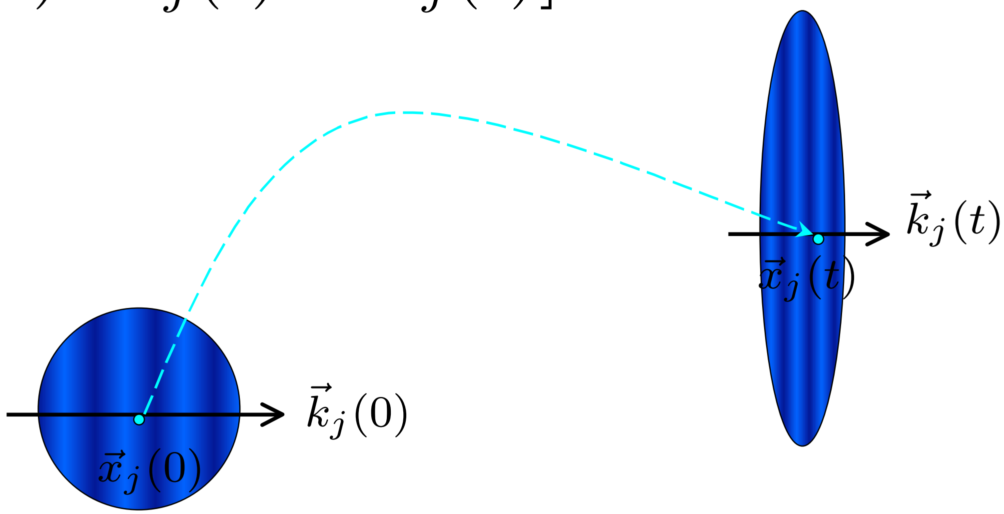
- Question: Given a certain flow $\vec{u}(\vec{x}, t)$, can we predict the decay rate γ_0?

1. R.T. Pierrehumbert, Chaos, Solitons and Fractals **4**, 1091 (1994)
2. Voth *el at.*, Phys. Fluids **15**, 2560 (2003)

Wave Packet Model

$$\varphi_j(\vec{x}_j(t)) = A_j(t)\sin[\vec{k}_j(t)\cdot\vec{x}_j(t) + \vartheta_j(t)]$$

$$\omega_j(t) = \langle\varphi_j^2\rangle$$



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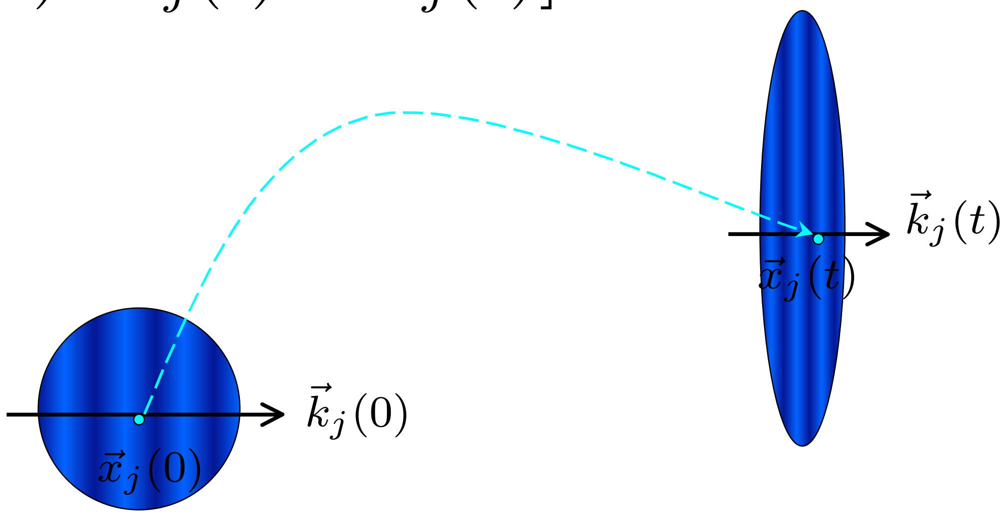

$$\phi(\vec{x},t) = \sum \varphi_j$$

$$C(t) \equiv \langle \phi^2 \rangle = \sum \omega_j(t)$$

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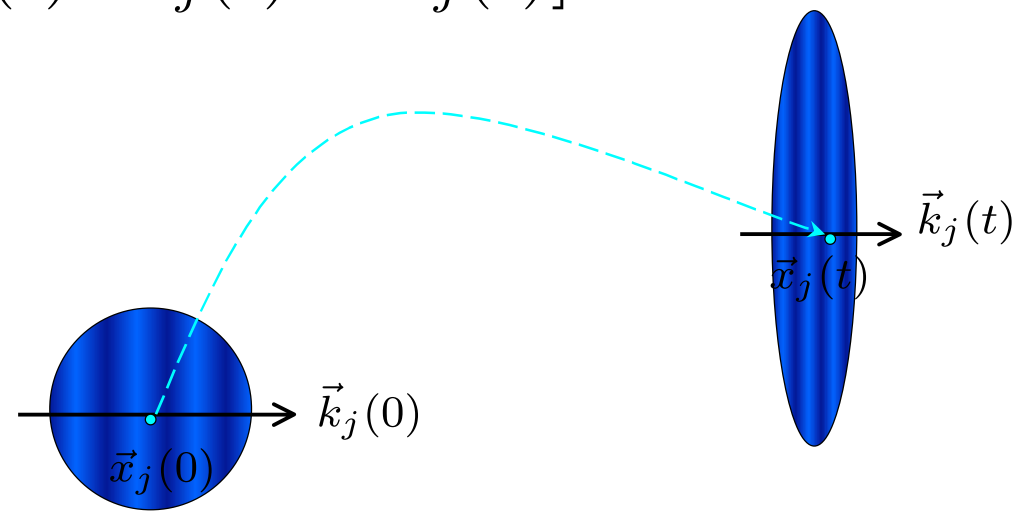


$$\phi(\vec{x},t) = \sum \varphi_j$$

$$C(t) \equiv \langle \phi^2 \rangle = \sum \omega_j(t)$$

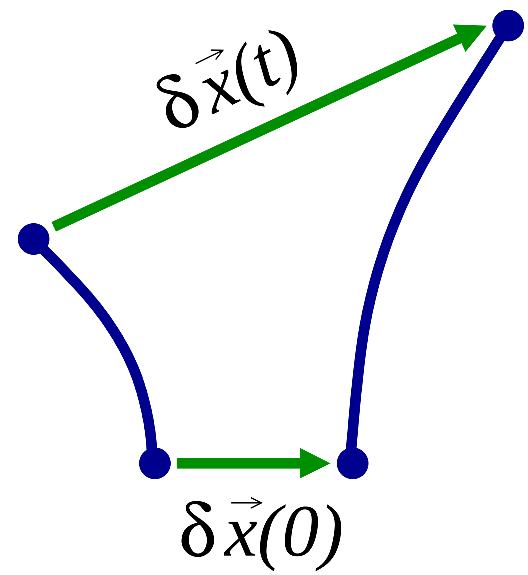
$$\frac{d\omega_j}{dt} = -2\kappa {k_j}^2 \omega_j$$

