Enstrophy-constrained stability analysis of β-plane Kolmogorov flow with drag

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Kolmogorov Flow

$$\zeta_t + u\zeta_x + v\zeta_y + \beta\psi_x = -\mu\zeta + \cos x + \nu\nabla^2\zeta$$

velocity: $(u, v) = (-\psi_y, \psi_x)$ *(2-D periodic domain)*

vorticity: $\zeta(x, y) = v_x - u_y = \nabla^2\psi$

- Kolmogorov flow: sinusoidal forcing *(single scale)*

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- μ = bottom drag
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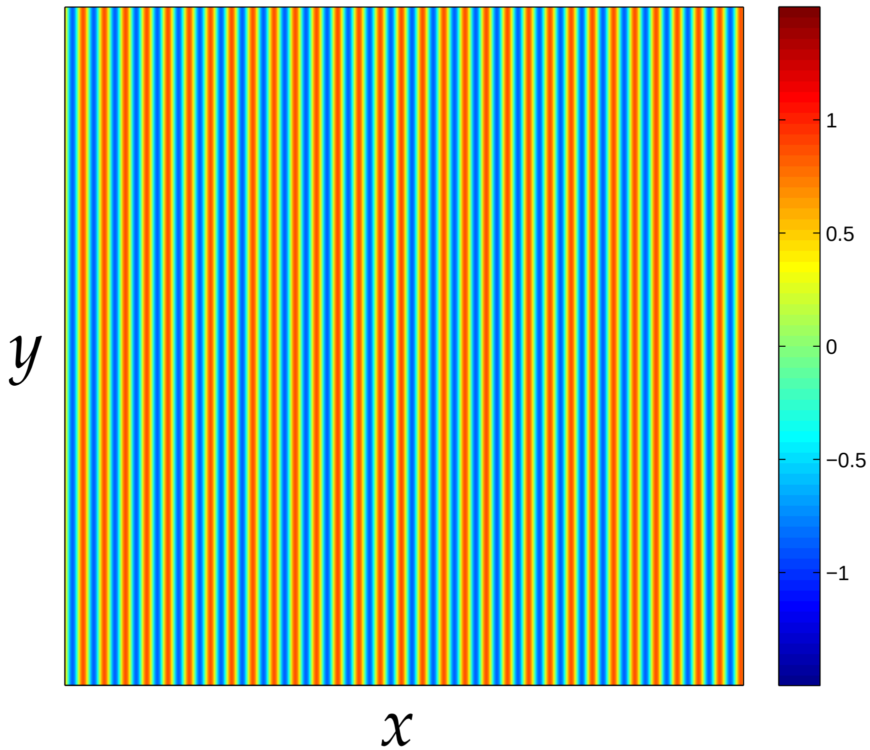
- μ = bottom drag
  - quasi-2D experiments: friction from the container walls or surrounding air
  - geophysical flows: Ekman friction
- β = gradient of Coriolis parameter along y
  - important in differentially rotating systems

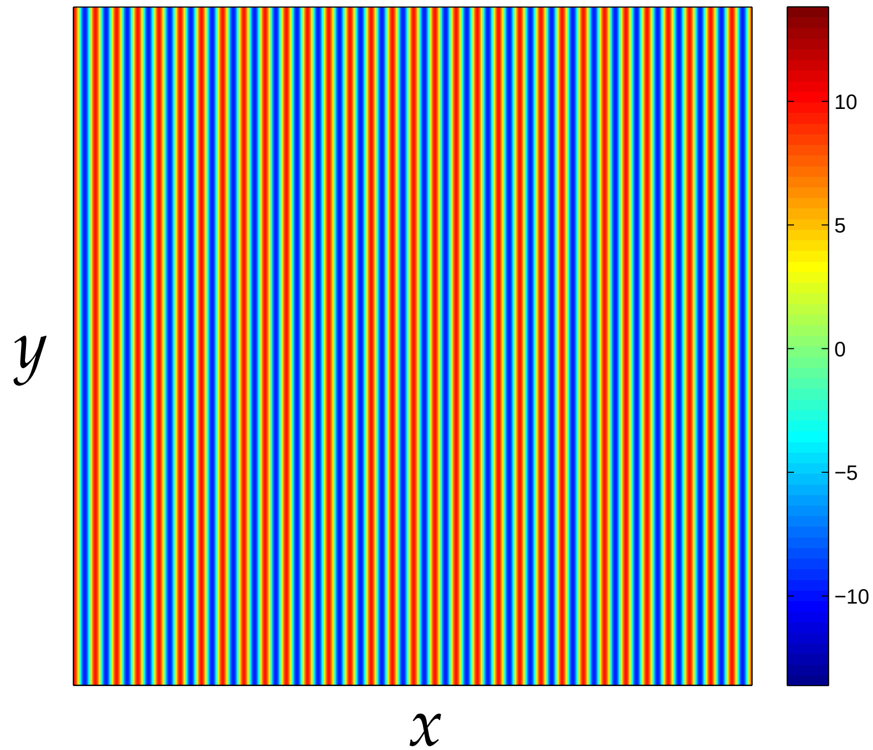
Stability of the Laminar Solution

$$\zeta_{\mathrm{L}}(x) = a\cos(x - x_\beta)$$

$$a = \frac{1}{\sqrt{\beta^2 + \mu^2}} \quad , \quad x_\beta = \tan^{-1}\frac{\beta}{\mu}$$



