Parametrization of moisture condensation using stochastic Lagrangian models

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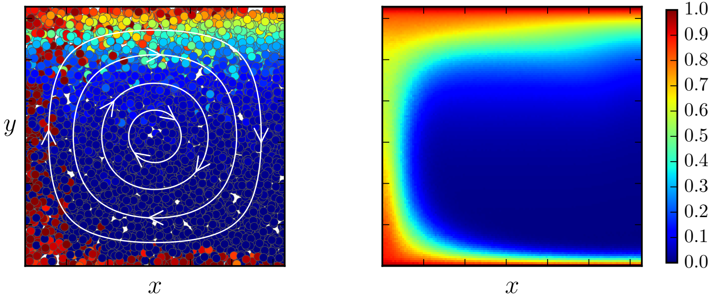


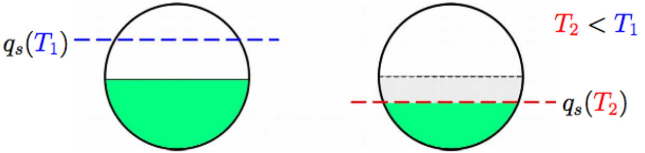
Feasibility project funded by the EPSRC network Research on Changes of Variability and Environmental Risk (ReCoVER). *Co-I: Geoff Vallis (Exeter), Jacques Vanneste (Edinburgh)*

Condensation of water vapour

- specific humidity of an air parcel:

$$Q = \frac{\text{mass of water vapour}}{\text{total air mass}}$$

- saturation specific humidity, $q_s(T)$
  - $Q = \begin{cases} q_s & \text{if} \quad Q > q_s \quad \text{(excessive moisture condensed)} \\ Q & \text{otherwise} \end{cases}$
  - $q_s(T)$ decreases with temperature T, hence generally decreases with height



Water vapour transport in the atmosphere

- specific humidity described by a continuous field $q(x, y, t)$
- evolution of $q(x, y, t)$ governed by a partial differential equation (advection–diffusion–condensation):

$$\frac{\partial q}{\partial t} + \vec{u} \cdot \nabla q = \kappa \nabla^2 q + S - C$$

$$C = \frac{1}{\tau_c}(q - q_s)\, \mathcal{H}(q - q_s)$$

κ: eddy diffusivity representing small-scale turbulence

Water vapour transport in the atmosphere

- specific humidity described by a continuous field $q(x, y, t)$
- evolution of $q(x, y, t)$ governed by a partial differential equation (advection–diffusion–condensation, fast condensation limit):

$$\frac{\partial q_*}{\partial t} + \vec{u} \cdot \nabla q_* = \kappa \nabla^2 q_* + S$$

$$C : q \to \min[q_*, q_s]$$

κ: eddy diffusivity representing small-scale turbulence

a set of equations for cloudy planetary boundary layer (Bougeault, JAS, 1981)

$$\frac{\partial u_i}{\partial t} = - \frac{\partial}{\partial x_\alpha}(u_\alpha u_i) - \frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \beta \delta_{3i} \theta_v + \nu \nabla^2 u_i + \epsilon_{ijk} \Omega_j u_k$$

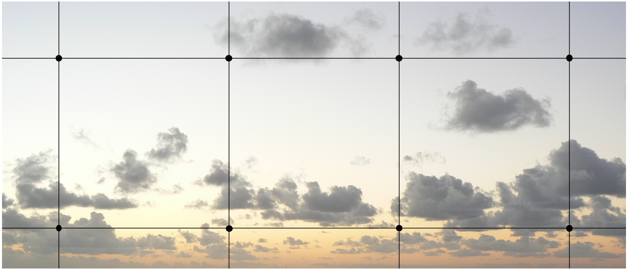
$$\frac{\partial \theta_l}{\partial t} = - \frac{\partial}{\partial x_\alpha}(u_\alpha \theta_l) + \nu_\theta \nabla^2 \theta$$

$$\frac{\partial q_w}{\partial t} = - \frac{\partial}{\partial x_\alpha}(u_\alpha q_w) + \nu_q \nabla^2 q$$

$$\theta = \theta_l + \left[\frac{p_0}{p_r(z)}\right]^k \frac{L}{c_p} q_l$$

$$q_l = \{q_w - q_s[\theta, p + p_r(z)]\} \times H\{q_w - q_s[\theta, p + p_r(z)]\}$$

