FRACTAL PATTERNS IN CHAOTIC FLUID MIXING

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INTRODUCTION

- We study passive scalar advection-diffusion in chaotic fluid flows.
- An every day example: mixing cream in coffee.

- Other examples: the ozone in the atmosphere, pollutants in the ocean, plankton in the upper ocean, and the mixing of paints.

BACKGROUND

- The density of the passive scalar, $\phi(\vec{x},t)$, obeys:

$$\partial\phi/\partial t + \vec{v}\cdot\vec{\nabla}\phi = \kappa\nabla^2\phi$$

Where κ : molecular diffusion coefficient
$\vec{v}(\vec{x},t)$: fluid velocity field

- We simulate a particular incompressible chaotic fluid flow given by the velocity field:

$$\vec{v}(\vec{x},t)=\begin{cases}\hat{x}_0\,U\cos[(2\pi y/L_f)+\alpha_n] & \text{for } nT\le t<(n+1/2)T\\ \hat{y}_0\,U\cos[(2\pi x/L_f)+\beta_n] & \text{otherwise}\end{cases}$$

Where α_n and β_n : random numbers between $[0, 2\pi]$
L_f : spatial period of the flow
T : time period of the flow

- Integrating the velocity field over one time period T, we obtain an iterative map expressing the position of the fluid elements (x_n, y_n) at time $n = t/T$:

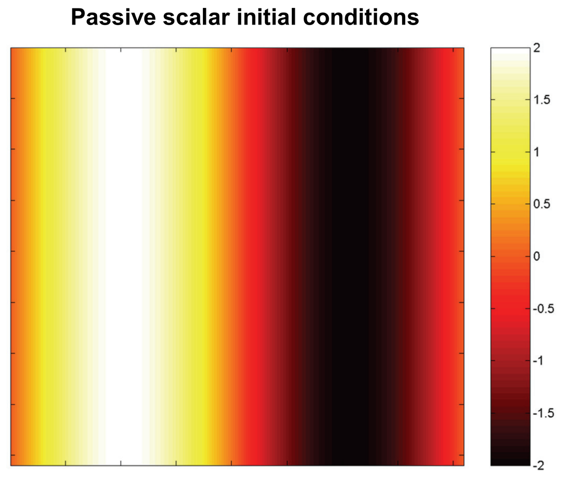
$$x_{n+1} = \{x_n + (UT/2)\cos[(2\pi y_n/L_f)+\alpha_n]\} \bmod L_d$$
$$y_{n+1} = \{y_n + (UT/2)\cos[(2\pi x_{n+1}/L_f)+\beta_n]\} \bmod L_d$$

L_d = M L_f : spatial period of the passive scalar
M : positive integer

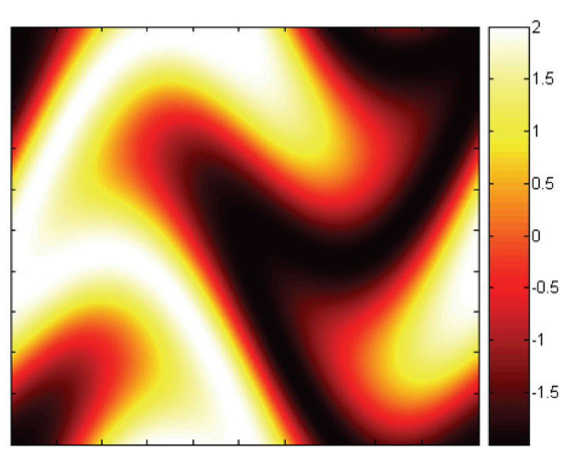
NUMERICAL SIMULATIONS

- We numerically determine the values of Φ on a 2-D grid with periodic boundary conditions (periodicity= M L_f) as a function of time n.