$\mu = 0.5 \quad \beta = 1.0$



$\mu = 0.1 \quad \beta = 0.0$

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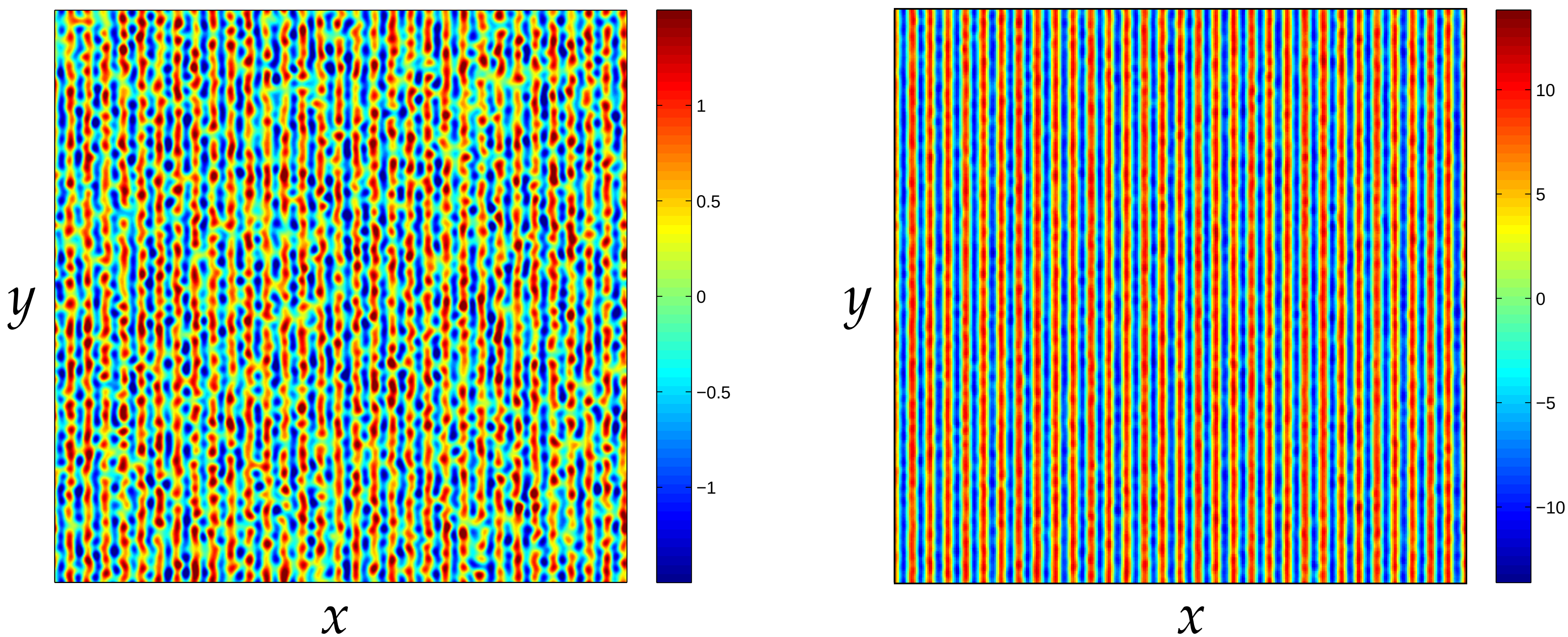

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Stability of the Laminar Solution

$$\zeta_{\mathsf{L}}(x) = \color{red}{a}\color{black} \cos(x - x_\beta)$$

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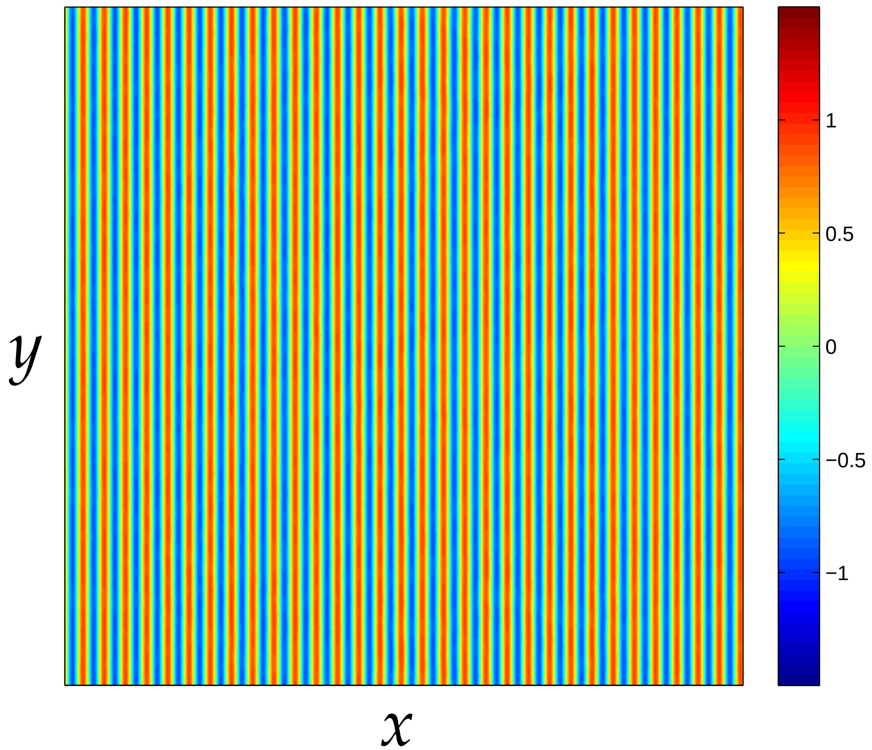

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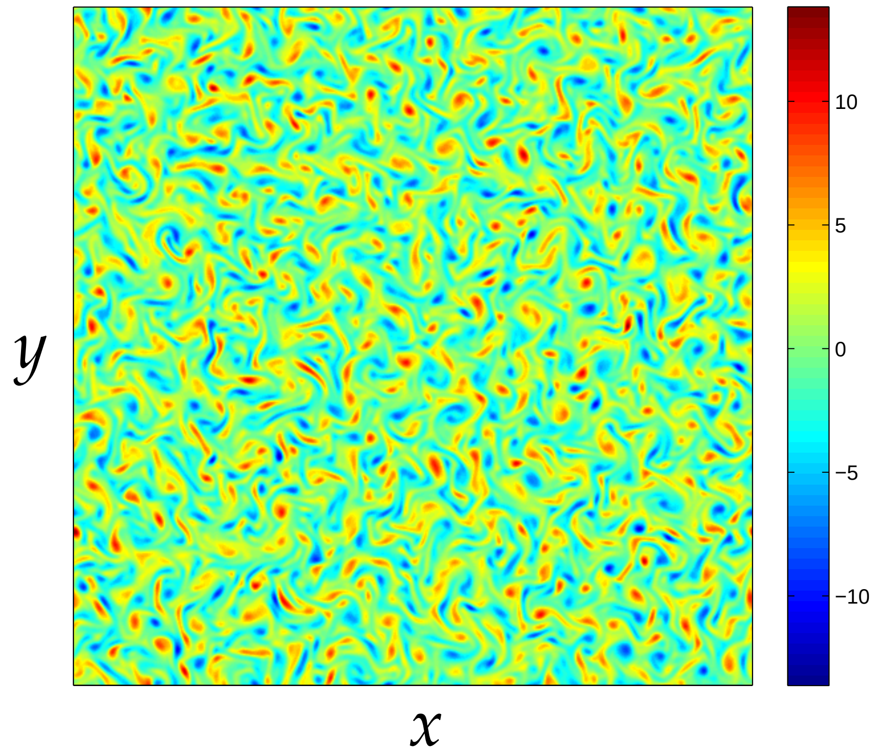
stable

