Planktonic Population in a Spatially Variable Environment

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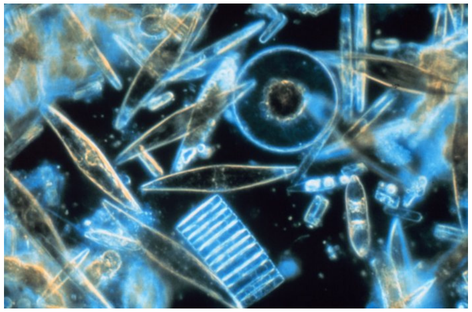
What is Plankton?

- tiny open-water bacteria, plants or animals that have *limited or no swimming ability*
- transported through the water by currents and tides



Phytoplankton

Macrozooplankton

Zooplankton

One gallon of water from the Chesapeake Bay can contain more than 500,000 zooplankton and one drop may contain thousands of individual phytoplankton!!

Why Study Plankton Population?

- carbon fixation: CO_2 concentration in atmosphere would be doubled without plankton
- indicators of nutrient level and other water quality conditions
- base of the food chain that support commercial fisheries
- plankton blooms that deplete oxygen

Population Model

$$\frac{\partial P}{\partial t} + \underbrace{\vec{u} \cdot \nabla P}_{\textit{advection}} = \underbrace{\gamma P}_{\textit{growth}} - \underbrace{\eta P^2}_{\textit{saturation}} + \underbrace{\kappa \nabla^2 P}_{\textit{small-scale diffusion}}$$

Population Model

$$\frac{\partial P}{\partial t} + \vec{u} \cdot \nabla P = \gamma P - \eta P^2 + \kappa \nabla^2 P$$

advection *growth* *saturation* *small-scale diffusion*

2D Lagrangian Chaotic Flow (U, k, τ)

$$\vec{u}(\vec{x},t) = \begin{cases} \sqrt{2}U\cos[ky+\alpha(t)]\,\hat{i}\ , & n\tau \leq t < (n+\frac{1}{2})\tau \\ \sqrt{2}U\cos[kx+\beta(t)]\,\hat{j}\ , & (n+\frac{1}{2})\tau \leq t < (n+1)\tau \end{cases}$$

- Spatially uniform γ and η: $P \rightarrow \eta/\gamma$

Population Model

$$\frac{\partial P}{\partial t} + \vec{u}\cdot\nabla P = \gamma(\vec{x})P - \eta P^2 + \kappa\nabla^2 P$$

advection

growth

saturation

small-scale diffusion

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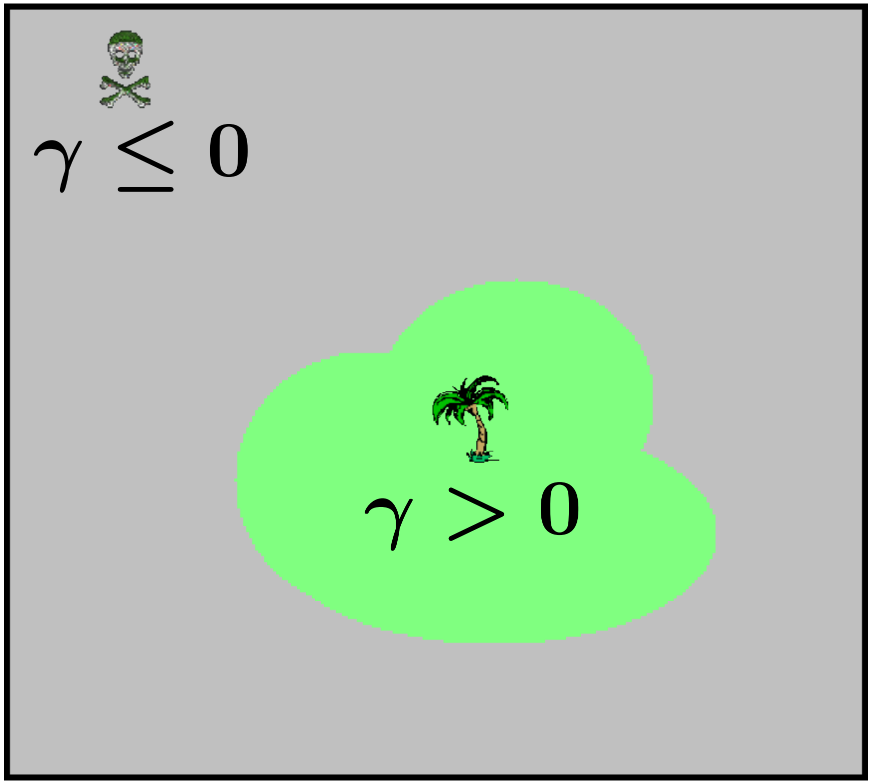
- Spatially uniform γ and η: $P \rightarrow \eta/\gamma$
- Environmental variability : $\gamma = \gamma(\vec{x})$

