Transition in the geomagnetic secular variation time scale below the CMB

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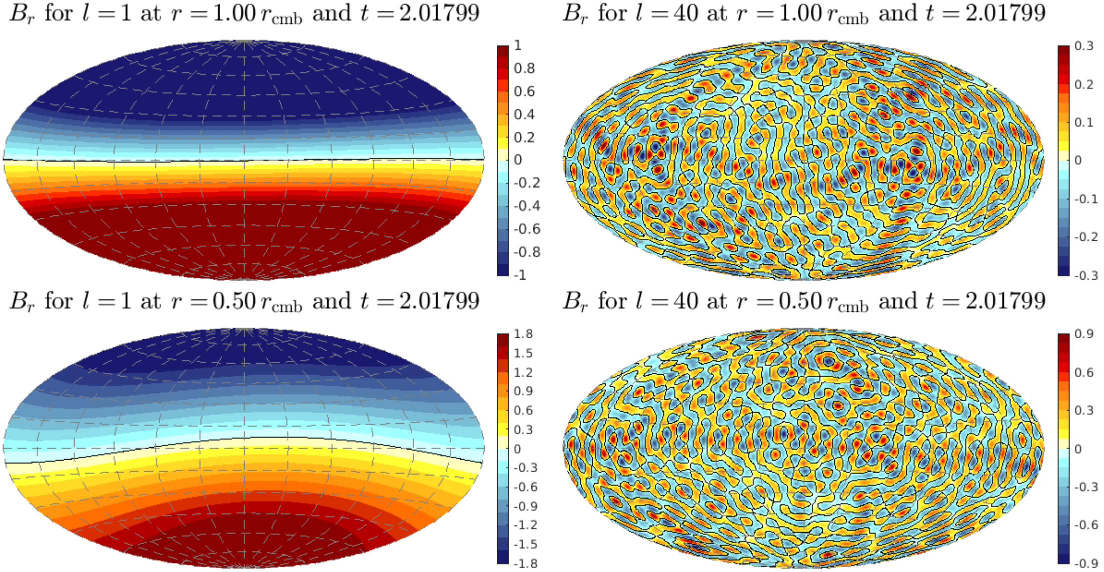
Geomagnetic secular variation at different spatial scales and depths



Spectra: to study properties at different spatial scales

Spectrum—a function of spherical harmonic degree l. For a potential field ($\boldsymbol{B} = -\nabla\Phi$) specified by the Gauss coefficients (g_{lm}, h_{lm}), the following spectra are defined for $r \geqslant r_{\text{cmb}}$:

(1) Lowes spectrum *(magnetic energy per spherical harmonic degree l)*

$$R(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[\, g_{lm}^2(t) + h_{lm}^2(t) \,\right] \qquad (a = \text{Earth's radius})$$

(2) Secular variation spectrum

$$R_{\text{sv}}(l,r,t) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left[\, \dot{g}_{lm}^2(t) + \dot{h}_{lm}^2(t) \,\right]$$

(3) Secular variation time-scale spectrum

$$\tau_{\text{sv}}(l,t) = \sqrt{\frac{R}{R_{\text{sv}}}} = \sqrt{\frac{\sum_{m=0}^{l} \left(g_{lm}^2 + h_{lm}^2\right)}{\sum_{m=0}^{l} \left(\dot{g}_{lm}^2 + \dot{h}_{lm}^2\right)}}$$

Secular variation time-scale spectrum $\tau_{\rm sv}$

$$\tau_{\rm sv}(l,t) = \sqrt{\frac{R}{R_{\rm sv}}} \quad (r \geqslant r_{\rm cmb})$$

- Characteristic time scale of magnetic field structures with spatial scale characterised by l
- Numerical simulations and *some* satellite data support the simple power-law:

$$\tau_{\rm sv}(l) \sim l^{-1}$$

(Christensen and Tilgner 2004, Holme and Olsen 2006, Lesur et al. 2008, Lhuillier et al. 2011)

- $\tau_{\rm sv}$ is sometimes interpreted as a characteristic time scale of the magnetohydrodynamics in the outer core
- **Question:** $\tau_{\rm sv}$ is defined using the Gauss coefficients derived from $\boldsymbol{B}$ observed at the Earth's surface. Do $\tau_{\rm sv}$ and the scaling law $\tau_{\rm sv} \sim l^{-1}$ describe the time variation of $\boldsymbol{B}$ inside the outer core?

