Two Dimensional Turbulence with Drag: Wavenumber Energy Spectrum and Intermittency

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Two-Dimensional Turbulence

- Soap film flows
- Rotating fluids
- Magnetically forced stratified flows
- Plasma in the equatorial ionosphere
- Geophysical flow: heat transport in the Earth's atmosphere, mixing of chemical species in the polar stratosphere

Navier-Stokes momentum equation:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{v} + \vec{f}$$

We focus on *small wave number* forcing.

In two dimensions, with vorticity defined as $\omega = \hat{z}\cdot\vec{\nabla}\times\vec{v}$,

$$\frac{\partial \omega}{\partial t} + \vec{v}\cdot\vec{\nabla}\omega = \nu\nabla^2\omega + S_\omega$$

$$\vec{\nabla}\cdot\vec{v} = 0 \qquad \text{(incompressibility)}$$

Passive scalar advection:

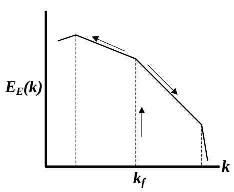
$$\frac{\partial \phi}{\partial t} + \vec{v}\cdot\vec{\nabla}\phi = \kappa\nabla^2\phi + S_\phi$$

Energy Cascade and Intermittency

Power spectrum

$$E_\omega(k) = \int \frac{d\vec{k}'}{(2\pi L)^2}\delta(k-k')|\tilde{\omega}(\vec{k}')|^2$$

In the inertial range, energy injected at k_f cascade to both smaller ($k>k_f$) and larger ($k<k_f$) scales and $E_E(k)\sim k^{-(3+\xi)}$:



Structure functions

$$S_{2q}(r) = \left\langle |\omega(\vec{x}+\vec{r})-\omega(\vec{x})|^{2q}\right\rangle_{\vec{x}} \sim r^{\zeta_{2q}} \qquad \text{(isotropic)}$$

- dimensional analysis $\Rightarrow \zeta_{2q} = q\zeta_2$ (normal scaling)
- $\omega(\vec{r})$ is intermittent:
  - uneven distribution of $\omega(\vec{r})$ over space, large values of $\omega(\vec{r})$ occupy small area fractions
  - $\omega(\vec{r})$ is not self-similar $\Rightarrow \zeta_{2q}\neq q\zeta_2$ (anomalous scaling)

Fractal dimensions

$$\text{measure}: p_i(\epsilon) = \frac{\int_{V_i}|\vec{\nabla}\omega|^2 d\vec{x}}{\int_V |\vec{\nabla}\omega|^2 d\vec{x}} \qquad V_i = \text{squares of sides } \epsilon$$

$$\sum_i p_i^q \sim \left(\frac{\epsilon}{L}\right)^{(q-1)D_q}$$

Is the measure p fractal? Is $D_q < 2$?

Classical Results

In the absence of drag, for two-dimensional turbulence

- $E_E(k)\sim k^{-3}$ $(k>k_f)$, *i.e.* $\xi = 0$ (with logarithm correction)
- $\zeta_{2q} = q\zeta_2 = 0$, *i.e.* no intermittency

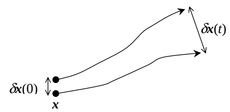
Drag is an important physical effect in most situations involving two-dimensional turbulence.

What is the effect of a linear drag force on ξ and ζ_{2q}?

$$\frac{\partial \omega}{\partial t} + \vec{v}\cdot\vec{\nabla}\omega = \nu\nabla^2\omega - \mu\omega + S_\omega$$

Our approach: relate ξ and ζ_{2q} to the distribution of the *finite-time Lyapunov exponent*

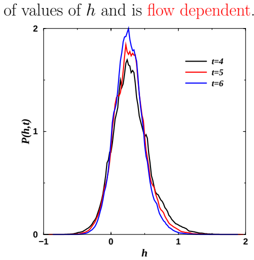
Finite-time Lyapunov Exponent



The finite-time Lyapunov exponent is defined as

$$h(\vec{x},t) = \frac{1}{t}\log\frac{|\delta\vec{x}(t)|}{|\delta\vec{x}(0)|}$$

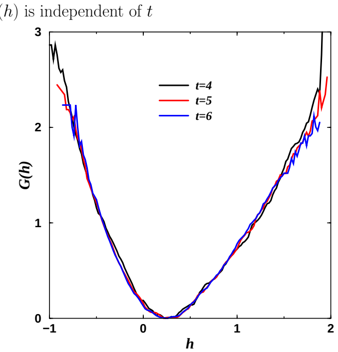
$P(h,t)$: probability density function that characterizes the distribution of values of h and is flow dependent.