Oasis and Desert

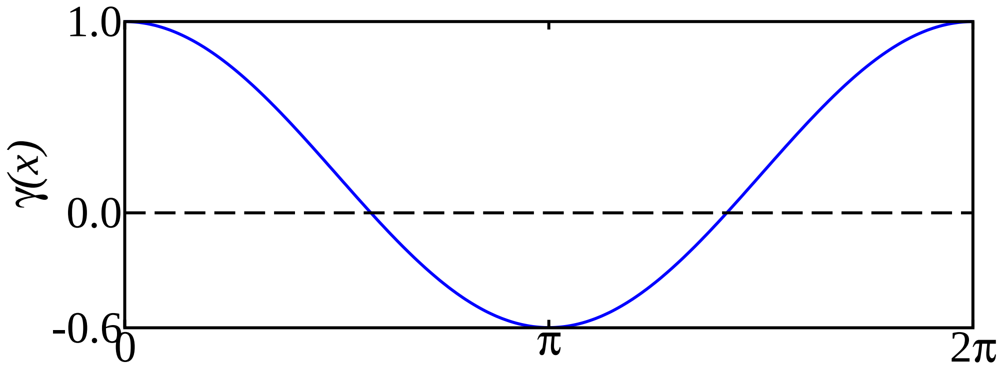
- interested in the situation where $\gamma(\vec{x}) > 0$ in some region (oasis) and $\gamma(\vec{x}) \leq 0$ in the complementary region (desert)