Generalisation to inside the dynamo region (outer core)

For a magnetic field $\boldsymbol{B}$ (not necessarily potential) at some r, expand in a set of vector spherical harmonics basis,

$$\boldsymbol{B}(r,\theta,\phi,t) = \sum_{lm} \left[q_{lm}(r,t)\hat{\boldsymbol{Y}}_{lm}(\theta,\phi) + s_{lm}(r,t)\hat{\boldsymbol{\Psi}}_{lm}(\theta,\phi) + t_{lm}(r,t)\hat{\boldsymbol{\Phi}}_{lm}(\theta,\phi) \right]$$

We define the magnetic energy spectrum $F(l,r,t)$ for any r:

$$\sum_{l=1}^{\infty} F(l,r,t) \equiv \frac{1}{4\pi} \oint |\boldsymbol{B}(r,\theta,\phi,t)|^2 \, \mathrm{d}\Omega = \sum_{l=1}^{\infty} \left[\frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|q_{lm}|^2 + |s_{lm}|^2 + |t_{lm}|^2 \right) (4 - 3\delta_{m,0}) \right]$$

Similarly, define the time variation spectrum $F_{\dot{B}}(l,r,t)$:

$$\sum_{l=1}^{\infty} F_{\dot{B}}(l,r,t) \equiv \frac{1}{4\pi} \oint |\dot{\boldsymbol{B}}(r,\theta,\phi,t)|^2 \, \mathrm{d}\Omega = \sum_{l=1}^{\infty} \left[\frac{1}{(2l+1)} \sum_{m=0}^{l} \left(|\dot{q}_{lm}|^2 + |\dot{s}_{lm}|^2 + |\dot{t}_{lm}|^2 \right) (4 - 3\delta_{m,0}) \right]$$

Then, the (time-averaged) **magnetic time-scale spectrum** is defined as:

$$\tau(l,r) = \left\langle \sqrt{\frac{F(l,r,t)}{F_{\dot{B}}(l,r,t)}} \right\rangle_t \quad \text{for any } r$$

- Outside the dynamo region: $F = R$, $F_{\dot{B}} = R_{\mathrm{sv}}$, $\tau = \tau_{\mathrm{sv}}$

A numerical model of geodynamo

Boussinesq, composition-driven, rotating convection of an electrically conducting fluid:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\frac{Pm}{Ek}\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\frac{Pm}{Ek}\nabla\Pi' + \left(\frac{RaPm^2}{Pr}\right)Cr\,\hat{\boldsymbol{r}} + \frac{Pm}{Ek}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + Pm\nabla^2\boldsymbol{u},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla^2\boldsymbol{B}$$

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{Pm}{Pr}\nabla^2 C - 1$$

$$\nabla \cdot \boldsymbol{u} = 0$$

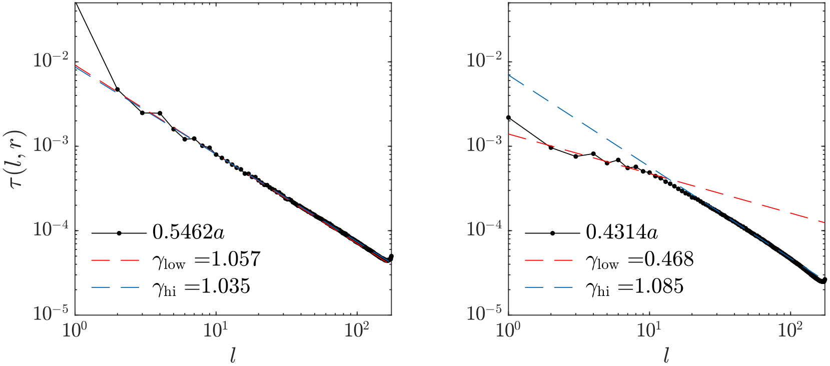
$$\nabla \cdot \boldsymbol{B} = 0$$

Boundary conditions: **no-slip** for $\boldsymbol{u}$, fixed-flux for C

Domain: a spherical shell $0.1912\,a \leqslant r \leqslant 0.5462\,a \equiv r_{\mathrm{cmb}}$ (or r_{o})

