Intermittency and Multifractality in Two-Dimensional Turbulence with Drag

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Turbulence: 2-D versus 3-D

3-D Turbulence

- Navier-Stokes momentum equation

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{v} + \vec{f} \qquad (\nabla\cdot\vec{v} = 0)$$

- Kolmogorov's Phenomenology $(r_d \ll r \ll L, Re \to \infty)$
  - mean energy dissipation rate $\langle\varepsilon\rangle$
  - structure functions

$$S'_{3p}(r) = \left\langle [\vec{v}(\vec{x}+\vec{r}) - \vec{v}(\vec{x})]^{3p}\right\rangle \sim r^{\zeta_{3p}}$$
$$\zeta_{3p} = p\zeta_3 \qquad (\zeta_3 = 1)$$

  - experiments *contradict* Kolmogorov's hypotheses

2-D Turbulence

- Vorticity equation $(\omega = \hat{z}\cdot\nabla\times\vec{v})$

$$\frac{\partial \omega}{\partial t} + \vec{v}\cdot\nabla\omega = \nu\nabla^2\omega + f_\omega \qquad (\nabla\cdot\vec{v} = 0)$$

- Kraichnan's Phenomenology (for forward cascade)
  - mean enstrophy dissipation rate $\langle\eta\rangle$ (enstrophy$=\frac{1}{2}\omega^2$)
  - structure functions

$$S_{2q}(r) = \left\langle |\omega(\vec{x}+\vec{r}) - \omega(\vec{x})|^{2q}\right\rangle \sim r^{\zeta_{2q}}$$
$$\zeta_{2q} = q\zeta_2 \qquad (\zeta_2 = 0) \qquad \text{linear in } q$$

  - recent experiment (magnetically forced stratified flow) *supports* Kraichnan's theory

Why care about 2-D Turbulence?


(V.Hessler)

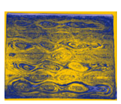
equatorial ionospheric plasma

ion-neutral collision


(NAOO)

geophysical flow

Ekman friction


(CNLD, U.Texas Austin)

soap film flow

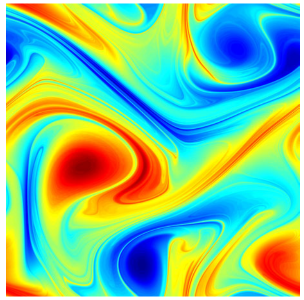
friction by surrounding air

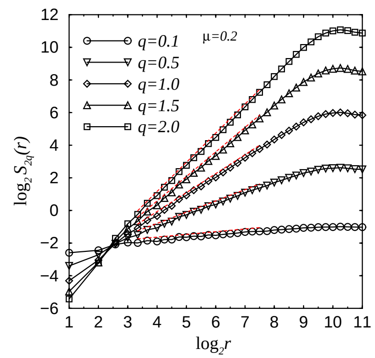
What is the effect of DRAG on 2-D turbulence?

Direct Numerical Simulation

$$\frac{\partial \omega}{\partial t} + \vec{v}\cdot\nabla\omega = -\nu\nabla^8\omega + S_\omega - \mu\omega$$

periodic boundary conditions	$\nu = 7.5\times10^{-25}$
domain size : $2\pi\times2\pi$	$f_\omega = \cos 2x$
grid size: 4096×4096	$\mu = 0.2$


snapshot of the vorticity field $\omega(x,y)$

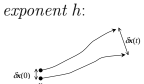


- **DRAG** $(\mu \neq 0) \Rightarrow$ **INTERMITTENCY** $(\zeta_{2q} \neq q\zeta_2)$

Theory for ζ_{2q}

- relate ζ_{2q} to the *finite-time Lyapunov exponent* h:

$$h(\vec{x},t) = \frac{1}{t}\log\frac{|\delta\vec{x}(t)|}{|\delta\vec{x}(0)|}$$

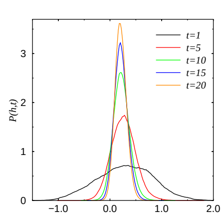


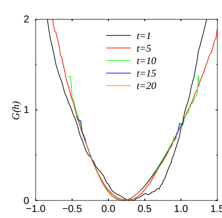
- probability density function of h, $P(h,t)$ with *large time asymptotic* form:

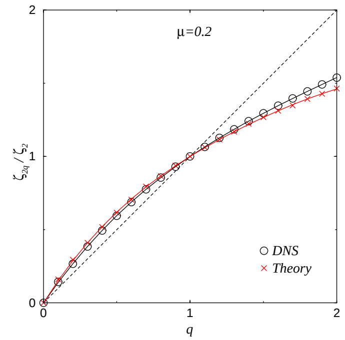
$$P(h,t) \sim \exp[-tG(h)]$$

- theoretical result for ζ_{2q}:

$$\boxed{\zeta_{2q} = \min\left\{\frac{G(h)}{h} + \frac{2q\mu}{h}\right\}}$$







Multifractal Dissipation

- 3-D Turbulence
  anomalous scaling of $S'_{3p}(r) \longleftrightarrow$ multifractality in ε
- 2-D Turbulence with Drag
  anomalous scaling of $S_{2q}(r) \longleftrightarrow$ multifractality in η


snapshot of $|\nabla\omega(x,y)|^2 \sim \eta$

Quantifying Multifractality

Divide the region R occupied by the fluid into grid of square boxes $R_i(\epsilon)$ of size ϵ,

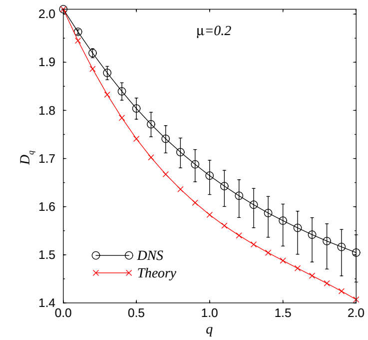
$$\text{Measure } p_i(\epsilon) = \frac{\int_{R_i(\epsilon)} |\nabla\omega|^2\, d\vec{x}}{\int_R |\nabla\omega|^2\, d\vec{x}}$$

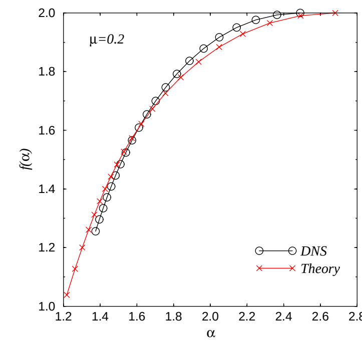
$$\text{Generalized Dimension } D_q \;:\; \sum_i p_i^q \sim \epsilon^{(q-1)D_q}$$

$$\text{Singularity Spectrum } f(\alpha) \;:\; p_i \sim \epsilon^{\alpha_i} \;\; ; \;\; N(\alpha) \sim \epsilon^{-f(\alpha)}$$

If the measure $p_i(\epsilon)$ is multifractal, then

- D_q varies with q
- $f(\alpha)$ is a non-trivial function of α





Relation between D_q and ζ_{2q}

At the dissipative scale r_d, the vorticity field is smoothed out by the action of viscosity, thus

$$\int_{R(r_d)} |\nabla\omega|^2 d\vec{x} \sim r_d^2 |\nabla\omega(\vec{x})|^2$$
$$|\nabla\omega(\vec{x})| \sim \frac{|\omega(\vec{x}+\vec{r}) - \omega(\vec{x})|}{r_d}$$

It then follows,

$$\boxed{D_q = 2 + \frac{\zeta_{2q} - q\zeta_2}{q-1}}$$

Conclusion

For two-dimensional turbulence with linear drag,

- the vorticity field is intermittent, $\zeta_{2q} \neq q\zeta_2$
- ζ_{2q} can be obtained in terms of the finite-time Lyapunov exponent and the drag coefficient μ
- the measure based on the viscous enstrophy dissipation is multifractal, $D_q \neq$ constant
- D_q is related to ζ_{2q}