Problem Set 7, Physics 622 (Fall 2001)
Solutions

1 $j_1 = \frac{3}{2}$ and $j_2 = \frac{3}{2}$ implies $j = 0, 1, 2, 3$. The eigenfunctions for j_1 and j_2 are the spherical harmonics $Y_{\frac{3}{2},m_1}$ and $Y_{\frac{3}{2},m_2}$ respectively. Let the eigenfunctions of the total angular momentum be $|jm\rangle$ where $m = m_1 + m_2$. We have the expansion

$$|jm\rangle = \sum_{m_1,m_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \; Y_{j_1=\frac{3}{2},m_1} Y_{j_2=\frac{3}{2},m_2}$$

(a) $j = 3$:
For $m = 3$, the only non-vanishing term in the expansion is $m_1 = m_2 = \frac{3}{2}$. The eigenstates for other values of m can be obtained by applying the lowering operator repeatedly to $|3,3\rangle$. The results are:

$$\begin{aligned}
|3,3\rangle &= Y_{\frac{3}{2},\frac{3}{2}} Y_{\frac{3}{2},\frac{3}{2}} \\
|3,2\rangle &= \frac{1}{\sqrt{2}}(Y_{\frac{3}{2},\frac{1}{2}} Y_{\frac{3}{2},\frac{3}{2}} + Y_{\frac{3}{2},\frac{3}{2}} Y_{\frac{3}{2},\frac{1}{2}}) \\
|3,1\rangle &= \frac{1}{\sqrt{5}}(Y_{\frac{3}{2},-\frac{1}{2}} Y_{\frac{3}{2},\frac{3}{2}} + \sqrt{3}\, Y_{\frac{3}{2},\frac{1}{2}} Y_{\frac{3}{2},\frac{1}{2}} + Y_{\frac{3}{2},\frac{3}{2}} Y_{\frac{3}{2},-\frac{1}{2}}) \\
|3,0\rangle &= \frac{1}{2\sqrt{5}}(Y_{\frac{3}{2},-\frac{3}{2}} Y_{\frac{3}{2},\frac{3}{2}} + 3\, Y_{\frac{3}{2},-\frac{1}{2}} Y_{\frac{3}{2},\frac{1}{2}} + 3\, Y_{\frac{3}{2},\frac{1}{2}} Y_{\frac{3}{2},-\frac{1}{2}} + Y_{\frac{3}{2},\frac{3}{2}} Y_{\frac{3}{2},-\frac{3}{2}} \\
|3,-1\rangle &= \frac{1}{\sqrt{5}}(Y_{\frac{3}{2},-\frac{3}{2}} Y_{\frac{3}{2},\frac{1}{2}} + \sqrt{3}\, Y_{\frac{3}{2},-\frac{1}{2}} Y_{\frac{3}{2},-\frac{1}{2}} + Y_{\frac{3}{2},\frac{1}{2}} Y_{\frac{3}{2},-\frac{3}{2}}) \\
|3,-2\rangle &= \frac{1}{\sqrt{2}}(Y_{\frac{3}{2},-\frac{3}{2}} Y_{\frac{3}{2},-\frac{1}{2}} + Y_{\frac{3}{2},-\frac{1}{2}} Y_{\frac{3}{2},-\frac{3}{2}}) \\
|3,-3\rangle &= Y_{\frac{3}{2},-\frac{3}{2}} Y_{\frac{3}{2},-\frac{3}{2}} \quad \blacksquare
\end{aligned}$$

(b) $j = 2$:
For $m = 2$, the two non-vanishing terms are $m_1 = \frac{1}{2}, m_2 = \frac{3}{2}$ and $m_1 = \frac{3}{2}, m_2 = \frac{1}{2}$. $|2,2\rangle$ can then be determined by requiring it to be orthogonal to $|3,2\rangle$ and the eigenstates for other values of m follows.