$Ra = 2.7 \times 10^8\,, Ek = 2.5 \times 10^{-5}\,, Pm = 2.5\,, Pr = 1$

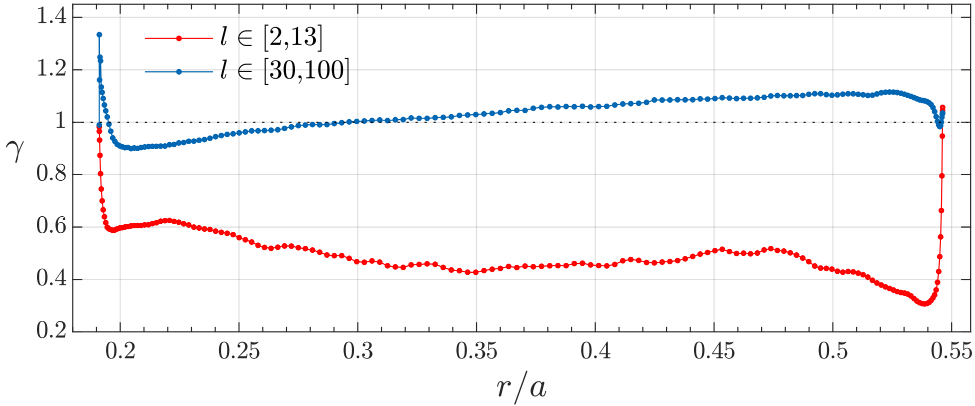
Magnetic time-scale spectrum $\tau(l,r)$ at different depth



For the large-scale modes (small l),

- At the surface: $\tau \sim l^{-1}$
- In the interior: $\tau \sim l^{-0.5}$, *the large-scale modes speed up in the interior!*
- This suggests $\tau_{\rm sv}(l)$ observed at the Earth's surface is unreliable as an estimate of time scales inside the outer core

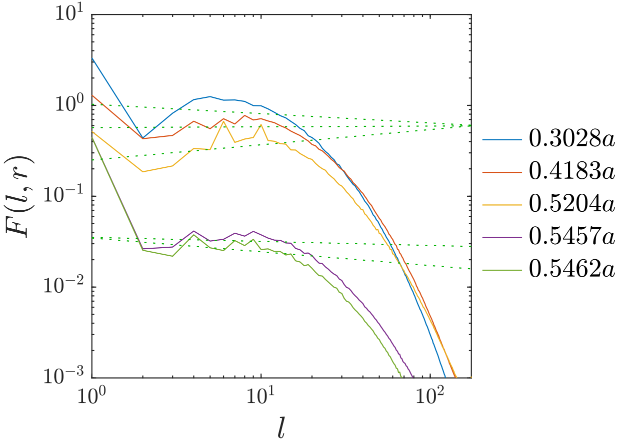
Transition in the scaling of τ: where does it occur?



- γ for the large-scale modes increases sharply within a boundary layer under CMB

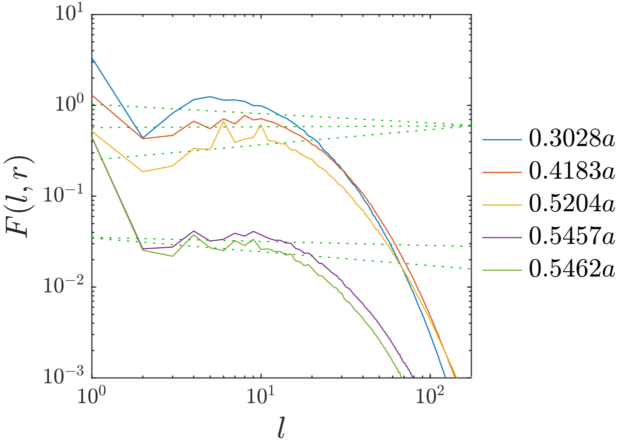
Focus on the large scales in following discussion ...

Change in the scaling of τ: who causes it?