- $\vec{k}_j(t)$ is determined by the stretching of fluid elements induced by the smooth velocity field $\vec{u}$

Characterizing Stretching

- Along a fluid trajectory,

$$\frac{d\vec{x}(t)}{dt} = \vec{u}(\vec{x}(t), t)$$



- Finite-time Lyapunov Exponent, h

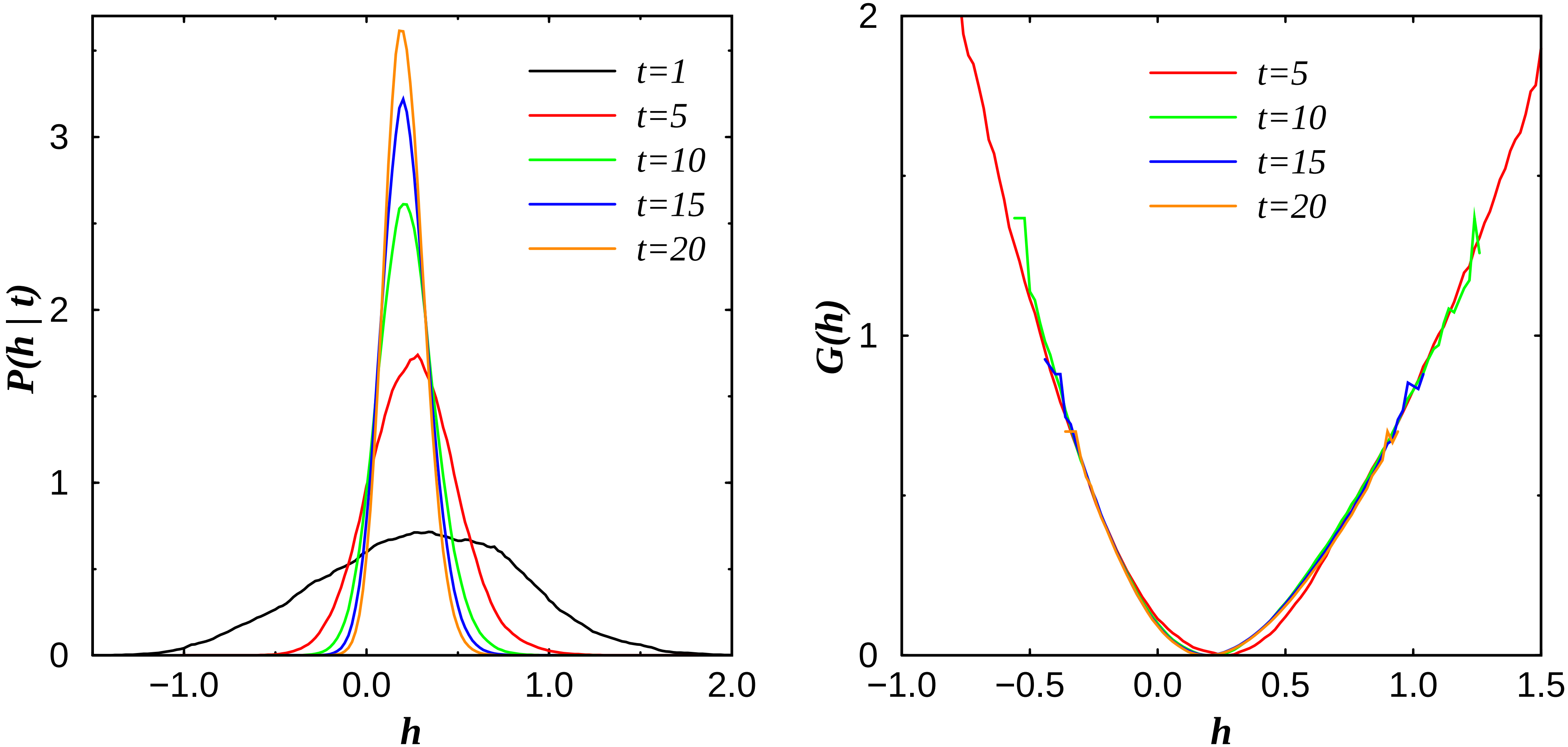
$$|\delta\vec{x}(t)| = |\delta\vec{x}(0)|e^{ht}$$

- Probability Distribution Function for h, $P(h\,|\,t)$

$$P(h\,|\,t) \sim \exp[-tG(h)]$$

(Reference: R.S. Ellis, “Entropy, Large Deviations and Statistical Mechanics”, 1985)

$P(h\,|\,t)$ and $G(h)$



t=1
t=5
t=10
t=15
t=20
P(h | t)
h
0
1
2
3
−1.0
0.0
1.0
2.0
t=5
t=10
t=15
t=20
G(h)
h
0
1
2
−1.0
−0.5
0.0
0.5
1.0
1.5

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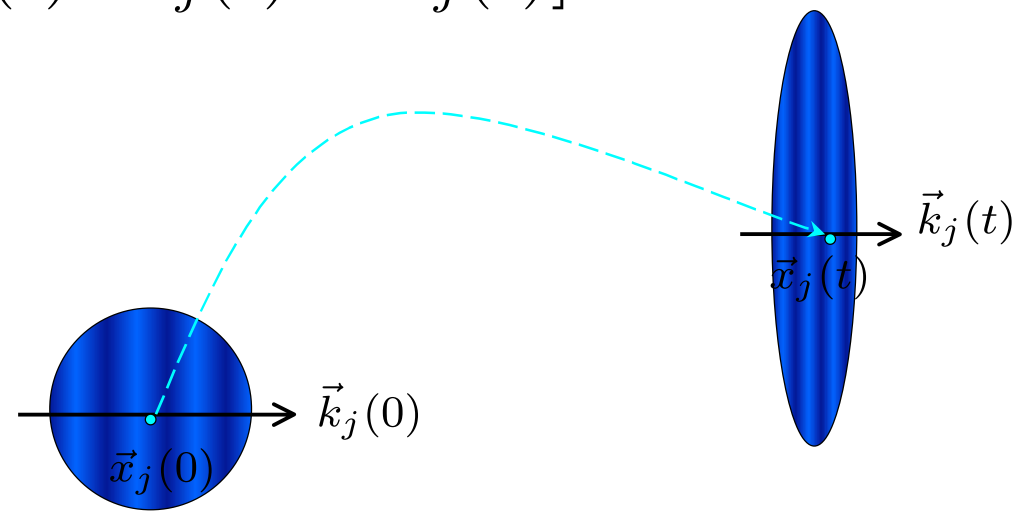
$$\omega_j(t) = \langle \varphi_j^2 \rangle$$