- two cases: (1) $\langle\gamma\rangle > 0$ and (2) $\langle\gamma\rangle < 0$

An Example

- doubly periodic domain
- domain size : $2\pi \times 2\pi$
- grid : 1024×1024



$$\gamma(\vec{x}) = 0.2 + 0.8\cos x$$

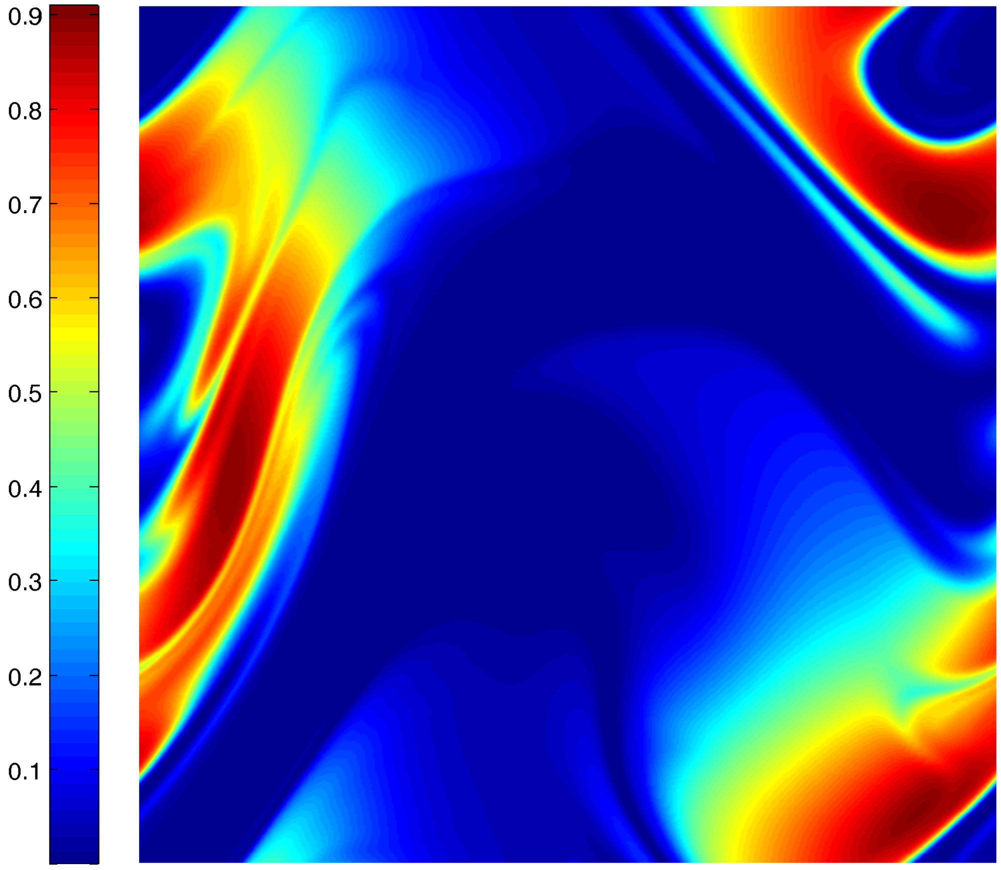
$$\eta = 1.0$$

$$\kappa \sim 10^{-4}$$

$$U = \pi$$

$$k = 1$$

$$\tau = 1/4$$



Biomass and Productivity

Biomass

$$\mathcal{B} = \langle P \rangle = \lim_{T\to\infty} \frac{1}{T A_\Omega} \int_0^T dt \int_\Omega d\vec{x}\; P(\vec{x}, t)$$

Productivity

$$\mathcal{P} = \langle \gamma P \rangle$$

For the above example: $\mathcal{B} \approx 0.24$ and $\mathcal{P} \approx 0.14$ (note $\langle \gamma \rangle = 0.2$)

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We shall obtain bounds on $\mathcal{B}$ and $\mathcal{P}$ in terms of γ and η

Bounds on $\mathcal{B}$ and $\mathcal{P}$

$$\frac{\partial P}{\partial t} + \vec{u} \cdot \nabla P = \gamma P - \eta P^2 + \kappa \nabla^2 P \qquad (*)$$

$$\langle (*) \rangle \qquad \langle \gamma P - \eta P^2 \rangle = 0$$

$$\mathcal{B} = \langle P \rangle + m \langle \gamma P - \eta P^2 \rangle \qquad (m > 0)$$

$$= \frac{m}{4\eta} \left\langle \left(\gamma + \frac{1}{m} \right)^2 \right\rangle - m\eta \langle (P - \#)^2 \rangle$$

Bounds on $\mathcal{B}$ and $\mathcal{P}$

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$$\mathcal{B} = \langle P \rangle + m \left\langle \gamma P - \eta P^2 \right\rangle \qquad (m > 0)$$

$$\leq \frac{m}{4\eta} \left\langle \left(\gamma + \frac{1}{m} \right)^2 \right\rangle$$

minimizing RHS with respect to m gives upper bound for $\mathcal{B}$,

$$\mathcal{B} \leq \frac{\sqrt{\langle \gamma^2 \rangle} + \langle \gamma \rangle}{2\eta}$$

Bounds on $\mathcal{B}$ and $\mathcal{P}$

$$\langle (*)/P \rangle \qquad \gamma - \eta \langle P \rangle + \kappa \langle |\nabla \ln P|^2 \rangle = 0$$

$$\boxed{\frac{\langle \gamma \rangle}{\eta} \leq \mathcal{B} \leq \frac{\sqrt{\langle \gamma^2 \rangle} + \langle \gamma \rangle}{2\eta}}$$