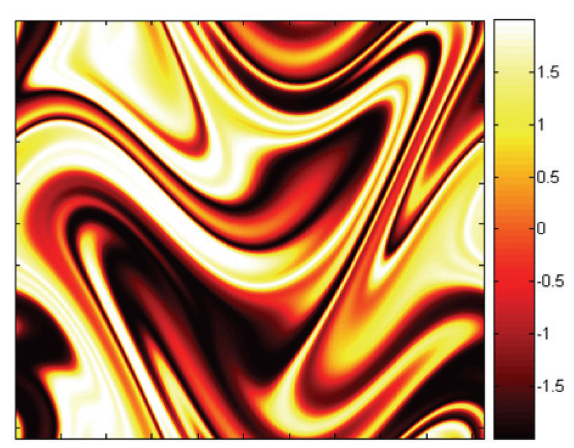
Passive scalar initial conditions



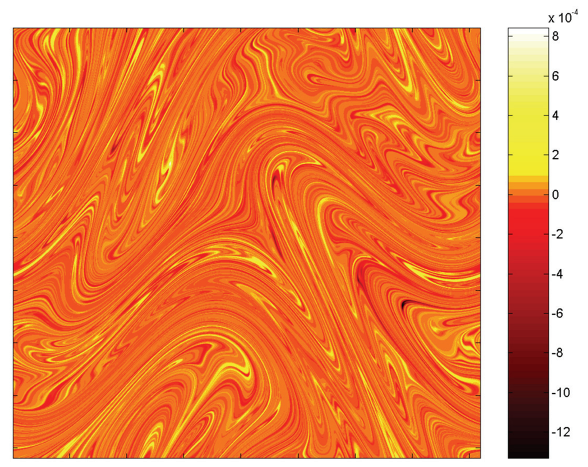
Passive scalar at n=1 with M=1



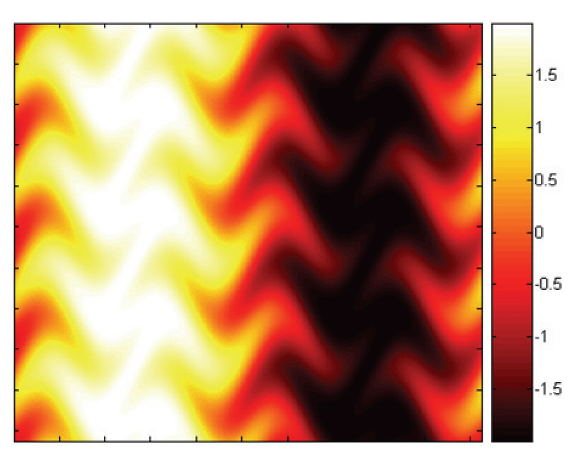
Passive scalar at n=5 with M=1



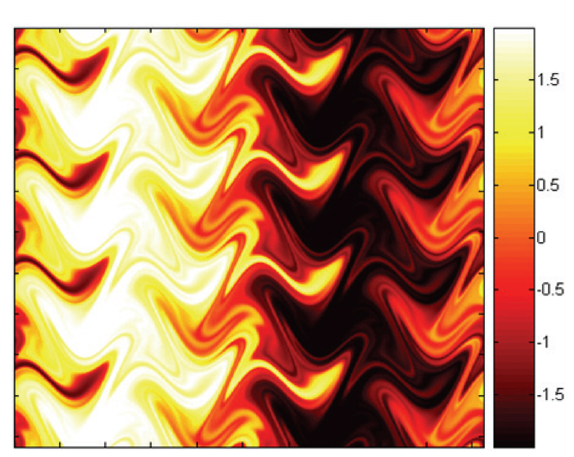
Passive scalar at n=125 with M=1



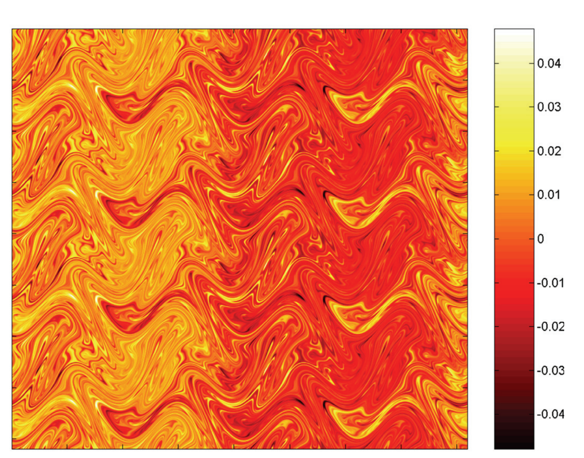
Passive scalar at n=1 with M=4



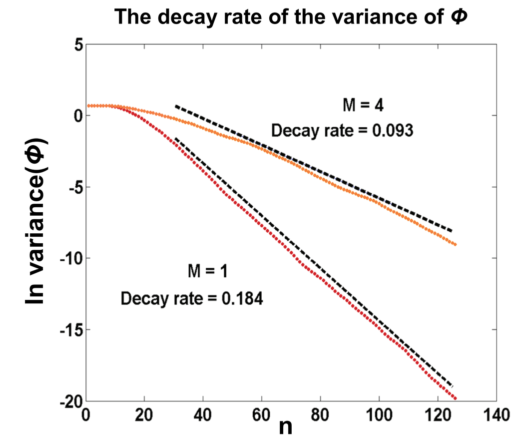
Passive scalar at n=5 with M=4



Passive scalar at n=125 with M=4



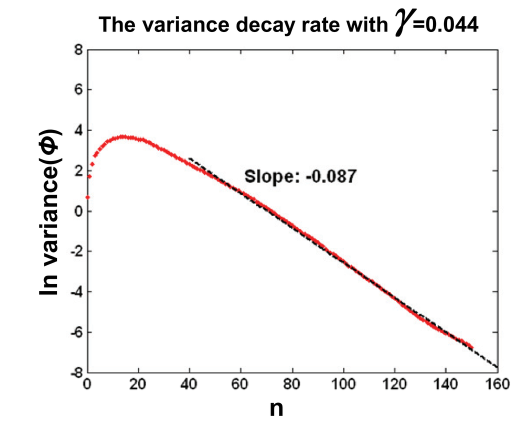
- As time increases, chaotic mixing causes Φ to develop variations on a finer and finer scale.
- Therefore, we consider the possibility that the density concentrates on a fractal set.



- As M increases, it takes longer for Φ to develop fine scale structure.
- It has previously been shown in several papers that the slower decay rate at larger M is controlled by processes at the largest possible length scales allowed by the system size (M L_f).
- Basically, variance slowly leaks from the largest length scales to shorter scales.
- We are interested in the case of a small decay rate.
- While the decay rate is determined by processes at long wavelength, fractal properties are short wavelength phenomena.
- Using large M is not feasible for the numerical simulations that we perform.