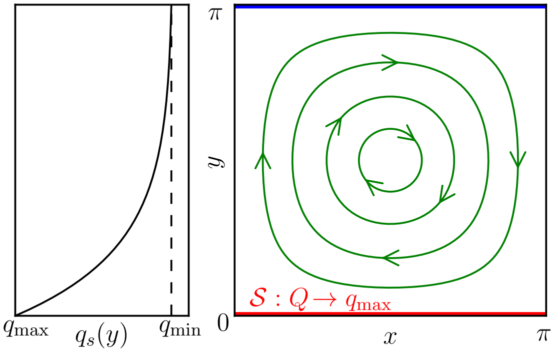
Subgrid-scale fluctuations



- governing PDEs describe the atmospheric state at all scales
- **weather/climate models**: numerical solutions of a discrete version of these equations on a coarse grid *(horizontal resolution* $\sim$ *kilometers)*
- humidity inside a grid box is represented by a single value of q
- missing subgrid-scale moisture variability has significant effects on the resolved scales—system tends to saturate!

Advection–condensation model

- bounded domain: $[0, \pi] \times [0, \pi]$, reflective boundaries
- large-scale cellular flow: $\psi = \sin x \sin y; \quad (u, v) = (-\psi_y, \psi_x)$
- small-scale turbulence
- $q_s(y) = q_{\max} \exp(-\alpha y)$: $q_s(0) = q_{\max}$ and $q_s(\pi) = q_{\min}$
- resetting source: $Q = q_{\max}$ if parcel hits $y = 0$

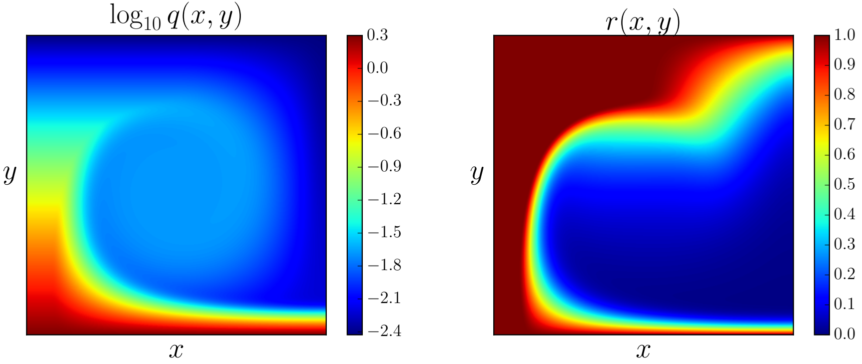


Eulerian field formulation: PDE

$$\frac{\partial q_*}{\partial t} + \vec{u} \cdot \nabla q_* = \kappa \nabla^2 q_*$$

$$C : q(x,y,t) \to \min[q_*(x,y,t), q_s(y)]$$

- boundary source: $q(x, y=0, t) = q_{\max}$
- relative humidity: $r(x,y,t) = \dfrac{q(x,y,t)}{q_s(y)}$