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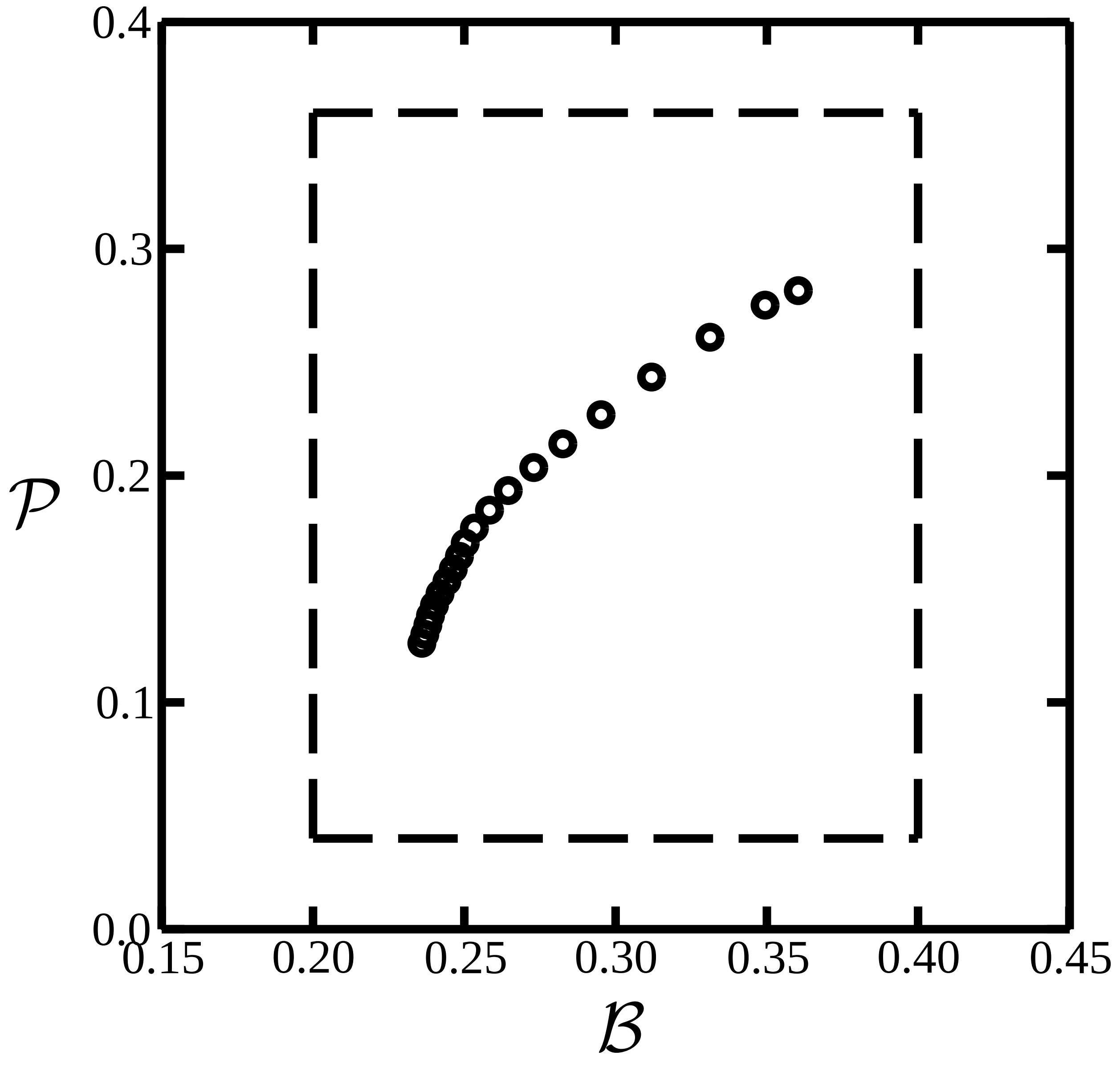
Cauchy-Schwarz Inequality: $\langle AB \rangle \leq \sqrt{\langle A^2 \rangle \langle B^2 \rangle}$,

$$\eta \langle P \rangle^2 \leq \eta \left\langle P^2 \right\rangle = \langle \gamma P \rangle \leq \sqrt{\langle \gamma^2 \rangle \langle P^2 \rangle}$$

$$\boxed{\frac{\langle \gamma \rangle^2}{\eta} \leq \mathcal{P} \leq \frac{\langle \gamma^2 \rangle}{\eta}}$$