$$\phi(\vec{x},t) = \sum \varphi_j$$

$$C(t) \equiv \langle \phi^2 \rangle = \sum \omega_j(t)$$



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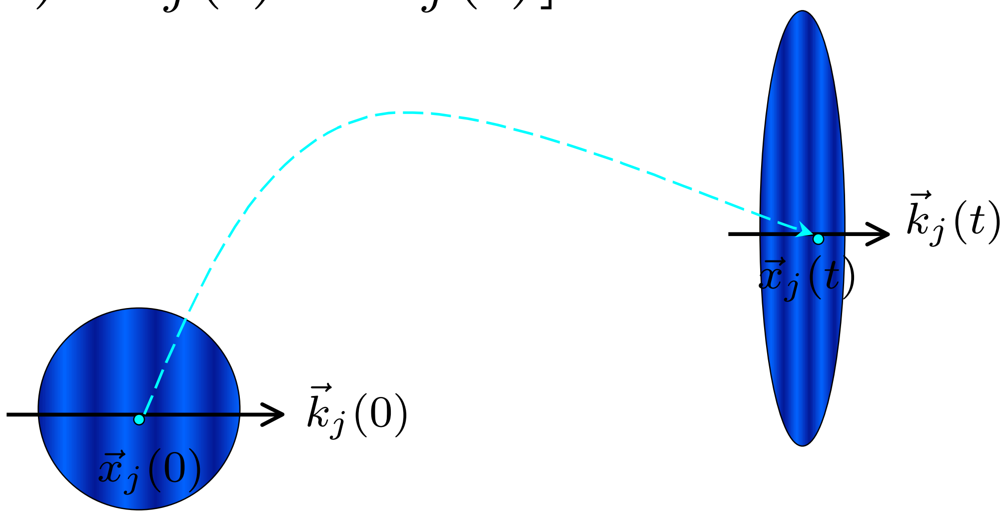


$$\frac{d\omega_j}{dt} = -2\kappa {k_j}^2 \omega_j$$

$$|\vec{k}_j(t)| \approx |\vec{k}_j(0)| \cos\theta \, e^{h_j t}$$

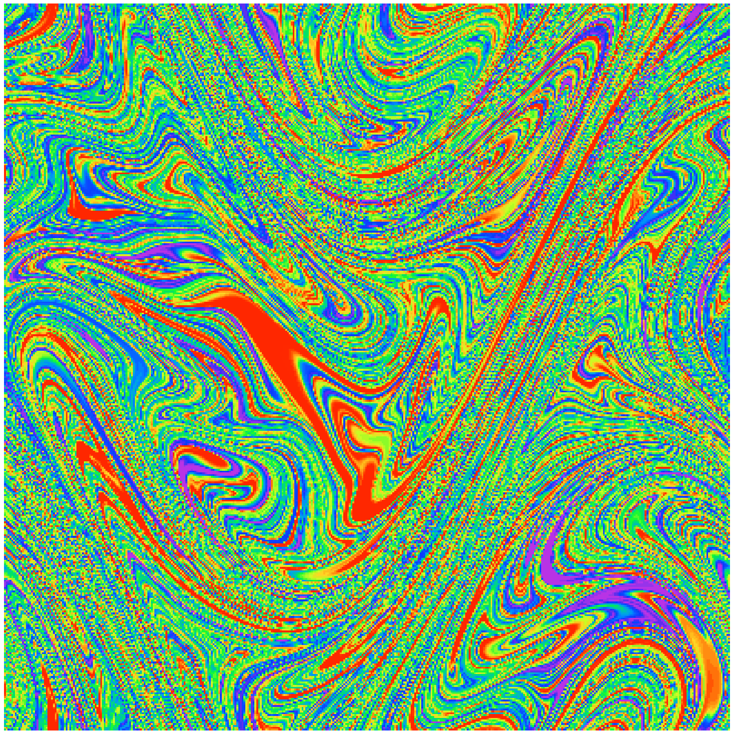
$$\boxed{\gamma_0 = \min_h [h + G(h)]}$$

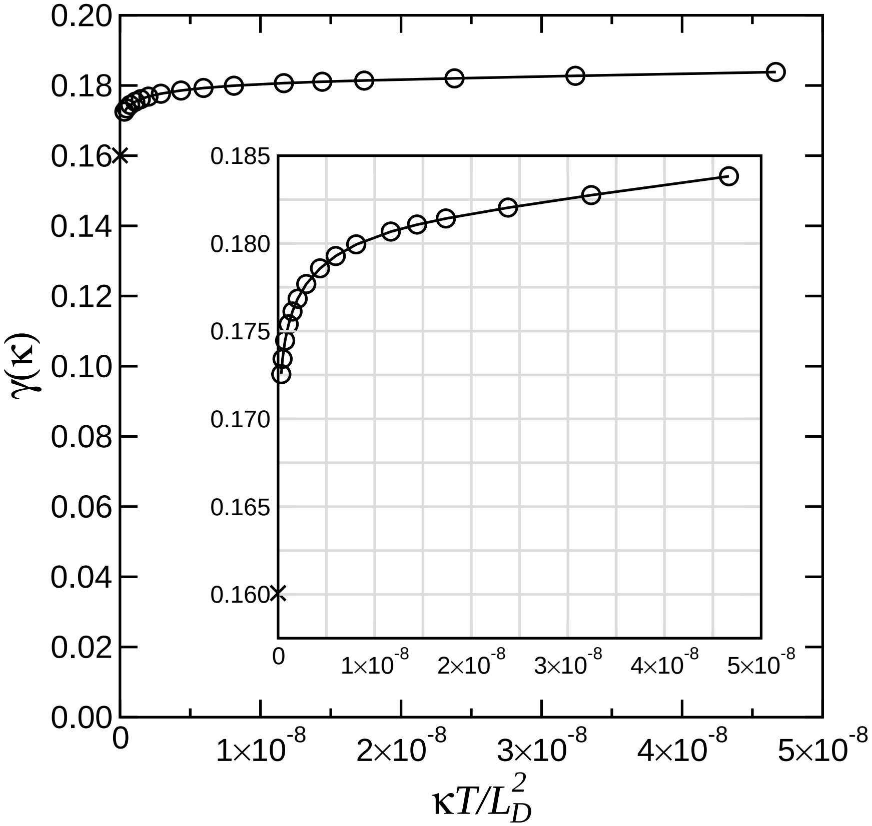
Antonsen *el at.*, Phys. Fluids **8**, 3094 (1996)