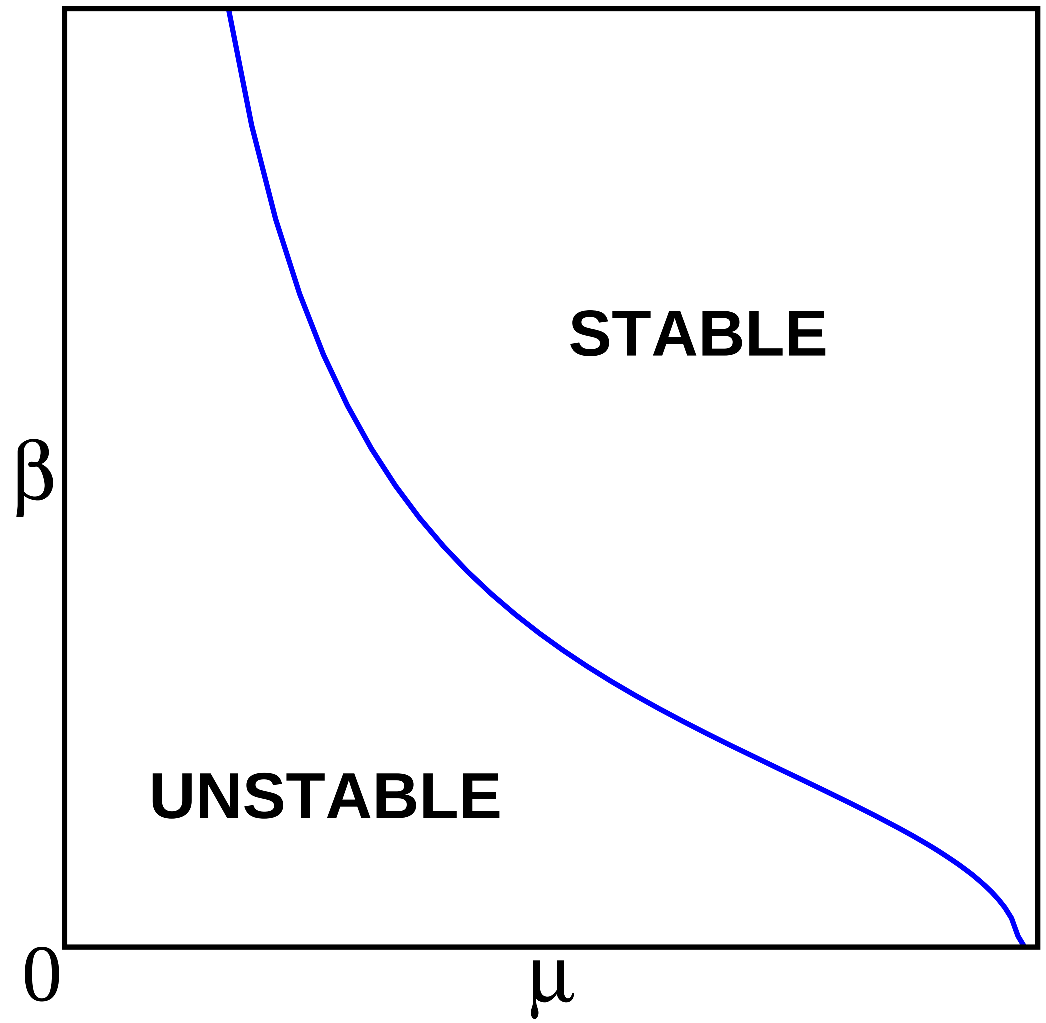


$\mu = 0.1 \quad \beta = 0.0$

unstable

Goal: Neutral Curve

$$\nabla^2\psi_t + J(\psi, \nabla^2\psi) + \beta\psi_x = -\mu\nabla^2\psi + \cos x$$



Stability Analysis

$$\psi(x,y,t) = \psi_{\mathrm{L}}(x) + \varphi(x,y,t)$$

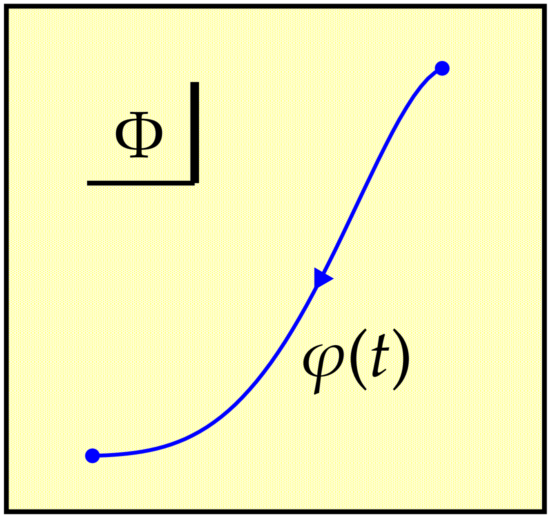
- Linear Instability
  - assume infinitesimal disturbance $\varphi \sim e^{-i\omega t}$
  - $\Im\{\omega\} > 0 \;\Rightarrow\; \psi_{\mathrm{L}}$ is unstable
  - gives sufficient condition for instability
- Global Stability *(Asymptotic Stability)*
  - φ is not assumed to be small
  - disturbance energy

$$E_{\varphi}(t) = \frac{1}{2}\left\langle |\nabla\varphi|^2 \right\rangle \to 0 \quad as \quad t \to \infty$$

  - gives sufficient condition for stability

Energy Method

$$\frac{dE_\varphi}{dt} = 2\left(a\mathcal{R}[\varphi] - \mu\right)E_\varphi$$



where
$$\mathcal{R}[\varphi] \equiv \frac{\left\langle \varphi_x \varphi_y \cos x \right\rangle}{\langle |\nabla\varphi|^2 \rangle}$$