Bounds on $\mathcal{B}$ and $\mathcal{P}$

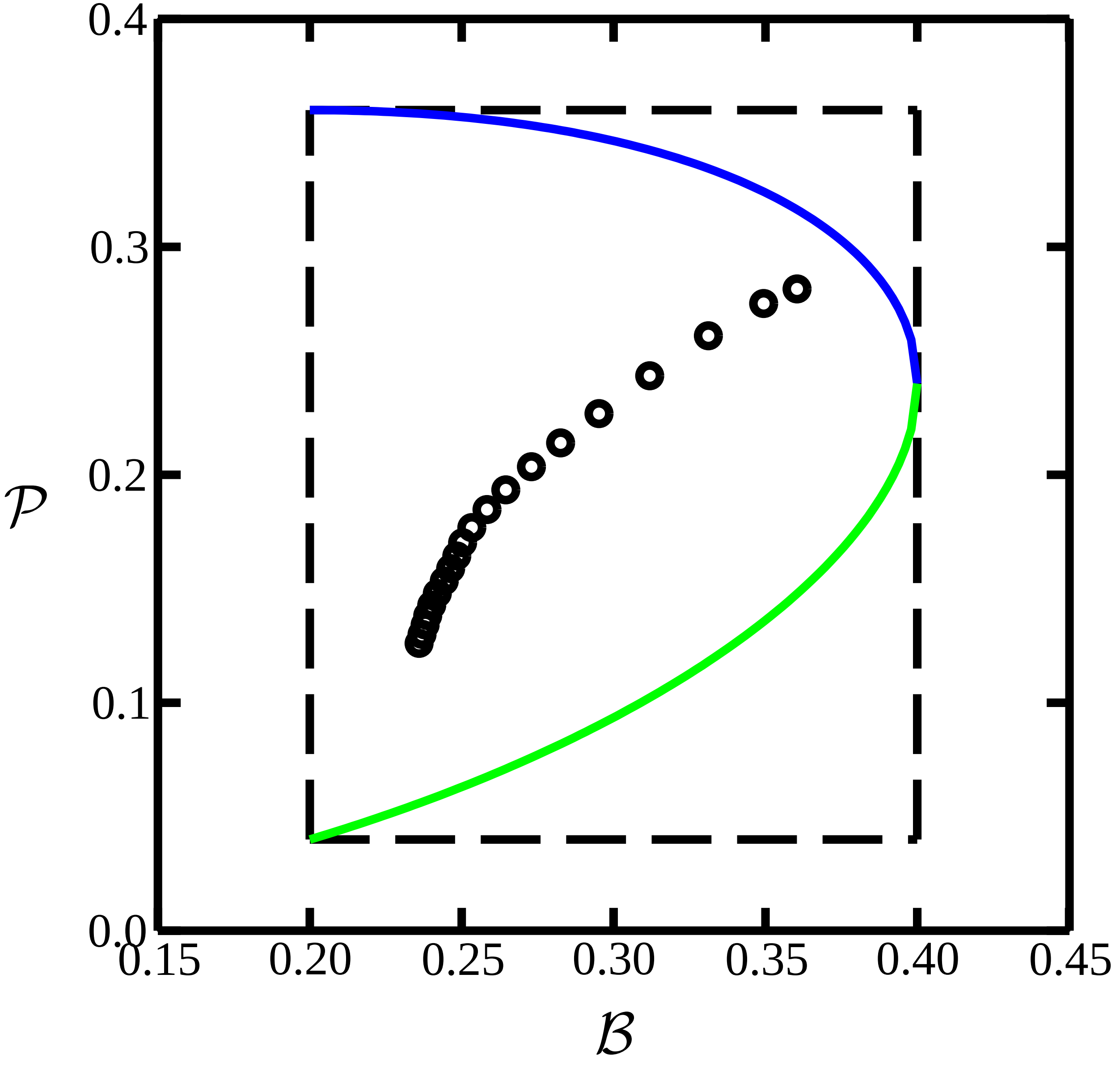
For our example with $\gamma(\vec{x}) = 0.2 + 0.8\cos x$ and $\eta = 1$,



Simultaneous Bounds

$$\mathcal{P} = \langle \gamma P \rangle + \alpha[\mathcal{B} - \langle P \rangle] + \beta[\langle \gamma P \rangle - \eta \left\langle P^2 \right\rangle]$$