Comparison with Numerics

Flow Model: $T = 1$, $U = \pi$ $(L_f = L_D = 2\pi)$

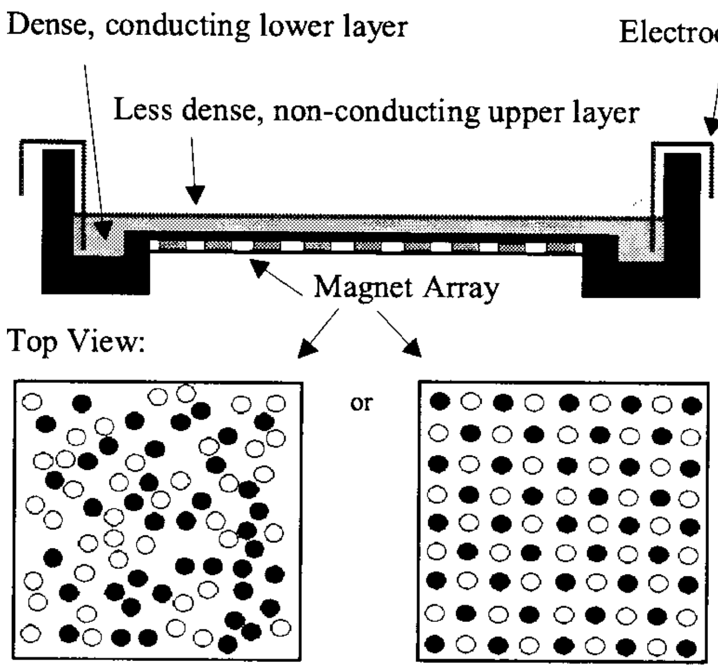
$$\vec{u}(\vec{x},t) = \begin{cases} U\cos(\frac{2\pi}{L_f}y + \alpha_n)\,\hat{i}\,, & nT \le t < (n+\frac{1}{2})T \\ U\cos(\frac{2\pi}{L_f}x + \beta_n)\,\hat{j}\,, & (n+\frac{1}{2})T \le t < (n+1)T \end{cases}$$

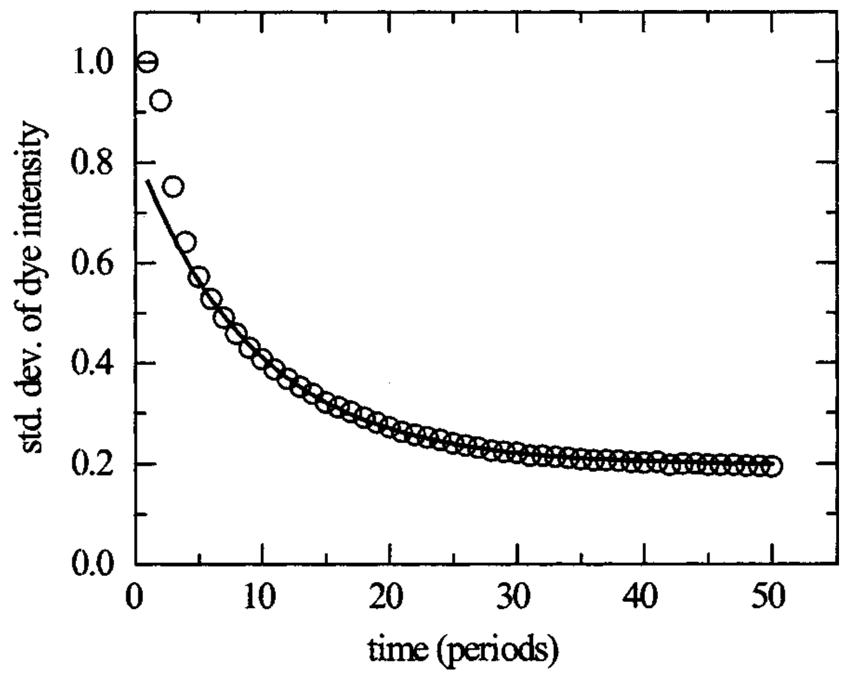




Laboratory Experiment

G. A. Voth, T.C. Saint, Greg Dobler, and J.P. Gollub, Phys. Fluids **15**, 2560 (2003)





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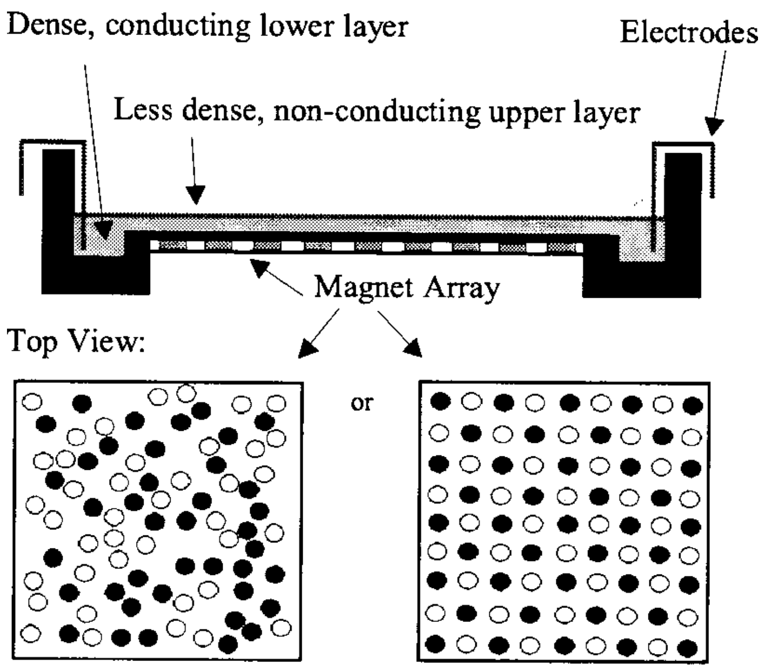
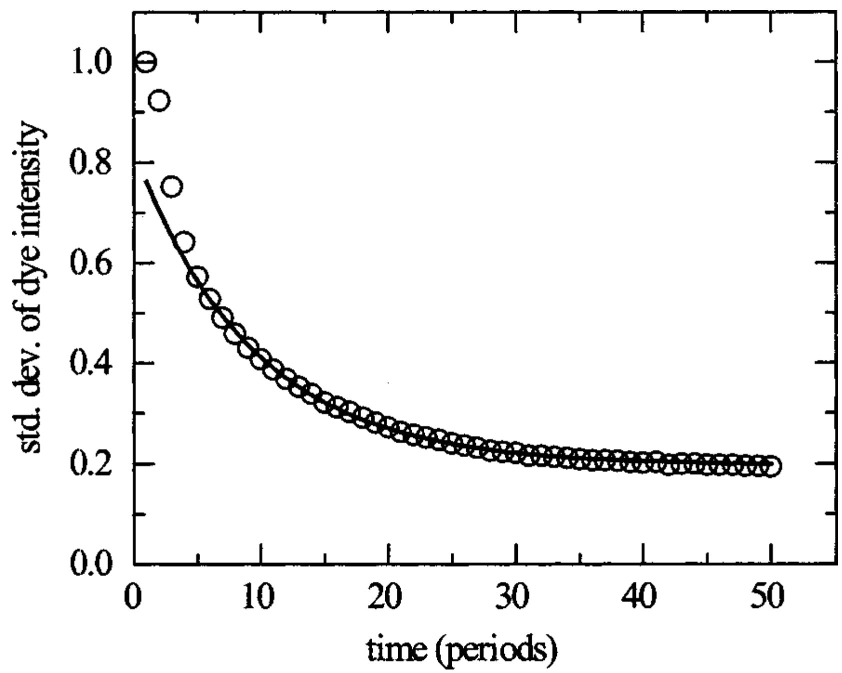




- measured decay rate is 10 times smaller than predicted γ_0!!
- Reason: the ratio L_D/L_f is an important factor

Variance Damping Mechanisms

$L_D \approx L_f$

- decay rate controlled by processes at small length scales (large k)
- γ_0 predicted by Lagrangian stretching theory *(short wavelength mechanism)*