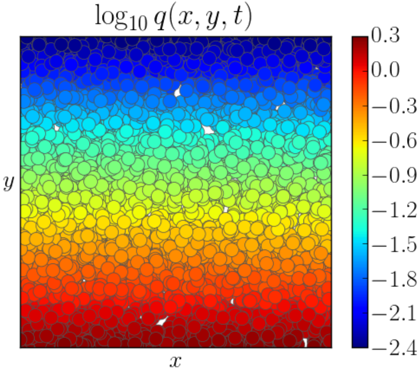


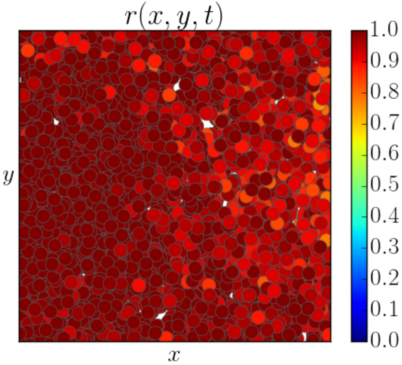
Stochastic Lagrangian model: SDE

$$\mathrm{d}X(t) = u(X,Y)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_1(t)$$
$$\mathrm{d}Y(t) = v(X,Y)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_2(t)$$
$$\mathrm{d}Q(t) = [S(Y) - C(Q,Y)]\mathrm{d}t$$

$$\psi = -\sin x \sin y$$
$$u = -\psi_y$$
$$v = \psi_x$$
$$\kappa = 10^{-2}$$

fast condensation $C : Q \to \min[\,Q\,, q_s(Y)\,]$

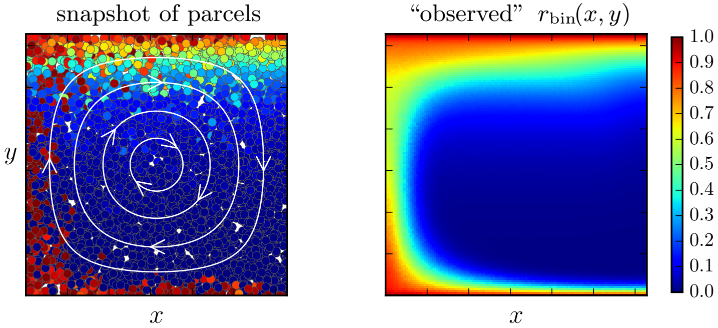




Tsang & Vanneste, Proc. R. Soc. A **473**, 20170196 (2017)

Eulerian versus Lagrangian formulation

- to facilitate comparison, divide the domain into small bins and average over parcels in each bin to produce a field $r_{\mathrm{bin}}(x, y)$: making a coarse-resolution "observation"



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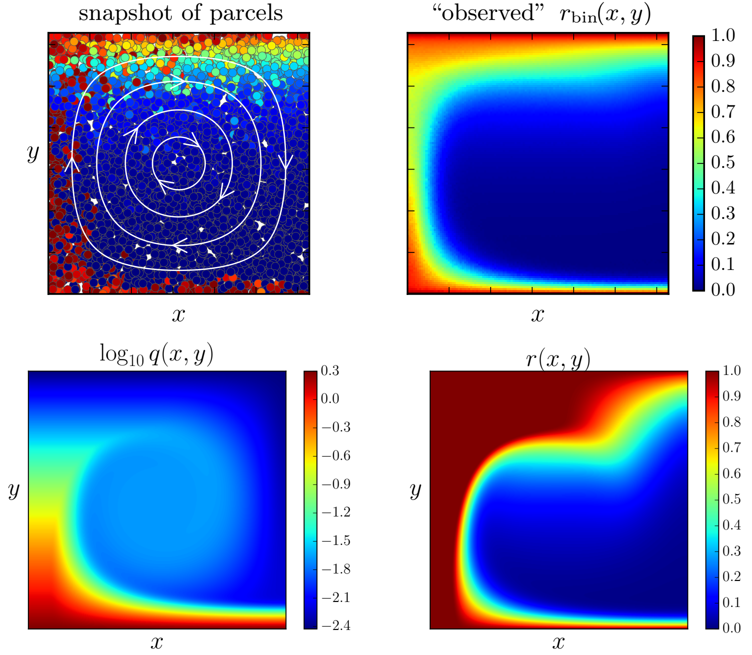
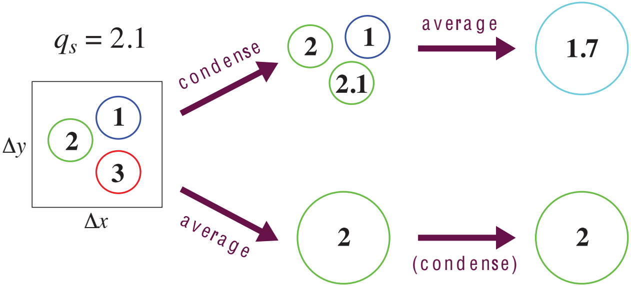


