Oscillatory double-diffusive convection in a rotating spherical shell

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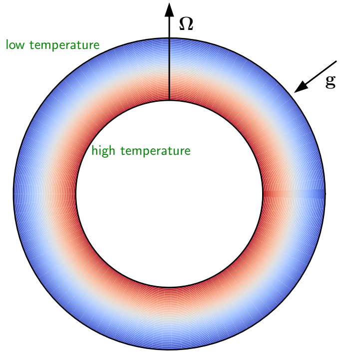
Pure thermal convection in a spherical shell

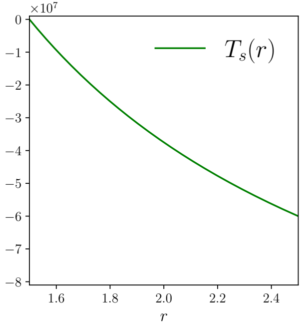


Boundary condition

fixed T gradient:

$$\left.\frac{\partial T}{\partial r}\right|_{r_i}, \left.\frac{\partial T}{\partial r}\right|_{r_o} < 0$$

Equilibrium profiles (at $\boldsymbol{u} = 0$)

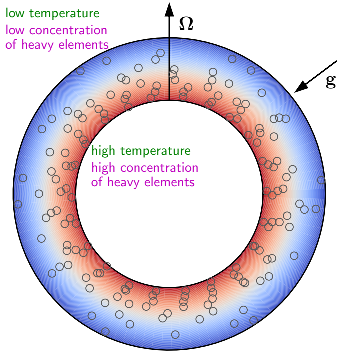


$\frac{\mathrm{d}T_s}{\mathrm{d}r} < 0$ *destabilising*

Consider a Boussinesq fluid in a rotating spherical shell of inner radius r_i and outer radius r_o

- thermal diffusivity: κ_T
- density: $\rho(T) = \rho_m[1 - \alpha_T(T - T_m)]$
- buoyancy frequency: $N^2 = -\frac{g}{\rho_m}\frac{\mathrm{d}}{\mathrm{d}r}\rho(T_s) \implies N^2 = g\alpha_T\frac{\mathrm{d}T_s}{\mathrm{d}r} < 0 \quad$ (top-heavy)

Oscillatory double-diffusive convection (ODDC) in a spherical shell



Boundary condition

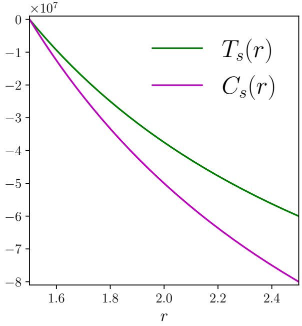
fixed T gradient:

$$\left.\frac{\partial T}{\partial r}\right|_{r_i}, \left.\frac{\partial T}{\partial r}\right|_{r_o} < 0$$

fixed C gradient:

$$\left.\frac{\partial C}{\partial r}\right|_{r_i}, \left.\frac{\partial C}{\partial r}\right|_{r_o} < 0$$

Equilibrium profiles (at $\boldsymbol{u} = 0$)



$\frac{\mathrm{d}T_s}{\mathrm{d}r} < 0$ *destabilising*

$\frac{\mathrm{d}C_s}{\mathrm{d}r} < 0$ *stabilising*

Consider a Boussinesq fluid in a rotating spherical shell of inner radius r_i and outer radius r_o

- composition diffusivity: $\kappa_C \ll \kappa_T$
- density: $\rho(T, C) = \rho_m[1 - \alpha_T(T - T_m) + \alpha_C(C - C_m)]$
- buoyancy frequency: $N^2 = -\frac{g}{\rho_m}\frac{\mathrm{d}}{\mathrm{d}r}[\rho(T_s, C_s)] \implies N^2 = g\alpha_T\frac{\mathrm{d}T_s}{\mathrm{d}r} - g\alpha_C\frac{\mathrm{d}C_s}{\mathrm{d}r}$