$$\begin{aligned}
|2,2\rangle &= \frac{1}{\sqrt{2}}(-Y_{\frac{3}{2},\frac{1}{2}} Y_{\frac{3}{2},\frac{3}{2}} + Y_{\frac{3}{2},\frac{3}{2}} Y_{\frac{3}{2},\frac{1}{2}}) \\
|2,1\rangle &= \frac{1}{\sqrt{2}}(-Y_{\frac{3}{2},-\frac{1}{2}} Y_{\frac{3}{2},\frac{3}{2}} + Y_{\frac{3}{2},\frac{3}{2}} Y_{\frac{3}{2},-\frac{1}{2}}) \\
|2,0\rangle &= \frac{1}{2}(-Y_{\frac{3}{2},-\frac{3}{2}} Y_{\frac{3}{2},\frac{3}{2}} - Y_{\frac{3}{2},-\frac{1}{2}} Y_{\frac{3}{2},\frac{1}{2}} + Y_{\frac{3}{2},\frac{1}{2}} Y_{\frac{3}{2},-\frac{1}{2}} + Y_{\frac{3}{2},\frac{3}{2}} Y_{\frac{3}{2},-\frac{3}{2}}) \\
|2,-1\rangle &= \frac{1}{\sqrt{2}}(-Y_{\frac{3}{2},-\frac{3}{2}} Y_{\frac{3}{2},\frac{1}{2}} + Y_{\frac{3}{2},\frac{1}{2}} Y_{\frac{3}{2},-\frac{3}{2}}) \\
|2,-2\rangle &= \frac{1}{\sqrt{2}}(-Y_{\frac{3}{2},-\frac{3}{2}} Y_{\frac{3}{2},-\frac{1}{2}} + Y_{\frac{3}{2},-\frac{1}{2}} Y_{\frac{3}{2},-\frac{3}{2}}) \quad \blacksquare
\end{aligned}$$

(c) $j = 1$:
Based on similar arguments, $|1,1\rangle$ consists of 3 terms and is orthogonal to $|3,1\rangle$ and $|2,1\rangle$, the eigenstates for $j = 1$ are as follows:

$$\begin{aligned}
|1,1\rangle &= \frac{1}{\sqrt{10}}(\sqrt{3}Y_{\frac{3}{2},-\frac{1}{2}}Y_{\frac{3}{2},\frac{3}{2}} + 2\,Y_{\frac{3}{2},\frac{1}{2}}Y_{\frac{3}{2},\frac{1}{2}} + \sqrt{3}Y_{\frac{3}{2},\frac{3}{2}}Y_{\frac{3}{2},-\frac{1}{2}}) \\
|1,0\rangle &= \frac{1}{20}(3\,Y_{\frac{3}{2},-\frac{3}{2}}Y_{\frac{3}{2},\frac{3}{2}} - Y_{\frac{3}{2},-\frac{1}{2}}Y_{\frac{3}{2},\frac{1}{2}} - Y_{\frac{3}{2},\frac{1}{2}}Y_{\frac{3}{2},-\frac{1}{2}} + 3\,Y_{\frac{3}{2},\frac{3}{2}}Y_{\frac{3}{2},-\frac{3}{2}}) \\
|1,-1\rangle &= \frac{1}{\sqrt{10}}(-\sqrt{3}Y_{\frac{3}{2},-\frac{3}{2}}Y_{\frac{3}{2},\frac{1}{2}} - 2\,Y_{\frac{3}{2},-\frac{1}{2}}Y_{\frac{3}{2},-\frac{1}{2}} + \sqrt{3}Y_{\frac{3}{2},\frac{1}{2}}Y_{\frac{3}{2},-\frac{3}{2}}) \quad \blacksquare
\end{aligned}$$

(d) $j = 0$:
Finally, $|0,0\rangle$ is orthogonal to $|3,0\rangle$, $|2,0\rangle$ and $|1,0\rangle$, therefore

$$|0,0\rangle = \frac{1}{2}(-Y_{\frac{3}{2},-\frac{3}{2}}Y_{\frac{3}{2},\frac{3}{2}} + Y_{\frac{3}{2},-\frac{1}{2}}Y_{\frac{3}{2},\frac{1}{2}} - Y_{\frac{3}{2},\frac{1}{2}}Y_{\frac{3}{2},-\frac{1}{2}} + Y_{\frac{3}{2},\frac{3}{2}}Y_{\frac{3}{2},-\frac{3}{2}}) \quad \blacksquare$$