$L_D \gg L_f$

- variance being “leaked” out of the longest wavelength mode (smallest k)
- decay rate limited by spatial diffusion on the large scales *(long wavelength mechanism)*

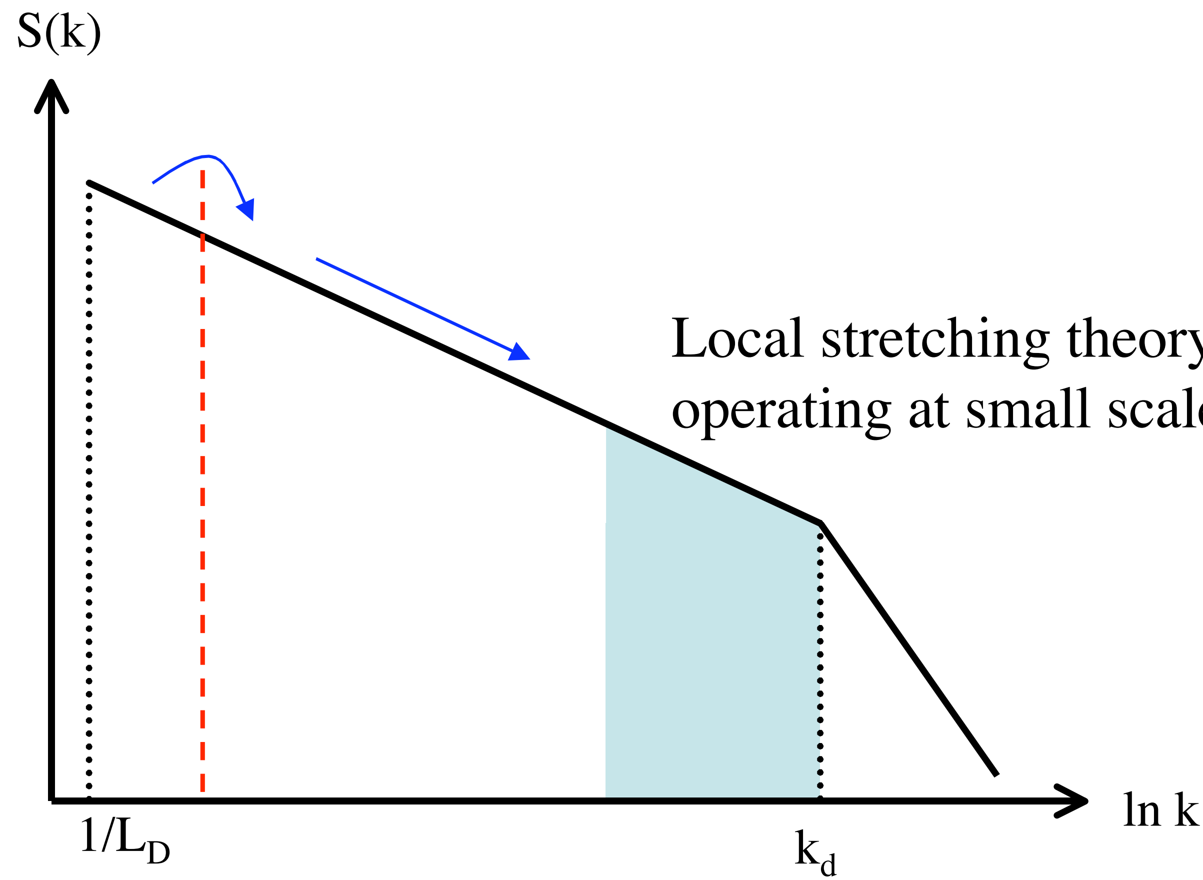
(D.R. Fereday, P.H. Haynes, A. Wonhas and J.C. Vassilicos, Phys. Rev. E **65** 035301(R), (2002))

Wavenumber Spectrum

$$S(k,t) = \int \frac{d\,\mathbf{k}'}{(2\pi)^2}\,\delta(k - |\mathbf{k}'|)\,\frac{|\tilde{\phi}(\mathbf{k}',t)|^2}{L_D^2} \sim S(k)e^{-\gamma(\kappa)t}$$

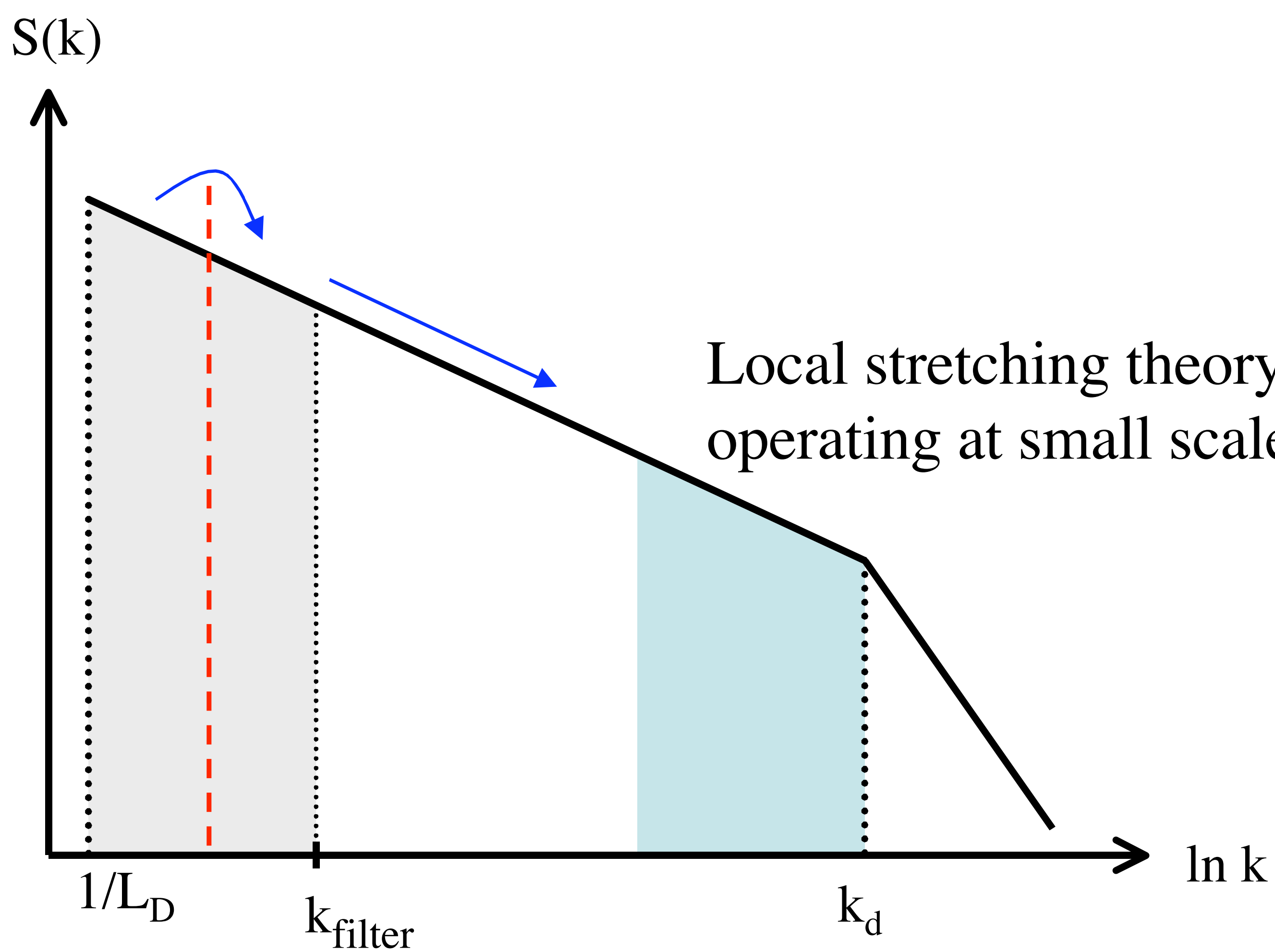
"strange eigenmode"