Why are PDE models so wet?

The coarse-graining process and the condensation process do not commute and local fluctuation is lost:



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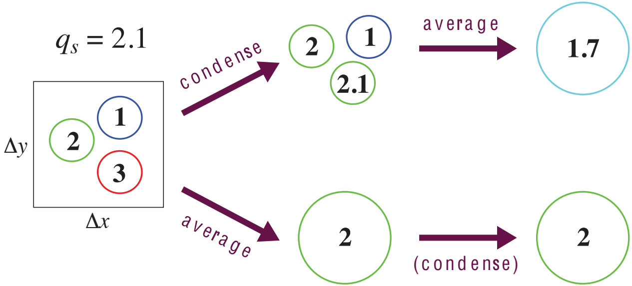

- Lagrangian formulation is generally computational expensive
- However, it is possible for the Eulerian model to mimic the Lagrangian model by parametrizing subgrid-scale condensation

An effective subgrid condensation

- In the stochastic Lagrangian model, (Q, X, Y) are governed by SDEs $\Rightarrow$ the joint PDF $P(q, x, y; t)$ satisfies the Fokker–Planck equation

$$\frac{\partial P}{\partial t} + \vec{u} \cdot \nabla P - \frac{\partial}{\partial q}(CP) = \kappa \nabla^2 P \qquad C = \tfrac{1}{\tau_c}(q - q_s)\,\mathcal{H}(q - q_s)$$

- $q_{\text{bin}}(x, y, t) \approx \overline{q}(x, y, t) = \int_{q_{\min}}^{q_{\max}} q' \, \hat{P}(q' | x, y; t) \, \mathrm{d}q'$

$$\frac{\partial \overline{q}}{\partial t} + \vec{u} \cdot \nabla \overline{q} = \kappa \nabla^2 \overline{q} - \int_{q_{\min}}^{q_{\max}} C(q', q_s) \hat{P}(q' | x, y; t) \, \mathrm{d}q'$$

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- In the Eulerian model, the coarse-grained field $q(x, y, t)$ satisfies

$$\frac{\partial q}{\partial t} + \vec{u} \cdot \nabla q = \kappa \nabla^2 q - C(q, q_s)$$

- Effective condensation to put back some local variability:

$$C(q, q_s) \to C_{\text{eff}}(q, q_s) = \int_{q_{\min}}^{q_{\max}} C(q', q_s) \Phi_*(q'|x, y; t) \, \mathrm{d}q'$$

$\Phi_*(q'|x, y; t)$ is an approximation to $\hat{P}(q'|x, y; t)$. ***How?***

Assumed PDF and matching moments

1. method of assumed PDF with small number of parameters: (β, a, σ)

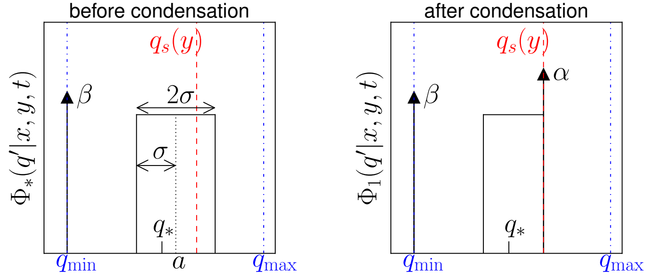
$$\Phi_*(q'|x,y,t) = \beta(x,y,t)\delta(q - q_{\min}) + \tilde{\Phi}_*(q'|x,y,t)$$