Governing equations of ODDC

Non-dimensional equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{2}{Ek}\hat{\boldsymbol{z}}\times\boldsymbol{u} = -\nabla\Pi + (\Theta - \xi)r\,\hat{\boldsymbol{r}} + \nabla^2\boldsymbol{u},$$

$$\nabla\cdot\boldsymbol{u} = 0,$$

$$\frac{\partial \Theta}{\partial t} + \boldsymbol{u}\cdot\nabla\Theta = \frac{Ra_T}{Pr}\left(\frac{\gamma}{1-\gamma}\right)^2\frac{u_r}{r^2} + \frac{1}{Pr}\nabla^2\Theta, \quad \Theta(\boldsymbol{x},t) = T(\boldsymbol{x},t) - T_s(r)$$

$$\frac{\partial \xi}{\partial t} + \boldsymbol{u}\cdot\nabla\xi = \frac{|Ra_C|}{Sc}\left(\frac{\gamma}{1-\gamma}\right)^2\frac{u_r}{r^2} + \frac{1}{Sc}\nabla^2\xi, \quad \xi(\boldsymbol{x},t) = C(\boldsymbol{x},t) - C_s(r)$$

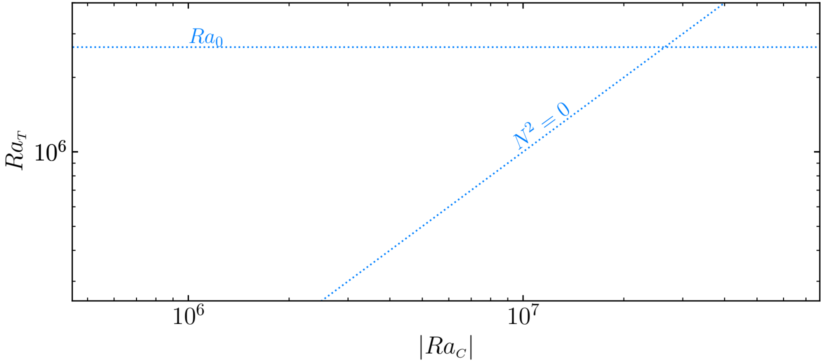
Dimensionless numbers:

$$\gamma = \frac{r_i}{r_o} = 0.6, \qquad Ek = \frac{\nu}{\Omega D^2} = 10^{-5}, \qquad Pr = \frac{\nu}{\kappa_T} = 0.3, \qquad Sc = \frac{\nu}{\kappa_C} = 3$$

$$Ra_T = \frac{g_o\alpha_T D^5}{r_o\nu\kappa_T}\left|T_s'(r_i)\right| \quad \text{and} \quad Ra_C = -\frac{g_o\alpha_C D^5}{r_o\nu\kappa_C}\left|C_s'(r_i)\right|$$

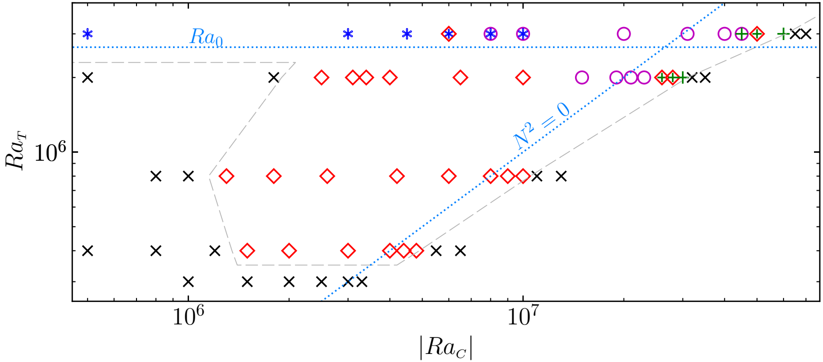
Numerical simulations using XSHELLS by Nathanaël Schaeffer (Université Grenoble Alpes).