$$S(k) = \langle S(k,t)/C(t)\rangle_t$$



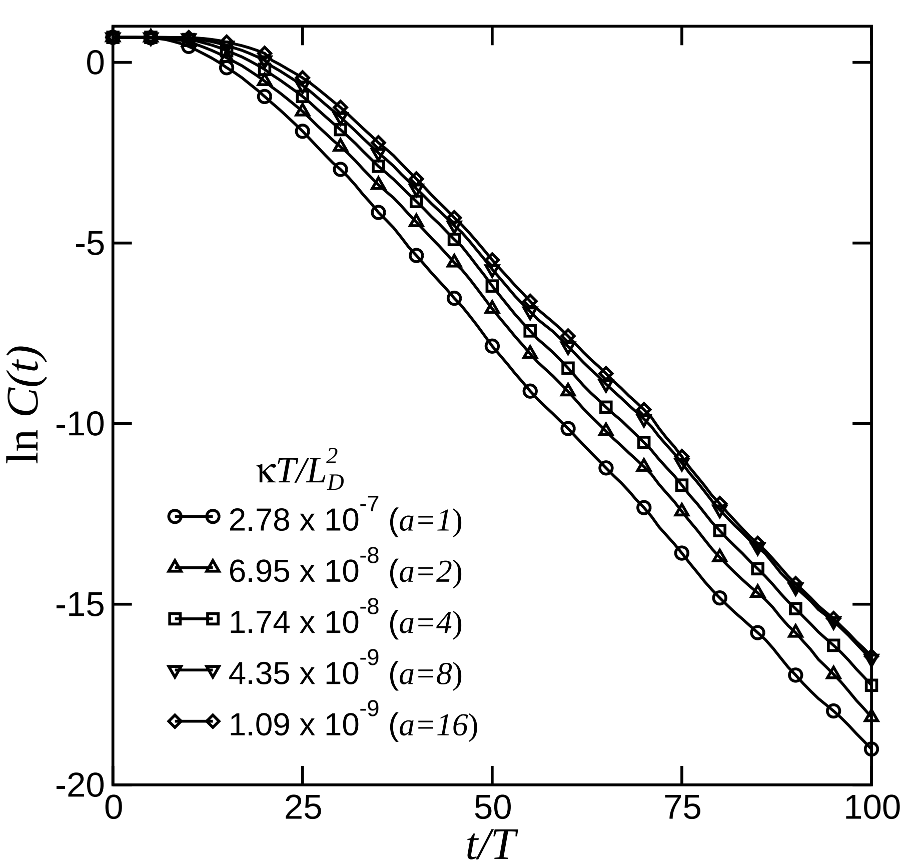
$L_D = L_f$

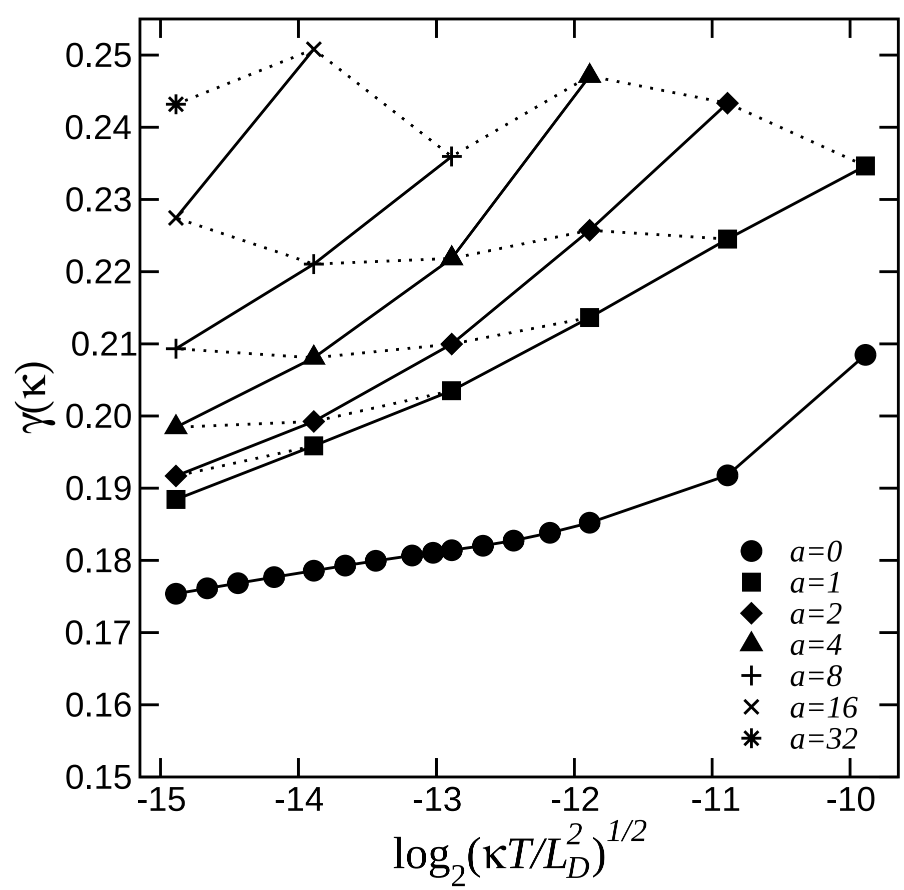
- for each time period $t > 20T$, remove all Fourier modes of ϕ with $|k_x|$ and $|k_y|$ less than $k_{filter} = a(2\pi/L_D)$