2(3.9) (a) From Eq.(3.2.52) in Sakurai, a general spin-$\frac{1}{2}$ state is given by

$$|\psi\rangle = \cos\frac{\beta}{2}e^{-i\alpha/2}|+\rangle + \sin\frac{\beta}{2}e^{i\alpha/2}|-\rangle$$

Knowing $\langle S_x\rangle$ and $\langle S_z\rangle$ means we have the following two equations:

$$\begin{aligned}
\langle S_x\rangle &= \frac{\hbar}{2}\sin\beta\cos\alpha \\
\langle S_z\rangle &= \frac{\hbar}{2}\cos\beta
\end{aligned}$$

which allow us to solve for β and $\cos\alpha$. To solve for α, thus determining the state vector completely, we need to know the sign of $\sin\alpha$ which is supplied by the sign of $\langle S_y\rangle$. $\blacksquare$

(b) Let ρ be the 2×2 density matrix. Given $[S_x]$, $[S_y]$ and $[S_z]$ with the normalization condition, we have the following four equations:

$$\begin{aligned}
\mathrm{tr}(\rho S_i) &= [S_i] \qquad i = x, y, z \\
\mathrm{tr}(\rho) &= 1
\end{aligned}$$

which allow us to solve for the four real elements of ρ. Notice that these equations are independent of the choice of basis. $\blacksquare$

3(3.11) From the definition of the density matrix Eq(3.4.9) in Sakurai, we see that ρ is of the form

$$\rho = \begin{pmatrix} a & b & c \\ b^* & d & e \\ c^* & e* & f \end{pmatrix}$$

where the diagonal elements a, d, f are real. With the normalization condition $a + d + f = 1$, there are 8 independent parameters. Therefore, to characterize the ensemble completely, we need 8 quantities which can be chosen as $[S_x]$, $[S_y]$, $[S_z]$, $[S_xS_y]$, $[S_yS_z]$, $[S_zS_x]$, $[S_x^2]$ and $[S_y^2]$. ■

4(3.26) We shall use Eq(3.10.27) in Sakurai to construct the required the tensors,

$$T_q^{(k)} = \sum_{q_1}\sum_{q_2}\langle k_1k_2; q_1q_2|k_1k_2; kq\rangle X_{q_1}^{(k_1)}X_{q_2}^{(k_2)}$$

(a) $k_1 = k_2 = k = 1$, hence

$$T_q^{(1)} = \sum_{q_1}\sum_{q_2}\langle 11; q_1q_2|11; 1q\rangle U_{q_1}^{(1)}V_{q_2}^{(1)}$$

where $U_q(V_q)$ is the qth component of a spherical tensor of rank 1 corresponding to vector $\mathbf{U}(\mathbf{V})$. The relations between U_q and $\mathbf{U}$ are given by

$$\begin{aligned} U_{+1} &= -\frac{U_x + iU_y}{\sqrt{2}} \\ U_{-1} &= \frac{U_x - iU_y}{\sqrt{2}} \\ U_0 &= U_z \end{aligned}$$

and similar relations hold between V_q and $\mathbf{V}$. Hence,

$$\begin{aligned} T_{-1}^{(1)} &= \frac{1}{2}[-(U_x - iU_y)V_z + (V_x - iV_y)U_z] \\ T_0^{(1)} &= \frac{i}{\sqrt{2}}[U_xV_y - U_yV_x] \\ T_1^{(1)} &= \frac{1}{2}[-(U_x + iU_y)V_z + (V_x + iV_y)U_z] \quad ■ \end{aligned}$$

(b) $k_1 = k_2 = 1, k = 2$, we have

$$\begin{aligned} T_{-2}^{(2)} &= \frac{1}{2}(U_x - iU_y)(V_x - iV_y) \\ T_{-1}^{(2)} &= \frac{1}{2}[(U_x - iU_y)V_z + (V_x - iV_y)U_z] \\ T_0^{(1)} &= \frac{1}{2\sqrt{6}}[-(U_x - iU_y)(V_x + iV_y) + 4U_zV_z - (U_x + iU_y)(V_x - iV_y) \\ T_1^{(2)} &= -\frac{1}{2}[(U_x + iU_y)V_z + (V_x + iV_y)U_z] \\ T_2^{(2)} &= \frac{1}{2}(U_x + iU_y)(V_x + iV_y) \quad ■ \end{aligned}$$