ODDC at low Rayleigh numbers



- $$N^2 = -\frac{Ra_T}{Pr}\frac{r_i^2}{r} + \frac{|Ra_C|}{Sc}\frac{r_i^2}{r}$$
- $N^2 < 0$: top-heavy, $N^2 > 0$: bottom-heavy
- Ra_0 = critical Rayleigh number for pure thermal convection
- pure thermal convection: unstable when $N^2 < 0$ and $Ra_T > Ra_0$

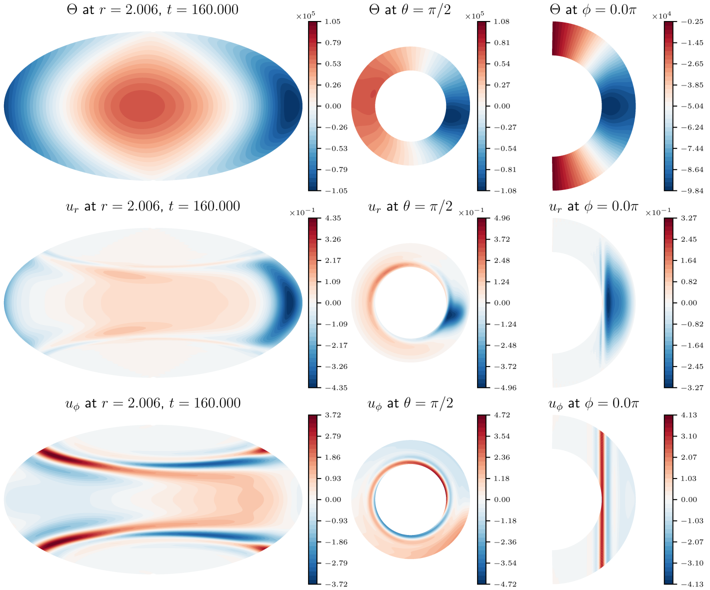
Phase diagram: $Ek = 10^{-5}$



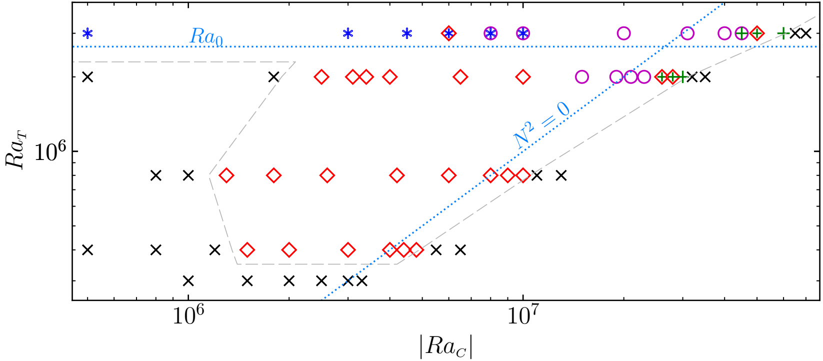
- Expected features:
  - at small $|Ra_C|$, similar to pure thermal convection
  - at large $|Ra_C|$, compositional effects stabilise the system
- Counter-intuitive features (intermediate $|Ra_C|$):
  - the system can become unstable even when $N^2 > 0$ (bottom-heavy)
  - sustained motion is possible at some $Ra_T < Ra_0$

$Ra_T = 8 \times 10^5\,(< Ra_0)\,,\ Ra_C = 1.8 \times 10^6$

- large-scale structures
- retrograde
- only exists at small Ek (rapid rotation)



Phase diagram: $Ek = 10^{-5}$



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  - sustained motion is possible at some $Ra_T < Ra_0$