$$f_1(\mathcal{B}) \leq \mathcal{P} \leq f_2(\mathcal{B})$$

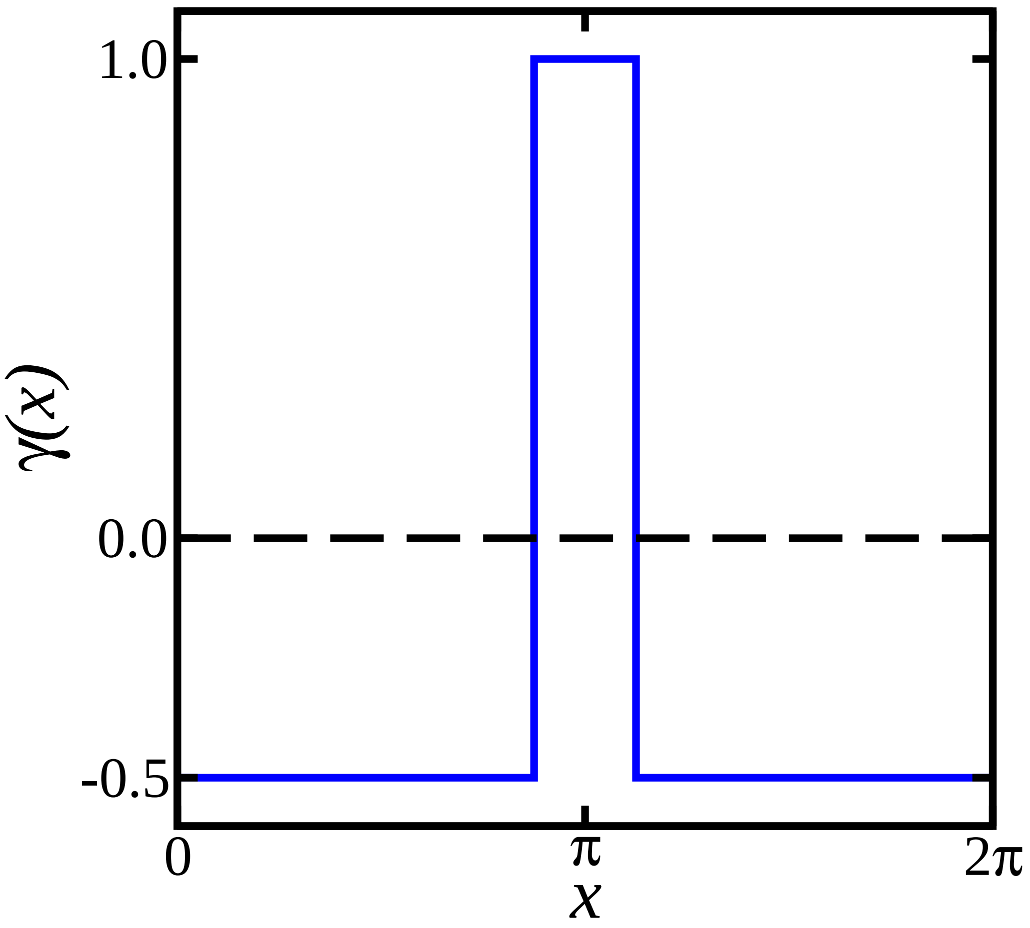


What happen when $\langle\gamma\rangle < 0$?