Now define
$$\mathcal{R}_* \equiv \max_{\varphi\in\Phi} \mathcal{R}[\varphi]$$

Φ : *set of all functions satisfying periodic boundary conditions*

Then,
$$E_\varphi(t) < E_\varphi(0)\, e^{2(a\mathcal{R}_* - \mu)t} \to 0 \quad \textit{if} \quad a\mathcal{R}_* - \mu < 0$$

Neutral condition

$$a = \frac{1}{\mathcal{R}_*}\mu \quad \Rightarrow \quad \boxed{\beta = \sqrt{\frac{\mathcal{R}_*^2}{\mu^2} - \mu^2}}$$

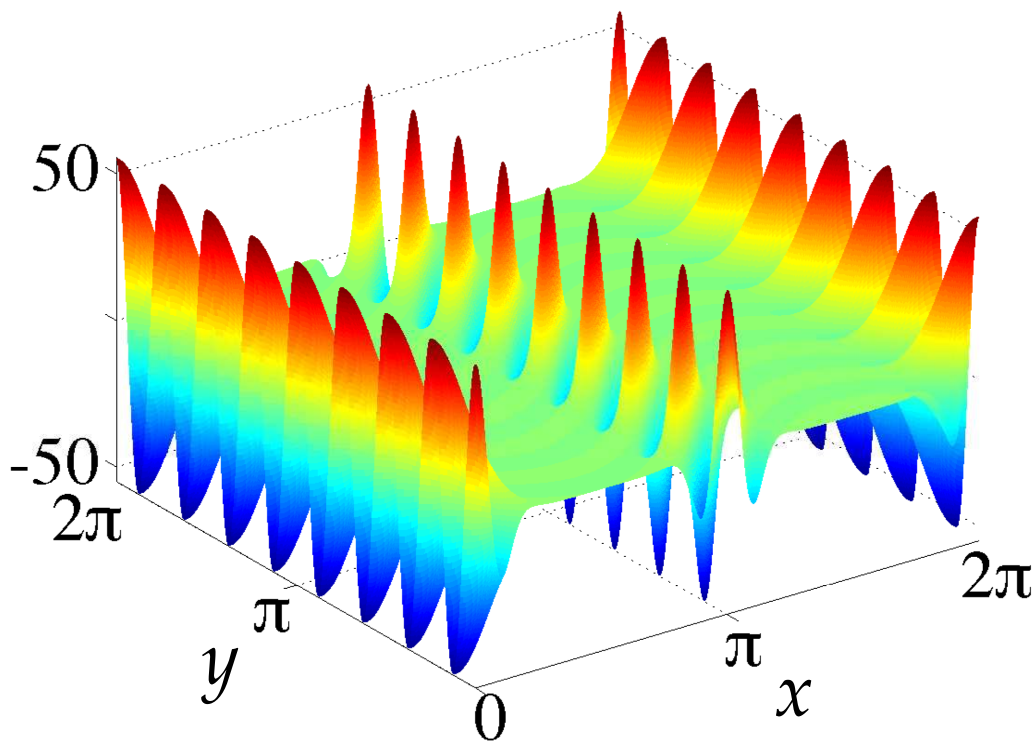
An Optimization Problem

Maximize: $\mathcal{R}[\varphi] \equiv \dfrac{\langle \varphi_x \varphi_y \cos x \rangle}{\langle |\nabla \varphi|^2 \rangle}$ over the set Φ.

Optimal solution

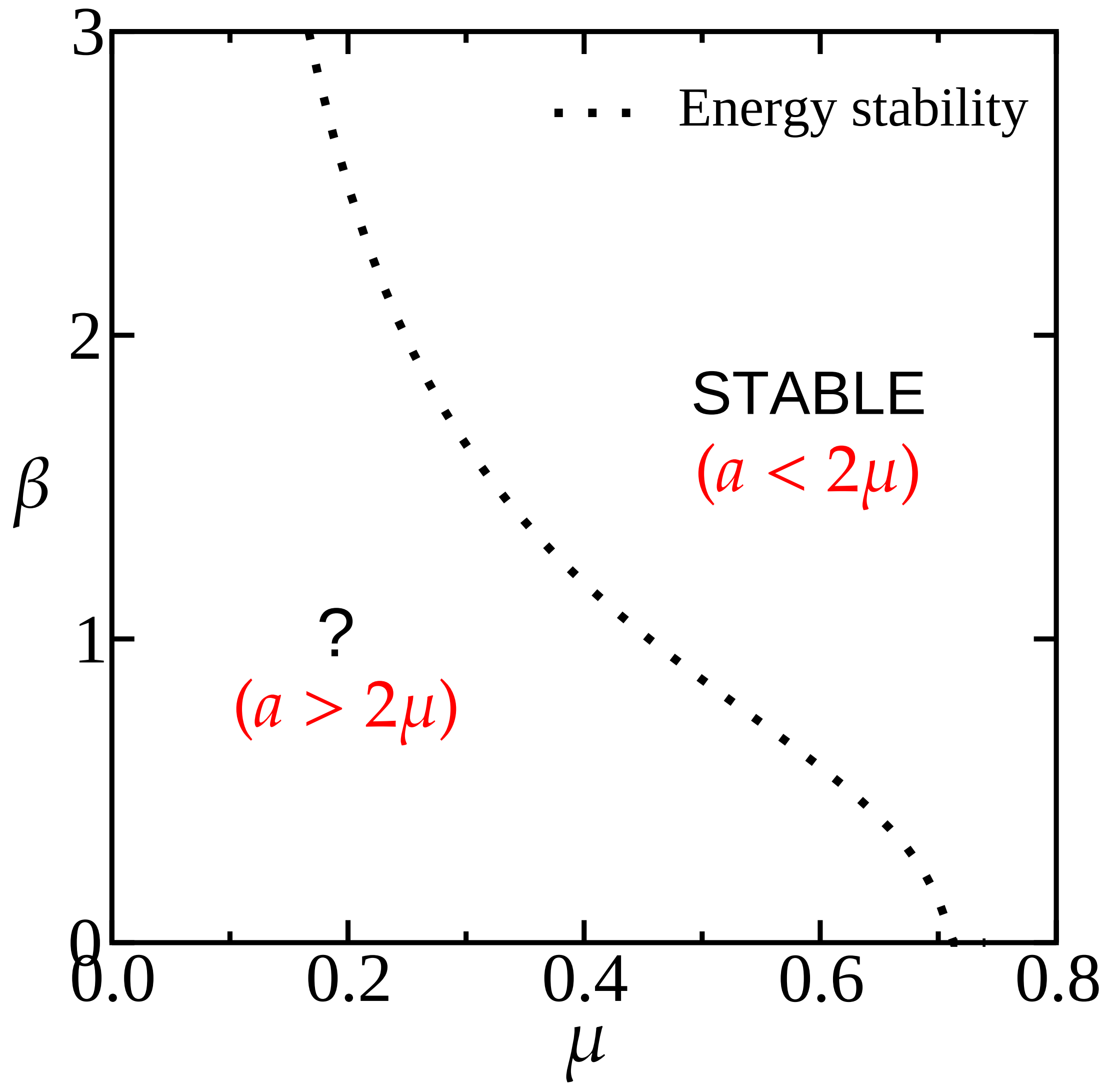
$$\mathcal{R}_* = \mathcal{R}[\varphi_*] = \frac{1}{2}$$