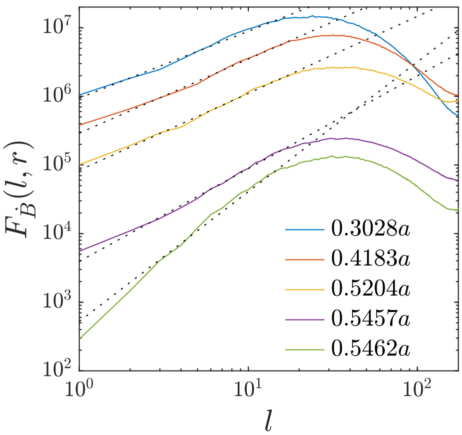
$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}}$$

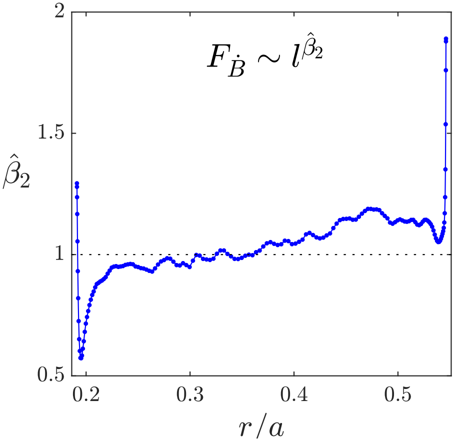
Change in the scaling of τ: who causes it?



$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{F_{\dot{B}}}} \sim F_{\dot{B}}{}^{-\frac{1}{2}}$$

Change in the scaling of τ: who causes it?





$$\tau \sim \sqrt{\frac{F}{F_{\dot{B}}}} \sim \sqrt{\frac{l^0}{F_{\dot{B}}}} \sim F_{\dot{B}}{}^{-\frac{1}{2}}$$

$$F_{\dot{B}} \sim l \implies \tau \sim l^{-0.5} \quad \text{(interior)}$$

$$F_{\dot{B}} \sim l^2 \implies \tau \sim l^{-1} \quad \text{(CMB)}$$

Transition in the scaling of τ: physical meaning

- Write $\boldsymbol{B}$ as the sum of a poloidal $\boldsymbol{B}_{\text{Pol}}$ and a toroidal $\boldsymbol{B}_{\text{Tor}}$ components:

$$\boldsymbol{B} = (B_r, B_\theta, B_\phi) = \underbrace{\nabla \times \nabla \times (P\,\boldsymbol{r})}_{\boldsymbol{B}_{\text{Pol}}} + \underbrace{\nabla \times (T\,\boldsymbol{r})}_{\boldsymbol{B}_{\text{Tor}}} \qquad (\boldsymbol{r} = r\,\hat{\boldsymbol{r}})$$

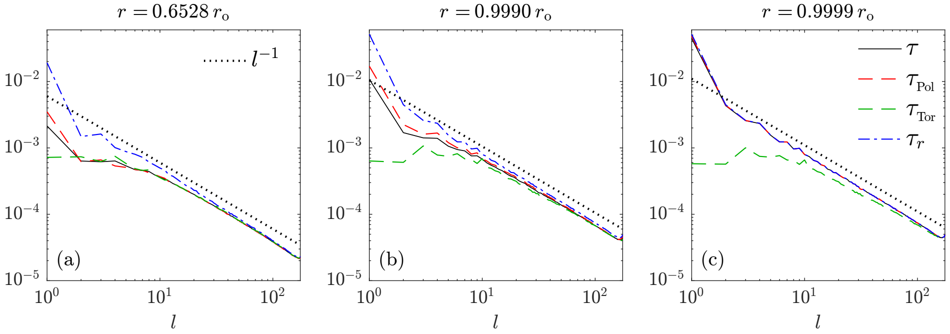
- Radial magnetic field is related to the poloidal potential: $B_r \longrightarrow P$
- Toroidal potential contributes to the horizontal magnetic field: $(B_\theta, B_\phi) \longrightarrow (P, T)$
- Define the three time scales:

$$\tau_{\text{Pol}} = \sqrt{\frac{\text{spectrum of } \boldsymbol{B}_{\text{Pol}}}{\text{spectrum of } \dot{\boldsymbol{B}}_{\text{Pol}}}}$$

$$\tau_{\text{Tor}} = \sqrt{\frac{\text{spectrum of } \boldsymbol{B}_{\text{Tor}}}{\text{spectrum of } \dot{\boldsymbol{B}}_{\text{Tor}}}}$$