- Thus we desire to use M=1 while simulating a situation with large M (i.e. slow damping) by incorporating an exponentially decaying source at the largest length scale:

$$\partial\phi/\partial t + \vec{v}\cdot\vec{\nabla}\phi = \kappa\nabla^2\phi + e^{-\gamma t}\sin(2\pi x/L_f)$$

Where
γ is the decay rate of the source



- By incorporating a source with exponential decay rate γ=0.044, we achieve a slow damping rate that mimics a simulation with M=4.

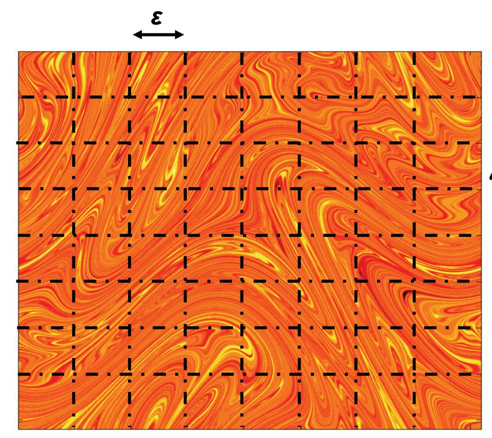
MAIN CONTRIBUTIONS OF OUR WORK

- We obtain high resolution numerical results for the fractal properties of the decaying scalar in the long time limit.
- We obtain theoretical predictions for the fractal properties.
- We compare the numerical and theoretical results.

- Our study of the fractal structure of the passive scalar is in terms of its "Spectrum of Fractal Dimension, D_q".
- D_q is a means of quantitatively characterizing the observed fine scale pattern behavior.

NUMERICAL APPROACH TO CALCULATING D_q

- To calculate D_q, we cover the space with a grid of boxes of size ε and determine the "measure", μ_i, of each box:



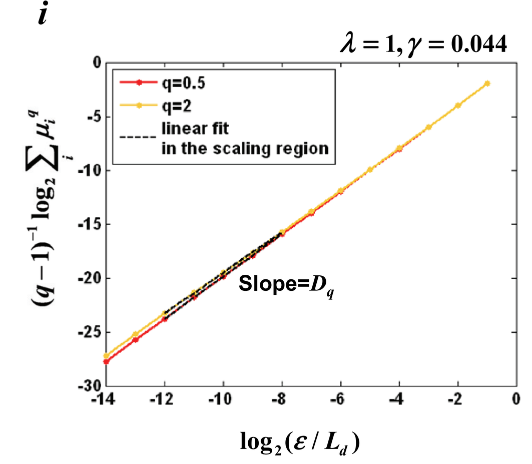
$$\mu_i = \frac{\int_{Box\ i} |\phi|^\lambda\, dxdy}{\int_{L_d\times L_d} |\phi|^\lambda\, dxdy}$$