$$\varphi_*(x, y) \approx \lim_{l \to \infty} \cos[l(y + \sin x)] \exp\left(\frac{l}{2} \cos 2x\right)$$



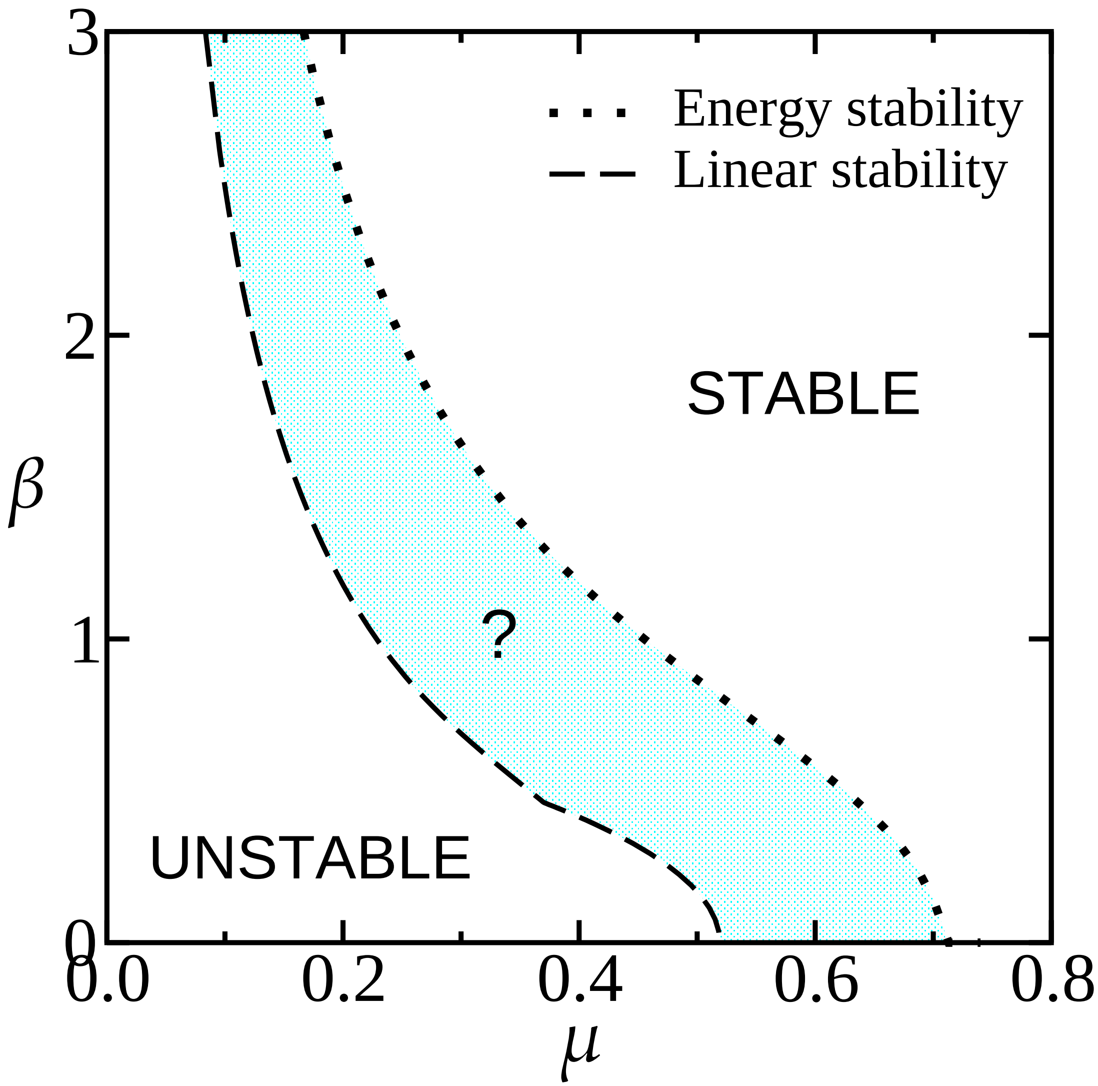
Energy Stability Curve

$$\beta = \sqrt{\frac{1}{4\mu^2} - \mu^2} \qquad (a = 2\mu)$$



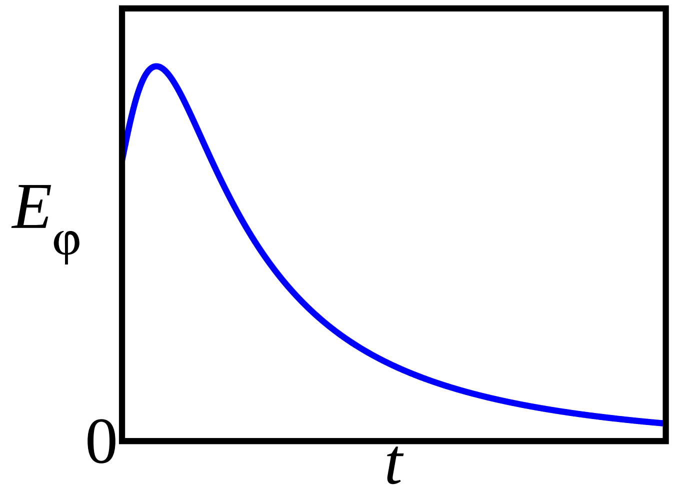
Energy Stability and Linear Stability Curve

$$\beta = \sqrt{\frac{1}{4\mu^2} - \mu^2} \qquad (a = 2\mu)$$



Limitations of the Energy Method

- requires $E_\varphi(t)$ to decrease monotonically for all φ, thus excludes transient growth of $E_\varphi(t)$