2. (a) from the Fokker–Planck equation, derive the equations governing the moments in the stochastic Lagrangian model $(m = 1, 2, \ldots, M)$

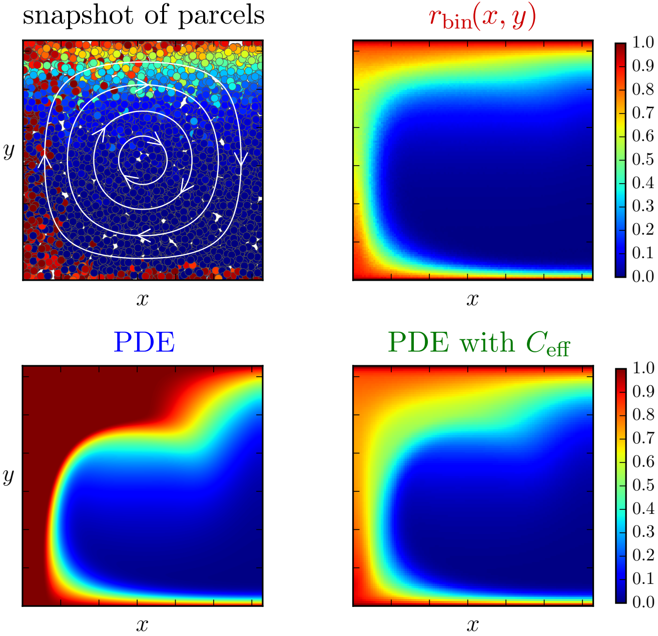
$$\mu_m(x,y,t) \equiv \int q'^m \hat{P}(q'|x,y,t_n)\mathrm{d}q',$$

   (b) to determine (β, a, σ), solve for μ_m and set

$$\int q'^m \Phi_*(q'|x,y,t_n)\mathrm{d}q' = \mu_m$$



Results: relative humidity



Condensation parametrization in atmospheric models

- the idea of representing subgrid-scale moisture variability by a probability distribution has been employed in weather and climate models since *Sommeria & Deardorff (1977)*,
  - such distributions are introduced in an ad hoc manner
  - the parameters in the PDF are determined using various forms of turbulent closures
- Here, using an advection–condensation model, we demonstrate:
  - mathematically how the Lagrangian formulation gives more realistic results than Eulerian coarse-resolution models by accounting for local variability.
  - an Eulerian model for a coarse-grained moisture field $q(\vec{x}, t)$ can mimic its Lagrangian counterpart when an effective condensation is implemented

Based on the present results, we propose a strategy to build a condensation parametrization in atmospheric GCMs using stochastic Lagrangian models.

Condensation parametrization in atmospheric models

1. choose a suitable stochastic Lagrangian model to describe parcel trajectories in subgrid-scale atmospheric transport *(Lagrangian Modeling of the Atmosphere, 2013)*

2. fix the assumed PDF $\Phi_*(q'|x, y; t)$ in the effective condensation:

$$C_{\text{eff}} = \int_{q_{\min}}^{q_{\max}} C(q', q_s)\Phi_*(q'|x, y; t)\,\mathrm{d}q' \qquad C = \frac{1}{\tau_c}(q - q_s)\,\mathcal{H}(q - q_s)$$

   using information from the stochastic Lagrangian model, e.g. by matching moments

3. implement C_{eff} in the Eulerian PDE governing the moisture field in the atmospheric GCM

- a consistent unified approach: the stochastic Lagrangian model (i) provides the theoretical basis for the probabilistic effective condensation as well as (ii) fixes the assumed PDF

Tsang & Vallis, J. Atmo. Sci., accepted (2018)