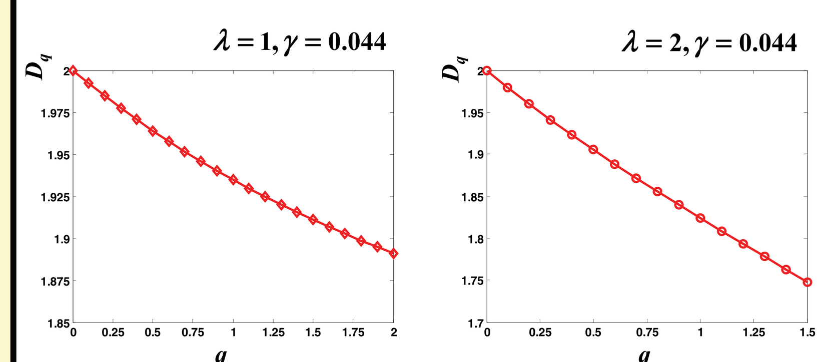
where $\lambda = 1\ or\ 2$

- D_q is then defined by:

$$\sum_i \mu_i^q \propto \varepsilon^{(q-1)D_q}$$

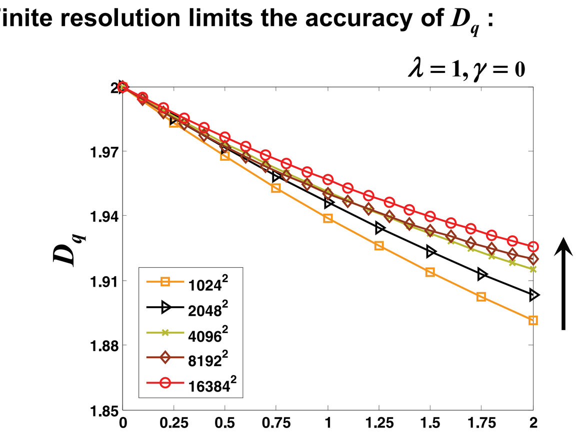
Diffusive cutoff $\ll \varepsilon \ll L_f$ scale





- Since D_q varies with q, by definition the decaying scalar has multifractal structure.
- For a nonfractal smooth density D_q would be 2 independent of q.

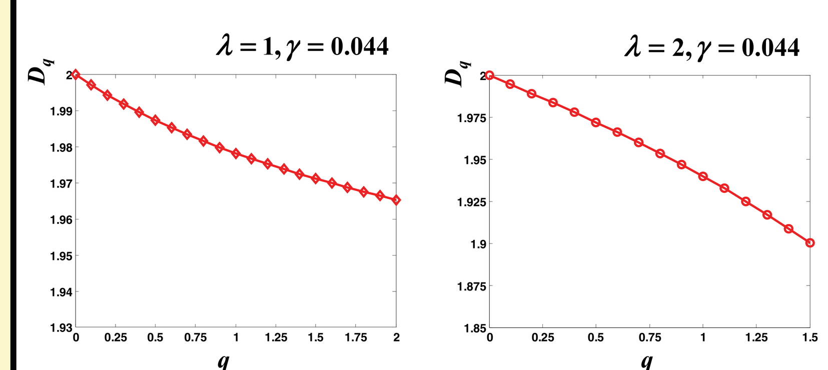
- Finite resolution limits the accuracy of D_q :



- With a constant source (γ= 0) D_q approaches 2.0 as the resolution is increased.