- the most efficient energy-releasing disturbance $\varphi_*(x, y)$ is unphysical: $l \to \infty$

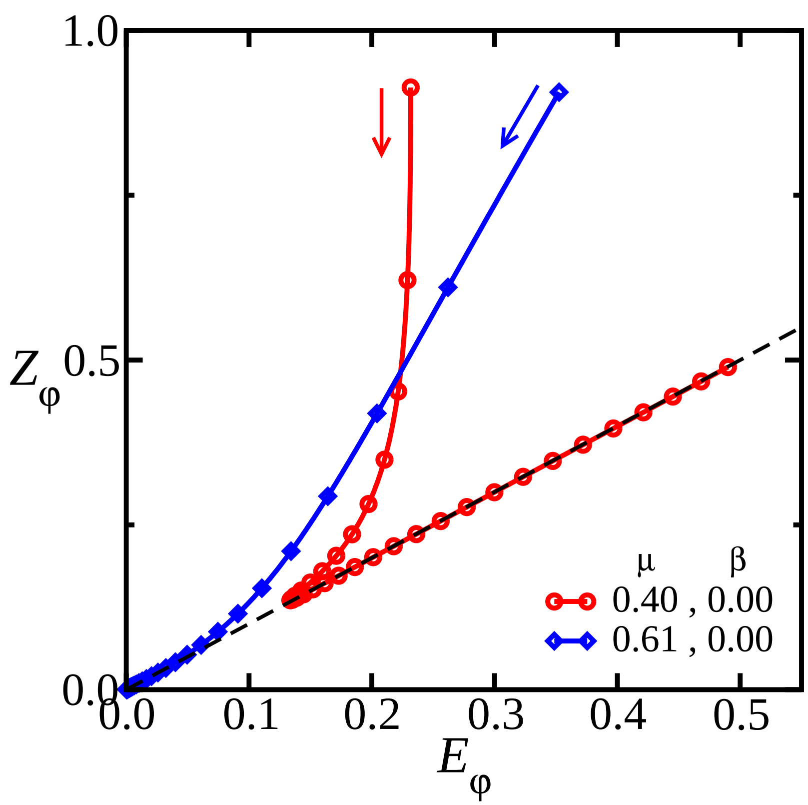
- a gap between the energy stability curve and the neutral curve from linear stability analysis

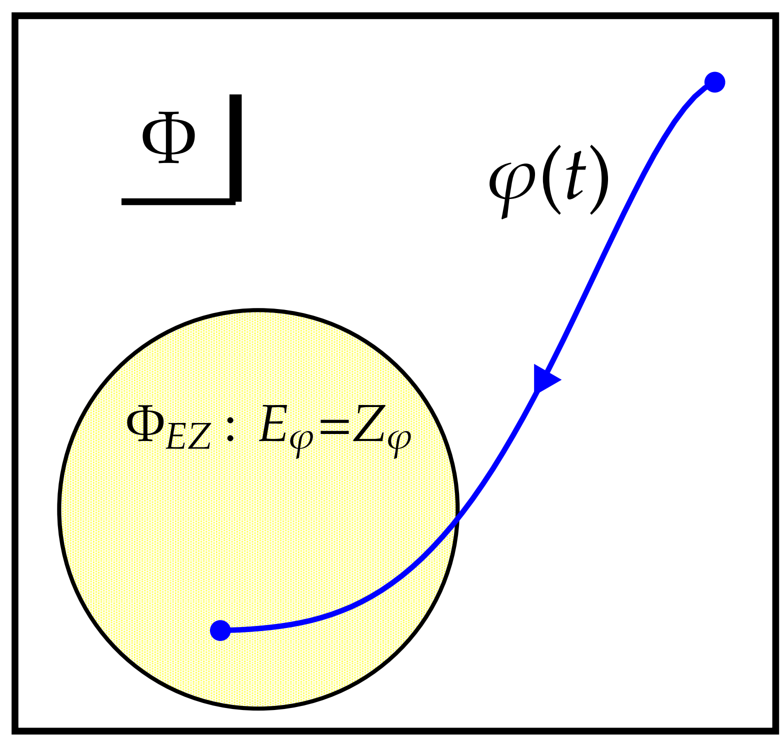
Energy-Enstrophy Balance

Disturbance enstrophy: $Z_\varphi = \frac{1}{2}\left\langle (\nabla^2\varphi)^2 \right\rangle$

$$\frac{d}{dt}(E_\varphi - Z_\varphi) = -2\mu(E_\varphi - Z_\varphi)$$

$$\boxed{E_\varphi = Z_\varphi \quad as \quad t \to \infty}$$





Optimization with Constraints

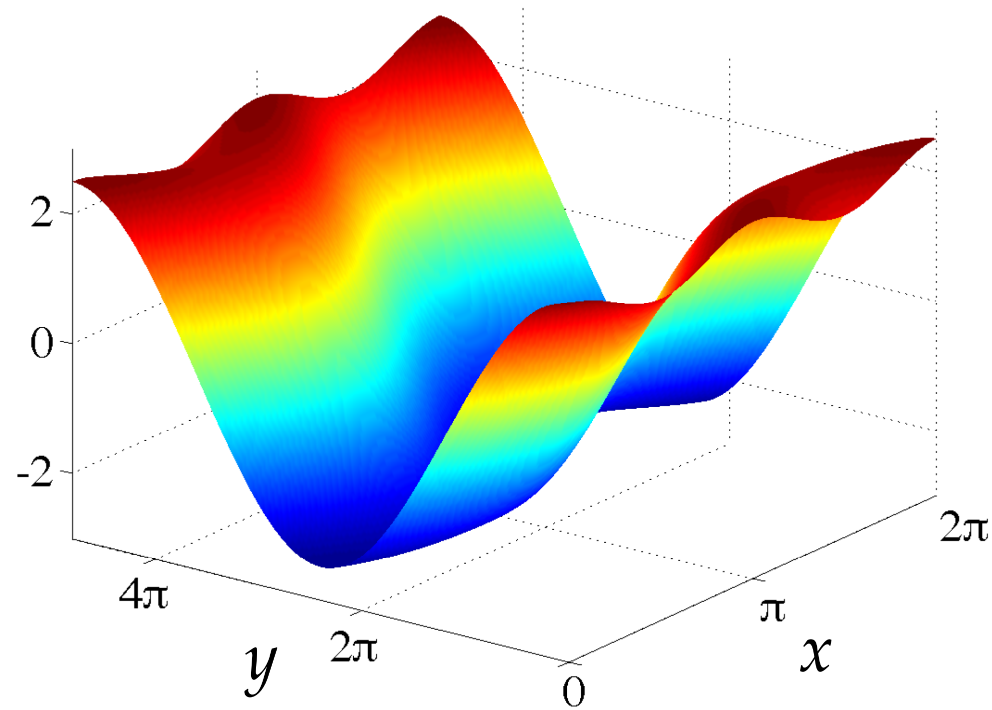
Maximize: $$\mathcal{R}[\varphi] \equiv \frac{\left\langle \varphi_x \, \varphi_y \cos x \right\rangle}{\langle |\nabla\varphi|^2 \rangle}$$