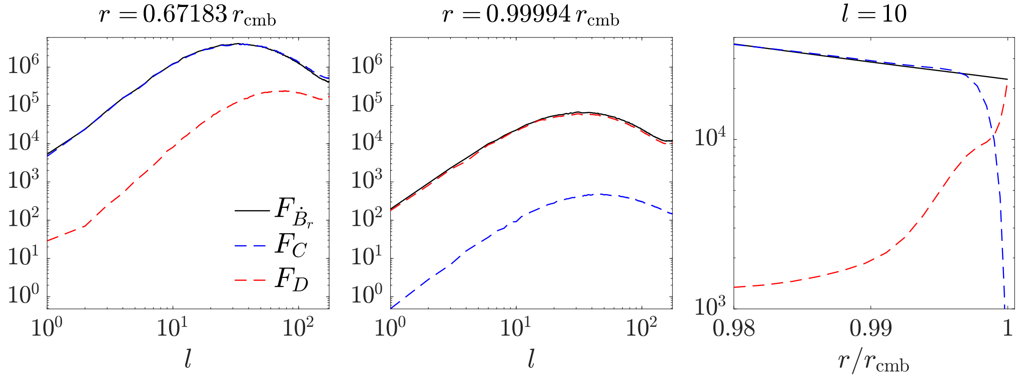
$$\tau_r = \sqrt{\frac{\text{spectrum of } B_r}{\text{spectrum of } \dot{B}_r}}$$

Transition in the scaling of τ: physical meaning



- In the interior (for the large scales):
  - $\tau_r \sim l^{-1}$ and $\tau_{\mathrm{Tor}} \sim l^{-0.5}$
  - $\tau \approx \tau_{\mathrm{Tor}} \ll \tau_r$
  - (B_θ, B_ϕ) vary faster than $B_r \implies$ $\dot{\boldsymbol{B}}$ is dominated by $(\dot{B}_\theta, \dot{B}_\phi)$
- τ_r has same scaling in the interior and at the surface, no abrupt transition for B_r
- Transition is a result of the magnetic boundary condition: $T = 0$ at $r = r_{\mathrm{cmb}}$

Process controlling the secular variation of B_r: induction vs. diffusion



$$\dot{B}_r = \hat{\boldsymbol{r}} \cdot [\nabla \times (\boldsymbol{u} \times \boldsymbol{B})] + \eta(\hat{\boldsymbol{r}} \cdot \nabla^2 \boldsymbol{B}) = C + D$$

- Interior: $\dot{B}_r \approx \hat{\boldsymbol{r}} \cdot [\nabla \times (\boldsymbol{u} \times \boldsymbol{B})]$, magnetic diffusion negligible
- At CMB: $\dot{B}_r \approx \eta(\hat{\boldsymbol{r}} \cdot \nabla^2 \boldsymbol{B})$, controlled by magnetic diffusion ($\because$ no-slip)
- Inside the CMB boundary layer: sharp transition for both C and D but $\dot{B}_r$ varies very weakly despite the switch in the controlling mechanism

Summary

- For the large scales, scaling of $\tau(l, r)$ with l at the CMB is different from that in the interior of the outer core:

$$\begin{aligned} \tau &\sim l^{-0.5}, && \text{in the interior} \\ \tau &\sim l^{-1}, && \text{at the CMB} \end{aligned}$$

- The transition in scaling occurs within a thin boundary layer under the CMB.
- $F_{\dot{B}}$ is responsible for the transition ($\tau = \sqrt{F/F_{\dot{B}}}$) which is enforced by the magnetic boundary condition at the CMB
- $\dot{\boldsymbol{B}} \sim \dot{B}_r$ at the CMB and $\dot{\boldsymbol{B}} \sim (\dot{B}_\theta, \dot{B}_\phi) \gg \dot{B}_r$ in the interior
- mechanism controlling $\dot{B}_r$ changes "stealthily" from induction to diffusion as $r \to r_{\mathrm{cmb}}$ from below