$L_D = L_f$

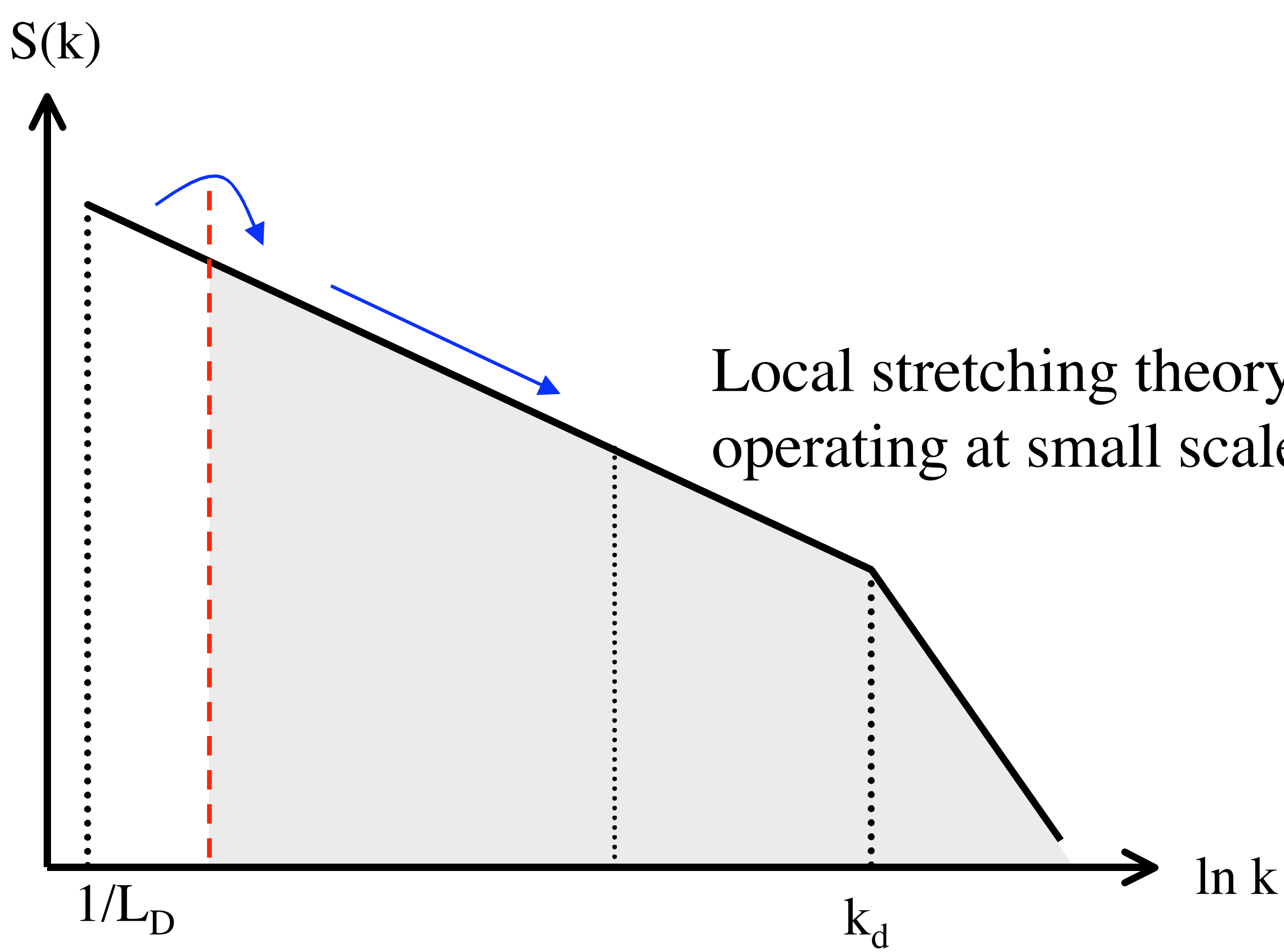
- decay rate (controlled by large k processes) is not affected by this filtering (fixed k_d/k_{filter})





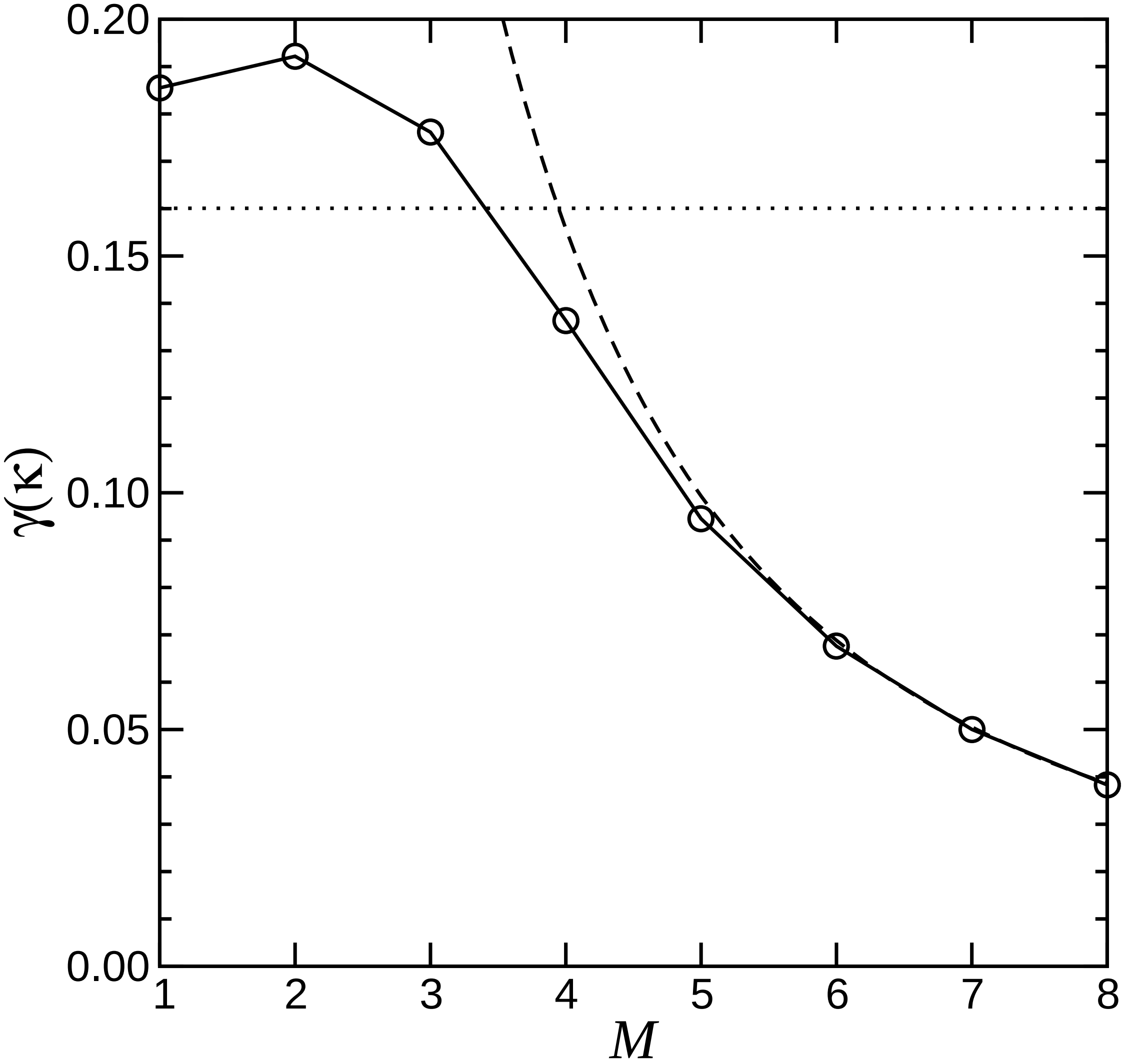
$L_D = ML_f \; (M > 1)$

- at each time step n, remove all but the lowest k mode (*i.e.* remove everything that leaks out of the lowest k mode)
- decay rate = rate of "leaking" from the lowest k mode