with constraint $$\langle |\nabla\varphi|^2 \rangle = \langle (\nabla^2\varphi)^2 \rangle$$

Optimal solution

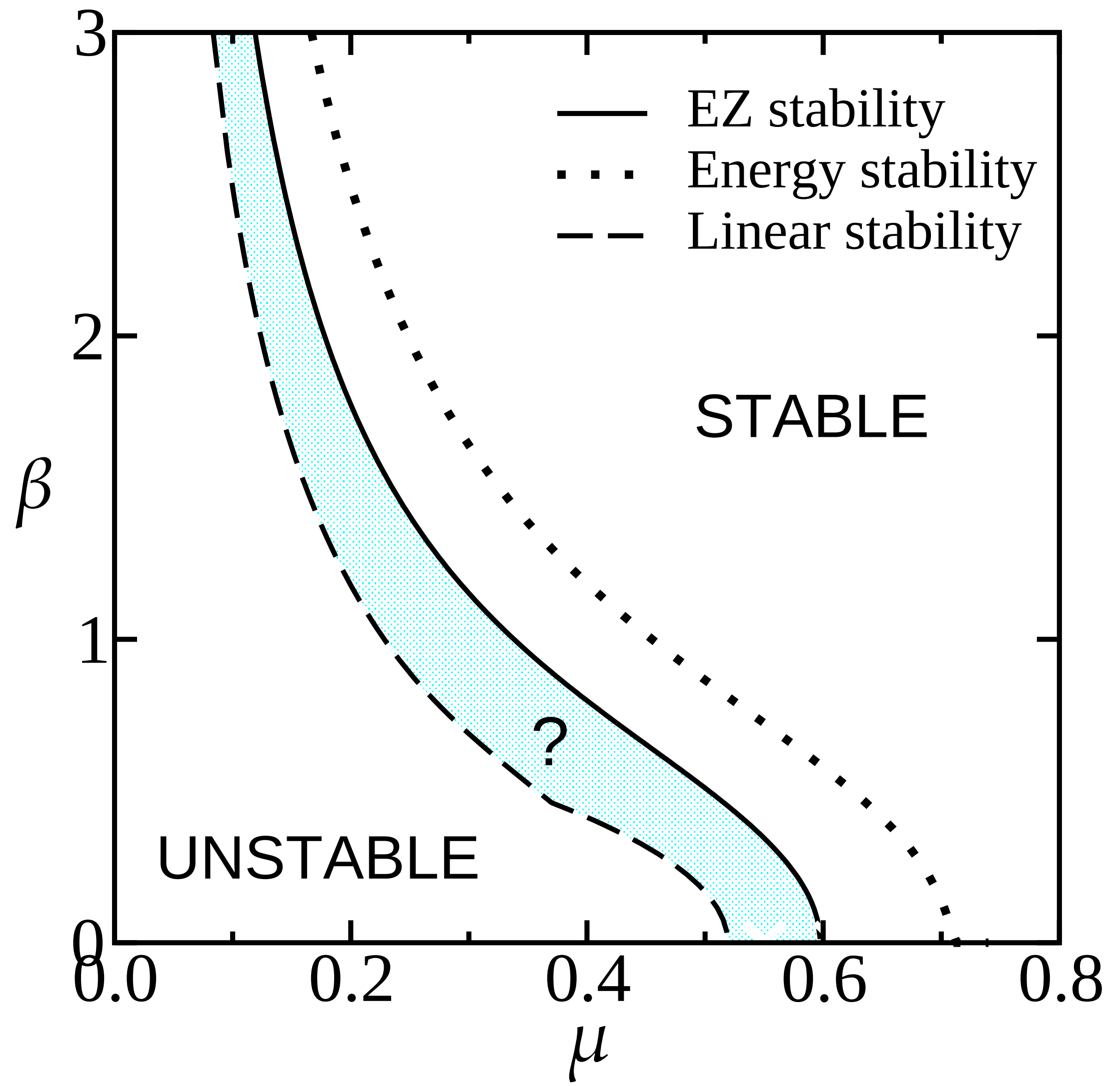
$$\mathcal{R}_* = \mathcal{R}[\varphi_*] = 0.3571$$

$$\varphi_*(x,y) = \mathfrak{R}\left\{e^{\mathrm{i}\,l\,y}\,\tilde{\varphi}(x)\right\} \quad \textit{with} \quad l \approx 0.4166$$