$Ra_T = 3 \times 10^6 > Ra_0$

$*$ $Ra_C = 5 \times 10^5$

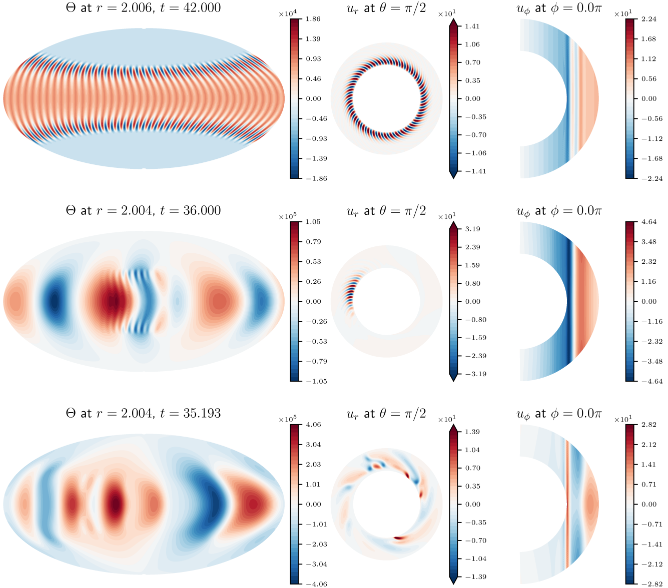
prograde

$\circledast$ $Ra_C = 8 \times 10^6$

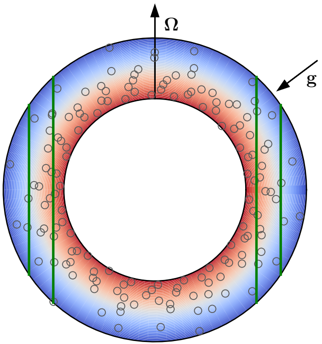
prograde/retrograde

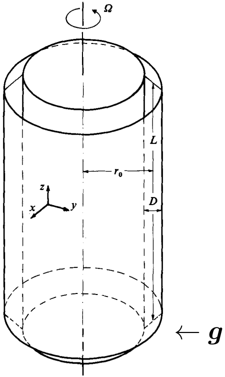
$\circ$ $Ra_C = 2 \times 10^7$

retrograde



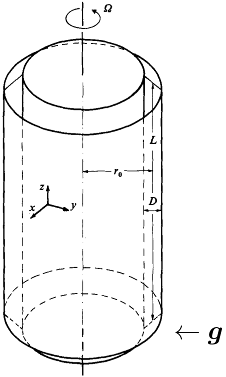
Thin cylindrical annulus model (Busse 1986)

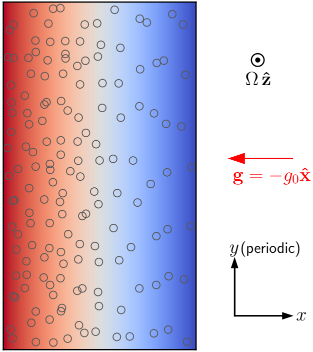




- cylinderical annulus with top and bottom tilted at a constant angle χ
- captures two pieces of physics of the spherical shell
  1. rotation
  2. curvature of the spherical geometry

Thin cylindrical annulus model (Busse 1986)





- rapid rotation: geostrophic balance at leading order (columnar structures)
- integration (average) over height ⇒ two-dimensional system
- thin annulus ⇒ Cartesian coordinate (x, y)

ODDC on a two-dimensional β-plane

$$(u, v) = \big(-\partial_y \psi, \partial_x \psi \big)$$

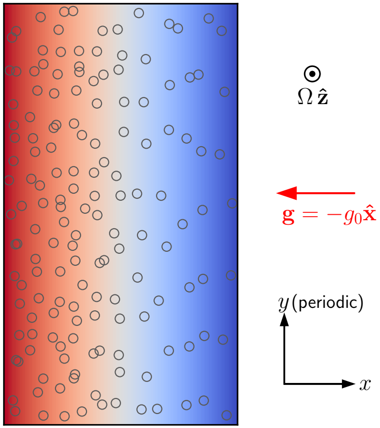
$$J(A, B) = \partial_x A \, \partial_y B - \partial_y A \, \partial_x B$$