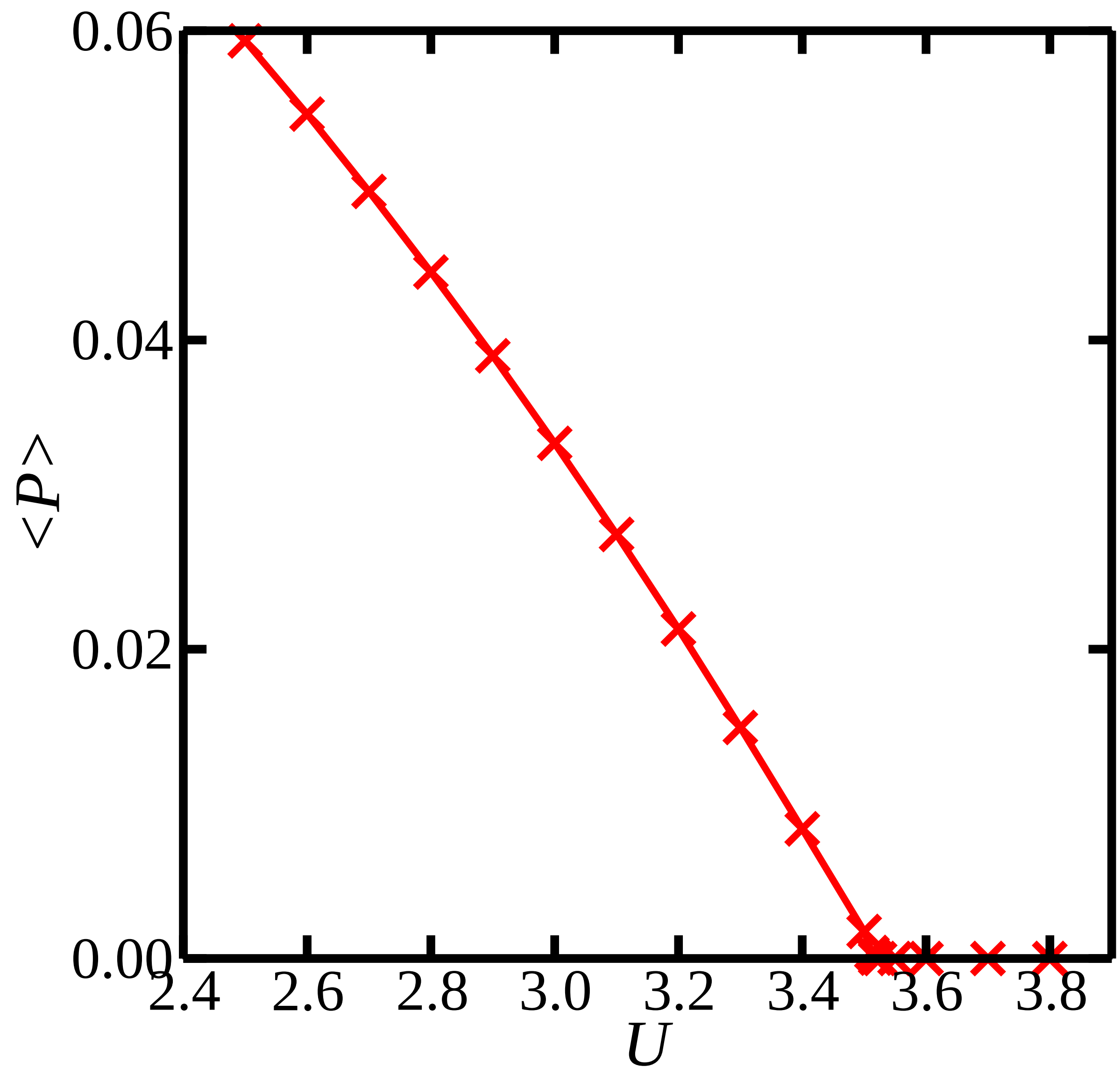
- Recall $\frac{\langle\gamma\rangle}{\eta} \leq \mathcal{B} \leq \frac{\sqrt{\langle\gamma^2\rangle}+\langle\gamma\rangle}{2\eta}$
- $\langle\gamma\rangle > 0$ implies population will never become extinct

What happen when $\langle\gamma\rangle < 0$?

- Recall $\frac{\langle\gamma\rangle}{\eta} \leq \mathcal{B} \leq \frac{\sqrt{\langle\gamma^2\rangle}+\langle\gamma\rangle}{2\eta}$
- $\langle\gamma\rangle > 0$ implies population will never become extinct
- When $\langle\gamma\rangle < 0$, extinction ($\mathcal{B} = 0$) is a possibility



Survival-Extinction Transition



- transition occurs at $U = U_c \approx 3.52$, prediction for U_c?

Eddy (Effective) Diffusivity

Characterize spreading using $\overline{X^2}$:

- a pure diffusive process

$$\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \phi \quad \Rightarrow \quad \overline{X^2} = 4\kappa t$$

- pure advection by our flow model

$$\frac{\partial \phi}{\partial t} + \vec{u} \cdot \nabla \phi = 0 \quad \Rightarrow \quad \overline{X^2} = \frac{U^2 \tau}{2} t$$

may parameterize the effect of $\vec{u}$ on large-scale ϕ by D,

$$\frac{\partial \phi}{\partial t} = {\color{red}D} \nabla^2 \phi \quad \text{with} \quad \boxed{D = \frac{U^2 \tau}{8}}$$

Theory for U_c

$$\frac{\partial P}{\partial t} + \vec{u} \cdot \nabla P = \gamma P - \eta P^2 + \kappa \nabla^2 P$$