Energy-Enstrophy (EZ) Stability

$$\beta = \sqrt{\frac{0.13}{\mu^2} - \mu^2} \qquad (a = 2.8\mu)$$



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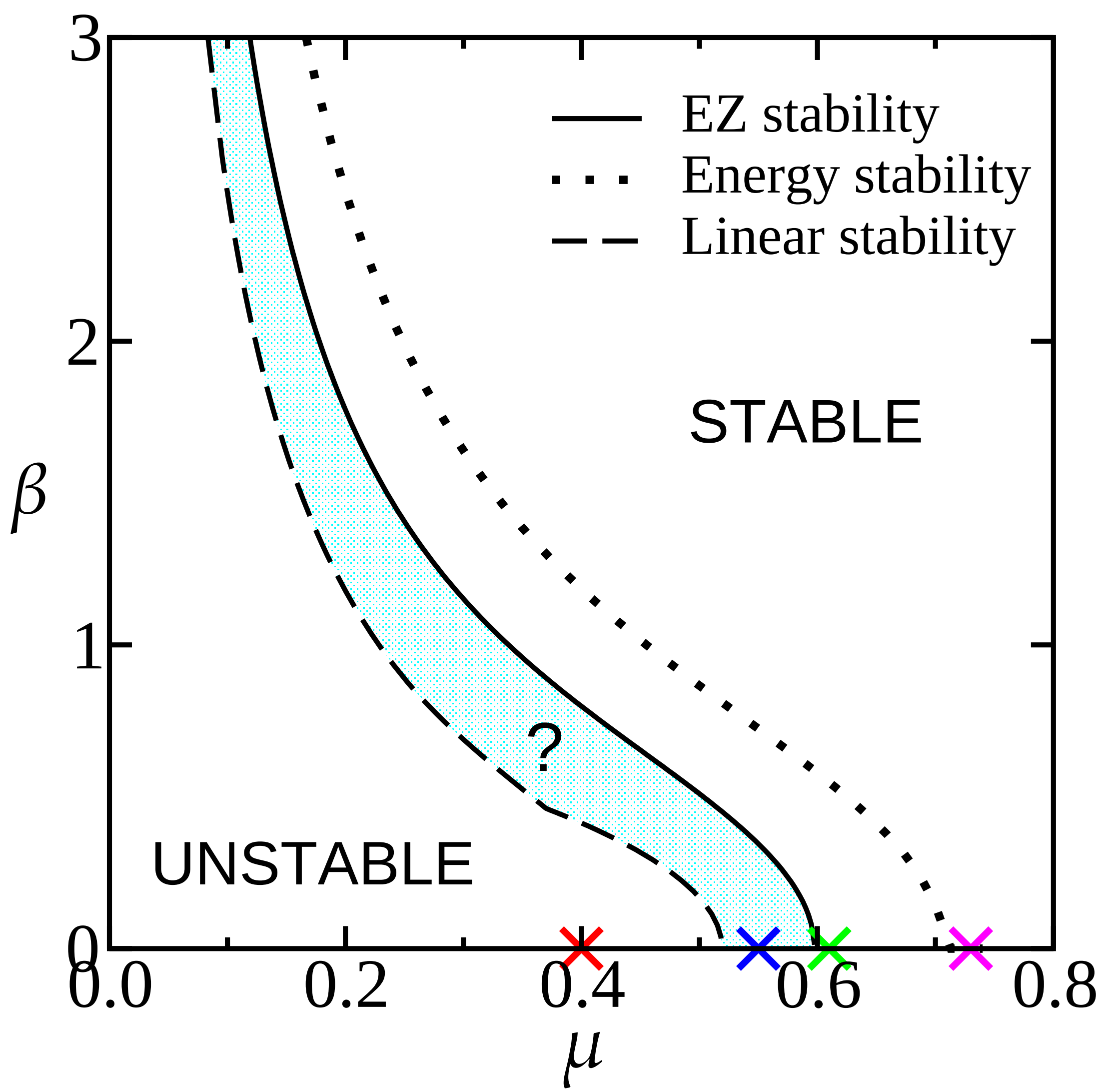

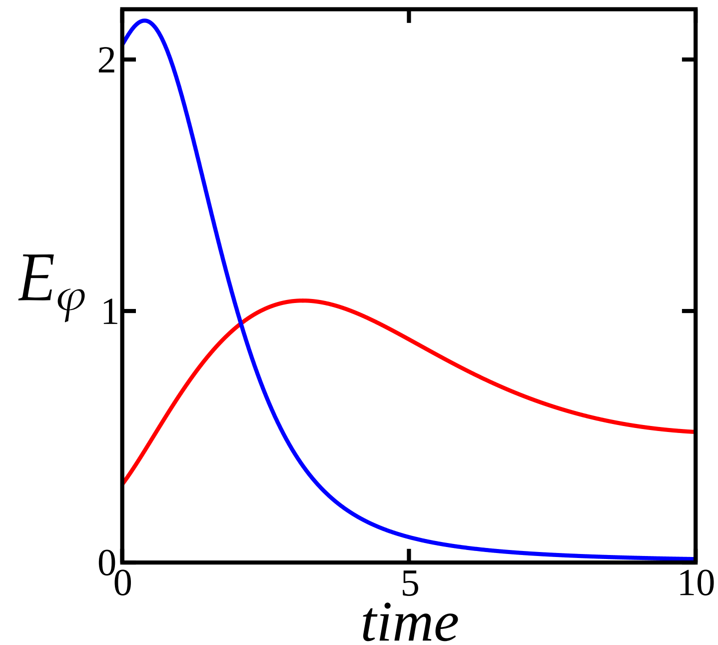

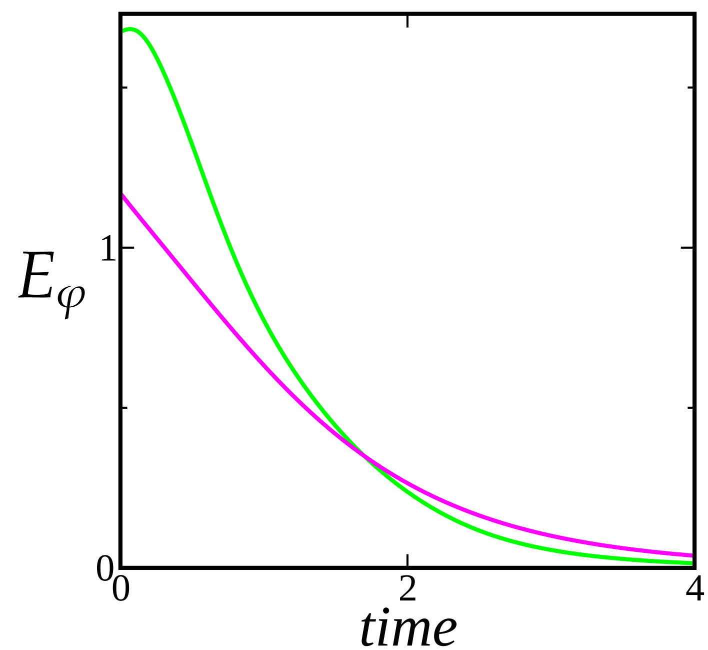

Summary

Based on the observation: $E_\varphi(t) = Z_\varphi(t) \;\; as \;\; t \to \infty$,
we develop the Energy-Enstrophy (EZ) stability method which

- allows transient growth in $E_\varphi(t)$ $(\varphi(t=0) \notin \Phi_{EZ})$
- identifies a physically realistic most-unstable disturbance
- lies closer to the linear stability neutral curve