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) - \beta \frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y} (\Theta - \xi) + \nabla^4 \psi$$

$$\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) = -\frac{Ra_T}{Pr} \frac{\partial \psi}{\partial y} + \frac{1}{Pr} \nabla^2 \Theta$$

$$\frac{\partial \xi}{\partial t} + J(\psi, \xi) = -\frac{|Ra_C|}{Sc} \frac{\partial \psi}{\partial y} + \frac{1}{Sc} \nabla^2 \xi$$

$$\beta = \frac{4D \, \tan \chi}{L \cdot Ek}$$



Linear stability analysis

Linearised equations:

$$\frac{\partial}{\partial t}\nabla^2\psi - \beta\frac{\partial\psi}{\partial y} = -\frac{\partial\Theta}{\partial y} + \frac{\partial\xi}{\partial y} + \nabla^4\psi$$

$$\frac{\partial\Theta}{\partial t} = -\frac{Ra_T}{Pr}\frac{\partial\psi}{\partial y} + \frac{1}{Pr}\nabla^2\Theta$$

$$\frac{\partial\xi}{\partial t} = -\frac{|Ra_C|}{Sc}\frac{\partial\psi}{\partial y} + \frac{1}{Sc}\nabla^2\xi$$

Eigenmodes:

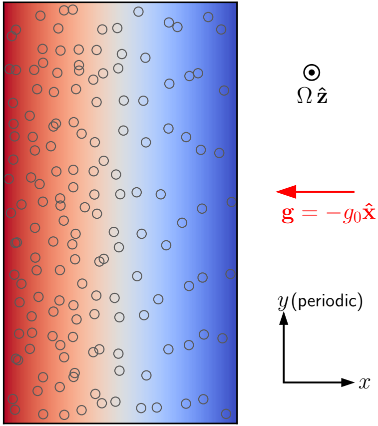
$$\psi(x,y,t) = \hat{\psi}\sin(kx)e^{ily}e^{\lambda t}$$

$$\Theta(x,y,t) = \hat{\Theta}\cos(kx)e^{ily}e^{\lambda t}$$

$$\xi(x,y,t) = \hat{\xi}\cos(kx)e^{ily}e^{\lambda t}$$

Complex growth rate:

$$\lambda = \sigma + i\omega \quad (\sigma,\omega \in \mathbb{R})$$



Maximum growth rate

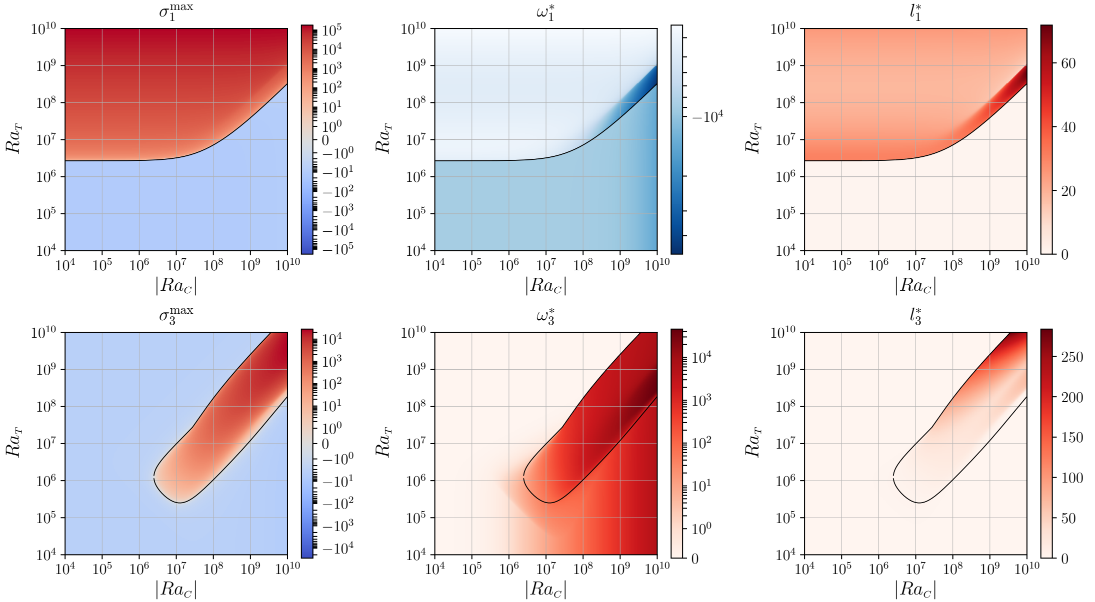
Solvability condition:

$$\lambda^3 + \left(\frac{Pr+1+\tau}{Pr}k_{\rm h}^2 + i\frac{\beta l}{k_{\rm h}^2}\right)\lambda^2 + \left[\frac{Pr(1+\tau)+\tau}{Pr^2}k_{\rm h}^4 + \frac{\tau|Ra_C| - Ra_T}{Pr}\frac{l^2}{k_{\rm h}^2} + i\frac{\beta(1+\tau)}{Pr}l\right]\lambda + \frac{\tau}{Pr^2}\left[k_{\rm h}^6 + (|Ra_C| - Ra_T)l^2 + i\beta k_{\rm h}^2 l\right] = 0.$$

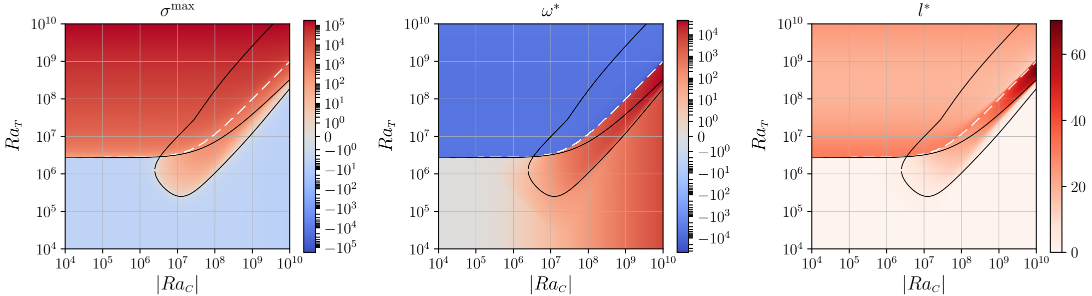
$$k = m\pi, \quad k_{\rm h}^2 \equiv k^2 + l^2, \quad m, n \in \mathbb{Z}\,; \quad \tau = Pr/Sc = 0.1, \quad \beta = 1.78 \times 10^5$$

- three roots: $\lambda_q = \sigma_q + i\omega_q, \quad q = 1, 2, 3$
- for each $(k, l), Ra_T, Ra_C$: calculate $\lambda_q(k, l; Ra_T, Ra_C)$, $q = 1, 2, 3$
- $\sigma_q^{\max}(Ra_C, Ra_T) = \max_{k,l}\left\{\sigma_q(k, l\,; Ra_C, Ra_T)\right\}$
- $\sigma^{\max}(Ra_C, Ra_T) = \max_q \left\{\sigma_q^{\max}\right\}$
- There is always a decaying solution: $\sigma_2 < 0$