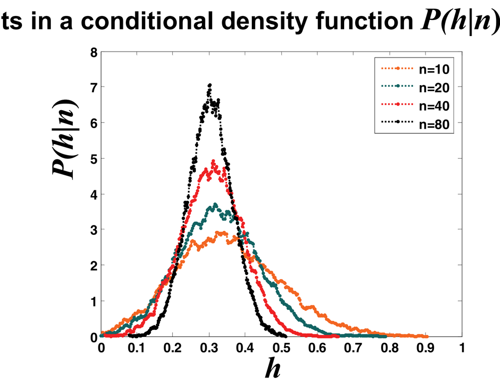
- At the 16384x16384 experiment, we correct for the finite resolution by adding the gap between 2.0 and D_q for γ= 0 to the dimension for γ= 0.044

Numerical results after correction for the finite resolution

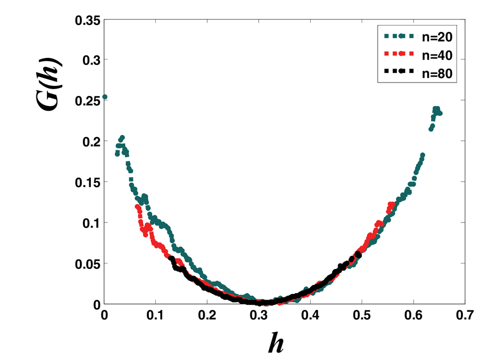


THEORETICAL APPROACH TO CALCULATING D_q

- We compute the Lyapunov exponent (h) of the trajectories of 16384 initial conditions uniformly distributed over the region of the flow.
- At finite time, the dependence of h on initial conditions results in a conditional density function $P(h|n)$:



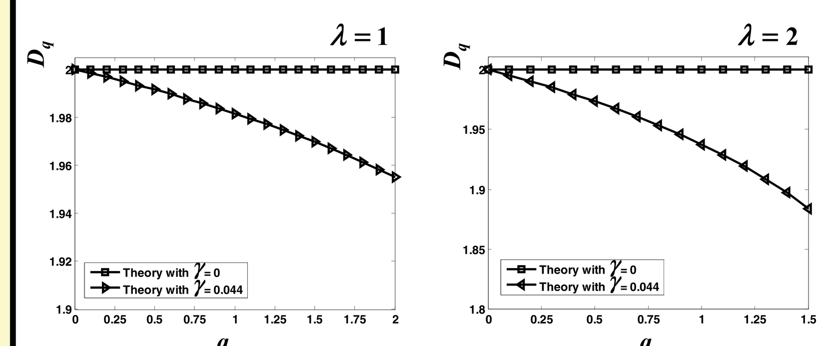
- A mean value of $\bar{h}$ = 0.30 ensures that the flow is chaotic as the trajectories of nearby fluid elements diverge exponentially in time.
- For large enough n: $P(h\,|\,n) \approx e^{-nG(h)}$



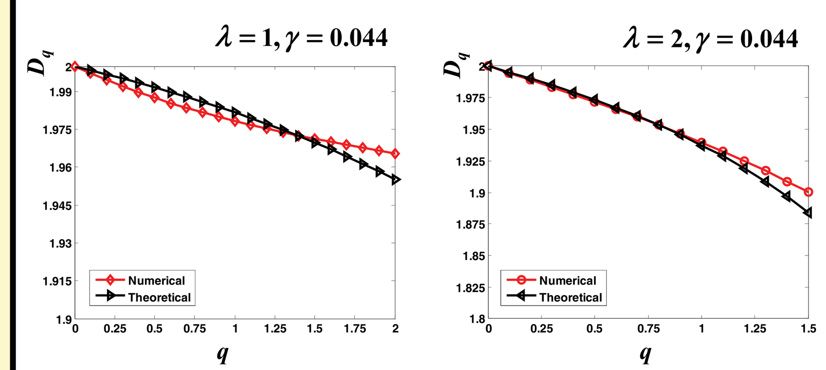
- The time-independent function $G(h)$ is concave up and has the minimum value zero, occurring at $h=\bar{h}$.
- A theoretical analysis (not given here) yields a result for D_q (with λ = 1) in terms of $G(h)$:

$$D_q = 2 - \frac{F(q)-qF(1)}{q-1} \quad \text{where} \quad F(q) = \max_{h\ge 0}\left\{\frac{q\gamma - G(h)}{h}\right\}$$

- A similar result is obtained for D_q with λ = 2



COMPARISON OF NUMERICAL AND THEORETICAL RESULTS



CONCLUSIONS

- The decaying passive scalar has fractal properties and is characterized by D_q.
- We have developed a theory which relates D_q to the distribution of finite-time Lyapunov exponents of the chaotic flow.
- We provide high resolution numerical evidence for our theoretical predictions.