For large t, $P(h,t)$ has the asymptotic form:

$$P(h,t)\sim \exp[-tG(h)]$$

- $G(\bar{h}) = G'(\bar{h}) = 0$ where $\bar{h} = \lim_{t\to\infty} h(t)$
- $G(h)$ is independent of t



Theoretical Results

Passive scalar

For a smooth velocity field $\vec{v}(\vec{x},t)$,

$$E_\phi(k) \sim k^{-(1+\xi)}$$

$$\text{where} \quad \xi = \min\left\{\frac{G(h)}{h} + \frac{2\mu}{h}\right\}$$

Vorticity

- $\tilde{\omega}(\vec{k})$ at large k is passively advected by the chaotic velocity field of large structures which determine h
- $\vec{v}$ is smooth if $E_E(k)\sim k^{-(3+\xi)}$ with $\xi > 0$

Apply passive scalar results to vorticity:

$$E_E(k) = \frac{E_\omega(k)}{k^2}\sim k^{-(3+\xi)}$$

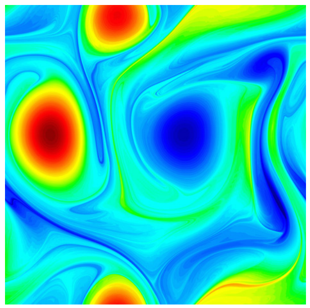
$$\zeta_{2q} = \min\left\{\frac{G(h)}{h} + \frac{2q\mu}{h}\right\}$$

Relation between D_q and ζ_{2q},

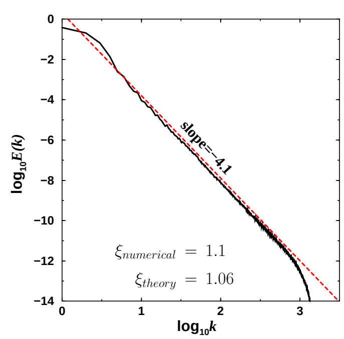
$$D_q = 2 + \frac{\zeta_{2q} - q\zeta_2}{q-1}$$

Numerical Simulations

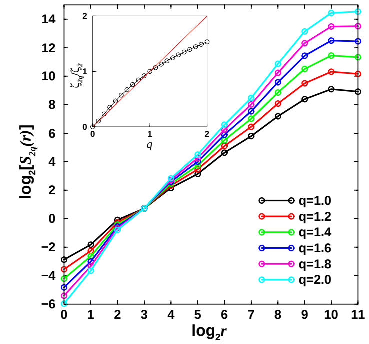
Grid size of simulation is 4096×4096, physical size is $2\pi\times 2\pi$. Hyperviscousity ∇^8 with $\nu = 7.5\times 10^{-15}$ is used. $\mu = 0.2$ and $S_\omega = \sin(2x)$. A snapshot of $\omega(\vec{x},t)$ is shown below:



Energy wavenumber Spectrum

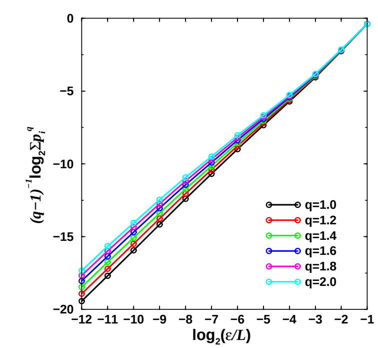


Structure Functions



q	1.0	1.2	1.4	1.6	1.8	2.0
ζ_{2q} (numerical)	1.24	1.40	1.54	1.67	1.79	1.90
ζ_{2q} (theory)	1.06	1.20	1.33	1.46	1.57	1.68

Fractal Dimensions



q	1.0	1.2	1.4	1.6	1.8	2.0
D_q (numerical)	1.69	1.62	1.56	1.51	1.47	1.43
D_q (theory)	–	1.58	1.53	1.49	1.45	1.43

D_q(theory) is calculated using values of ζ_{2q} obtained from the numerical simulations.

Conclusion

In two-dimensional turbulence with linear drag,

- $E_E(k)\sim k^{-(3+\xi)}$ with $\xi > 0$
- vorticity field becomes intermittent, $\zeta_{2q}\neq q\zeta_2$
- the measure of vorticity gradient becomes fractal, $D_q < 2$
- relate ξ and ζ_{2q} to the finite-time Lyapunov exponent and the drag coefficient μ
- relate D_q to ζ_{2q}

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