$L_D = ML_f\ (M > 1)$

- $\phi_{n+1} = [J_0(\eta)]^2\phi_n$ where $\eta = \pi UT/(ML_f)$
- leaking rate $= -\ln[J_0(\eta)]^4/T$ (dashed line)



Upper Bound on γ_0

$$S(k)e^{-\gamma_0 t} = \int_0^\infty dk'\, S(k') \left\langle \delta(k - k'|\cos\theta| e^{ht}) \right\rangle_{h,\theta}$$

Assuming $S(k) \sim k^{-\psi}$ (can generalize to anisotropic case), one can show

$$\gamma_0 = \min_h \left[h + G(h) - |\psi| h \right]$$

Two consequences:

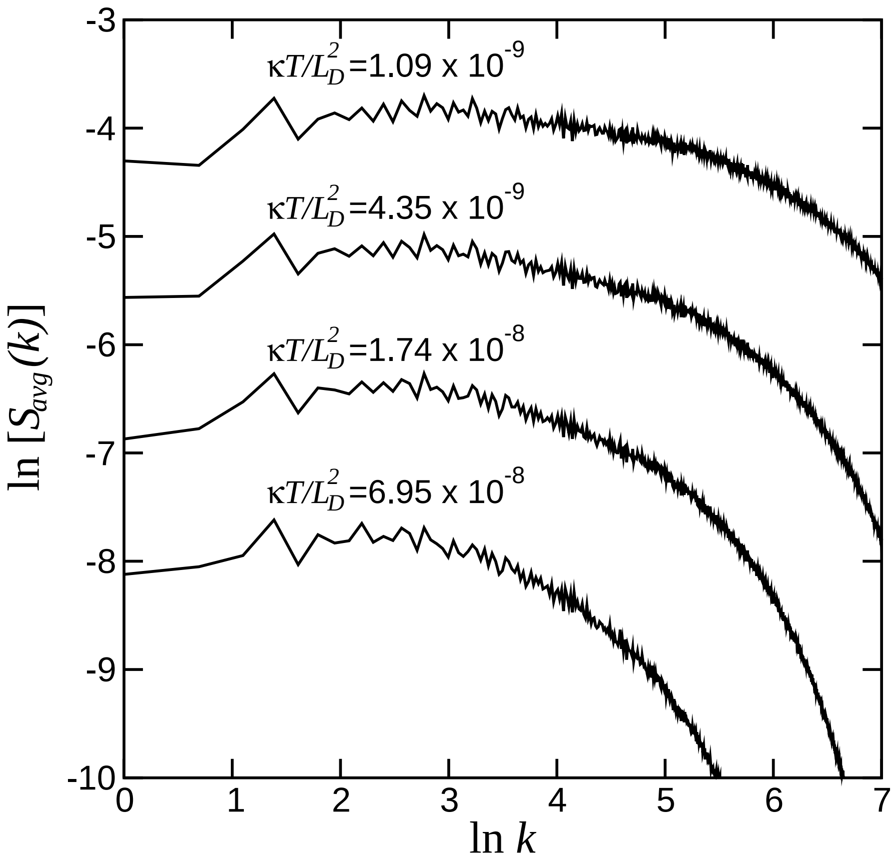
$$\boxed{\gamma_0 < \min_h \left[h + G(h) \right]}$$

$$\boxed{\psi = 1 + \min_h \left[\frac{G(h) - \gamma_0}{h} \right]}$$

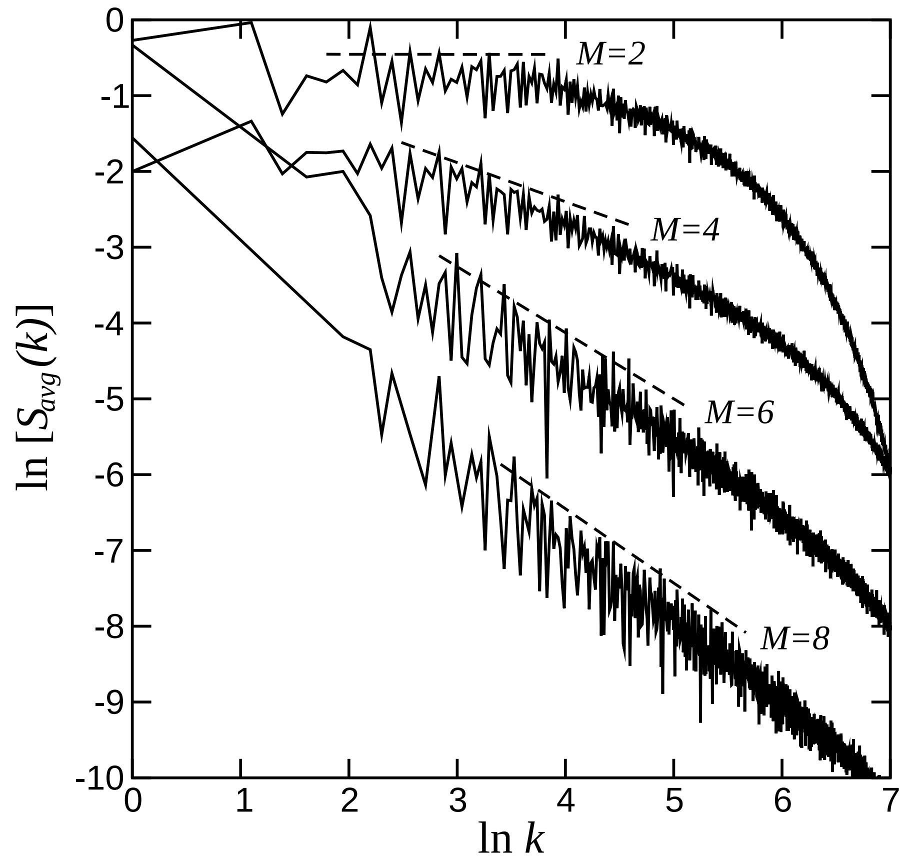
Wavenumber Spectra Exponent ψ

- short wavelength mechanism $\Rightarrow$ flat spectra
- long wavelength mechanism $\Rightarrow$ power-law spectra

$M = 1$



$M > 1$



Summary

- For $L_D \approx L_f$, short wavelength mechanism applies and γ_0 can be predicted using local stretching theory
- For $L_D \gg L_f$, long wavelength mechanism applies, γ_0 limited by the decay of the longest wavelength mode
- Decay rate predicted by the local stretching theory provides an upper bound on γ_0
- Long wavelength mechanism gives a power-law power spectrum, $k^{-\psi}$ with $\psi > 0$, short wavelength mechanism gives a flat power spectrum $(\psi = 0)$

Tsang, Antonsen and Ott, Phys. Rev. E **71***, 066301 (2005)*