Results: $\sigma_1^{\max}$ and $\sigma_3^{\max}$

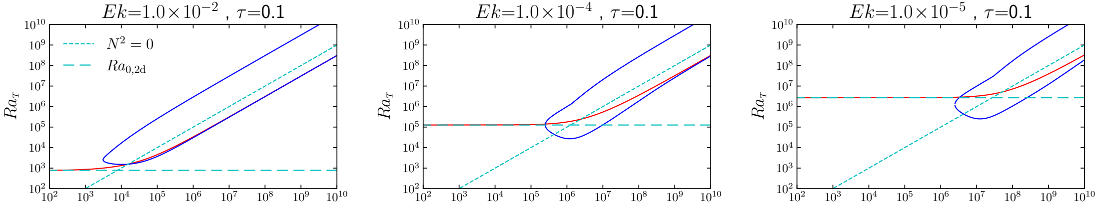


Understanding the nonlinear spherical results using $\sigma^{\max}$



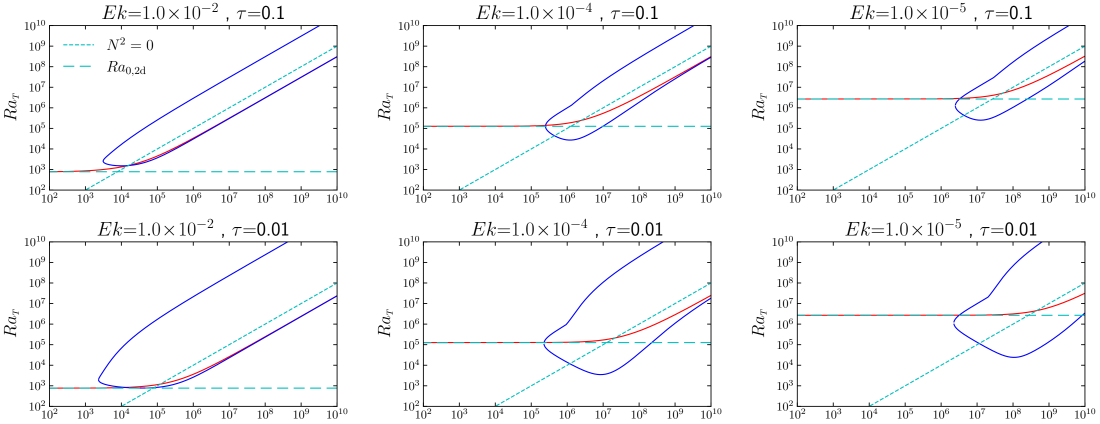
- $\lambda_1 = \sigma_1 + i\omega_1$: short-wavelength, prograde mode
  - unstable when $Ra_T > Ra_{0,2d}$ = critical Ra_T for pure thermal convection
  - $\sim$ pure thermal convection (modified by compositional effects)
- $\lambda_3 = \sigma_3 + i\omega_3$: long-wavelength, retrograde mode
  - can exists at some $Ra_T < Ra_{0,2d}$ (strong rotation needed)
  - a 'genuine' double-diffusive effect
- there is a range of $|Ra_C|$ in which the two modes can coexists with minimal interaction

Effects of the Ekman number Ek and diffusivity ratio τ



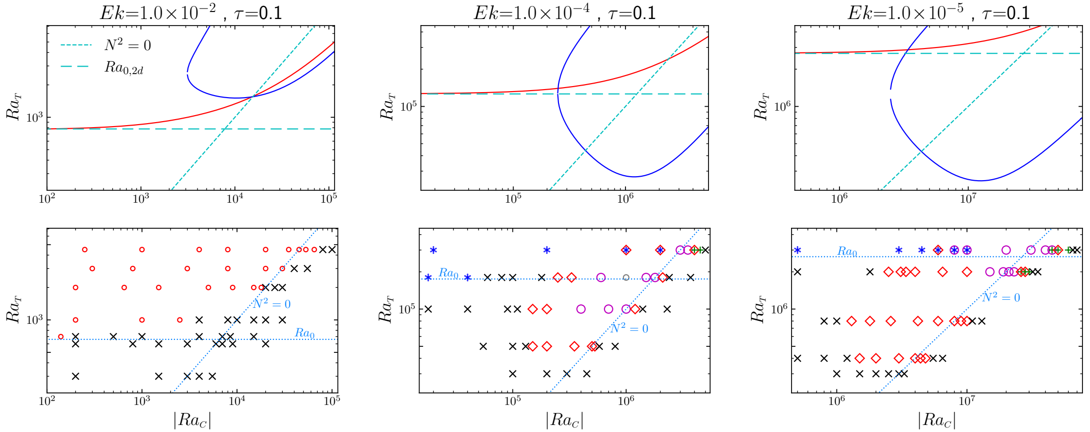
- unstable ‘tongue’ below $Ra_{0,2d}$ disappears at weak rotation (large Ek)

Effects of the Ekman number Ek and diffusivity ratio τ



- unstable ‘tongue’ below $Ra_{0,2d}$ disappears at weak rotation (large Ek)
- width of the ODDC unstable region increases as $\tau = \kappa_C/\kappa_T$ decreases

Role of rotation: linear annulus model and nonlinear spherical simulations



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