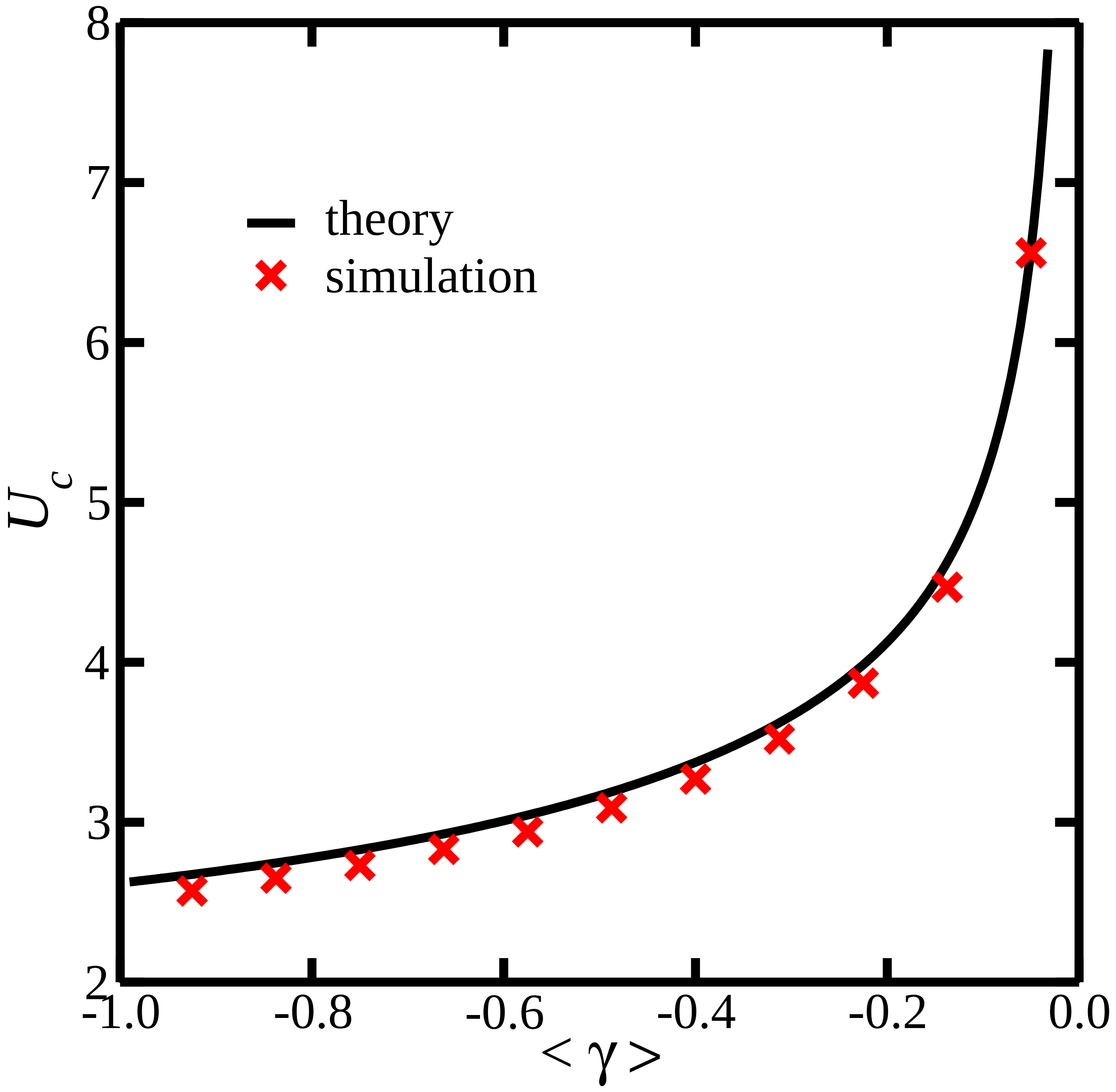
- consider small κ
- near transition: $\eta P^2 \approx 0$
- parameterization using eddy diffusivity D

$$\frac{\partial P}{\partial t} = \gamma P + D \nabla^2 P$$

Linear stability analysis about $P = 0$: $P = e^{st} \hat{P}$
maximum $s = 0 \Rightarrow$ extinction, thus critical D_c given by

$$\nabla^2 P + \frac{\gamma}{D_c} P = 0 \quad \text{and} \quad D_c = \frac{U_c^2 T}{8}$$

Theory vs. Simulation



Summary

- we study plankton population using an advection-diffusion logistic-growth model in a domain divided into oasis ($\gamma > 0$) and desert ($\gamma < 0$)

- for $\langle\gamma\rangle > 0$, population never extinct and we obtain bounds on the biomass and productivity

- for $\langle\gamma\rangle < 0$, population becomes extinct when the velocity is large, we make prediction to such critical velocity U_c