5(3.28) (a) Notice that the $Y_{2,m}$ are components of spherical irreducible tensor of rank 2, and in particular, we have

$$\begin{aligned} Y_2^0 &= \sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2} \\ Y_2^{\pm 1} &= \mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2} \\ Y_2^{\pm 2} &= \sqrt{\frac{15}{32\pi}}\frac{(x\pm iy)^2}{r^2} \end{aligned}$$

It is obvious then,

$$\begin{aligned} xy &= i\sqrt{\frac{2\pi}{15}}(Y_2^{-2}-Y_2^2)r^2 \\ xz &= \sqrt{\frac{2\pi}{15}}(Y_2^{-1}-Y_2^1)r^2 \\ x^2-y^2 &= \sqrt{\frac{8\pi}{15}}(Y_2^{-2}+Y_2^2)r^2 \quad \blacksquare \end{aligned}$$

(b)

$$\begin{aligned} Q &= e\langle \alpha jj|3z^2-r^2|\alpha jj\rangle \\ &= e\sqrt{\frac{16\pi}{5}}\langle \alpha j||r^2Y_2^0||\alpha j\rangle \\ &= e\sqrt{\frac{16\pi}{5}}\langle j2;j0|j2;jj\rangle\frac{\langle \alpha j||r^2Y_2||\alpha j\rangle}{\sqrt{2j+1}} \\ e\langle \alpha jj|(x^2-y^2)|\alpha jj\rangle &= e\sqrt{\frac{8\pi}{15}}\langle \alpha jj|r^2(Y_2^2+Y_2^{-2})|\alpha jj\rangle \\ &= e\frac{\langle \alpha j||r^2Y_2||\alpha j\rangle}{\sqrt{2j+1}}\sqrt{\frac{8\pi}{15}}[\langle j2j2|j2jm'\rangle+\langle j2j-2|j2jm'\rangle] \\ &= \frac{Q}{\sqrt{6}}\frac{\langle j2j-2|j2jm'\rangle}{\langle j2j0|j2jj\rangle} \quad \blacksquare \end{aligned}$$

6(3.29) Let $\phi_{\alpha\alpha} = \left(\frac{\partial^2\phi}{\partial\alpha^2}\right)_0$, $\alpha = x, y, z$. Recall that

$$\begin{aligned} S_x^2 &= \frac{S_+^2+S_-^2+S_+S_-+S_-S_+}{4} \\ S_y^2 &= -\frac{S_+^2+S_-^2-(S_+S_-+S_-S_+)}{4} \\ S_+S_- &= S_-S_+ = S_x^2+S_y^2 = S^2-S_z^2 \end{aligned}$$

we then have

$$
\begin{aligned}
&\phi_{xx}S_x^2 + \phi_{yy}S_y^2 + \phi_{zz}S_z^2 \\
= &\left(\frac{\phi_{xx}-\phi_{yy}}{4}\right)(S_+^2 + S_-^2) + \left(\frac{\phi_{xx}+\phi_{yy}}{2}\right)(S^2 - S_z^2) + \phi_{zz}S_z^2 \\
= &\left(\frac{\phi_{xx}-\phi_{yy}}{4}\right)(S_+^2 + S_-^2) + \left(\frac{\phi_{xx}+\phi_{yy}}{2}\right)(S^2 - S_z^2) + \phi_{zz}S_z^2 + \frac{\phi_{zz}}{2}(S^2 - S_z^2) - \frac{\phi_{zz}}{2}(S^2 - S_z^2) \\
= &\frac{\phi_{zz}}{2}(3S_z^2 - S^2) + \left(\frac{\phi_{xx}-\phi_{yy}}{4}\right)(S_+^2 + S_-^2) \qquad (\because \nabla^2\phi = 0)
\end{aligned}
$$

Therefore, $H = A(3S_z^2 - S^2) + B(S_+^2 + S_-^2)$ with

$$
\begin{aligned}
A &= \frac{eQ}{4s(s-1)\hbar^2}\phi_{zz} \\
B &= \frac{eQ}{8s(s-1)\hbar^2}(\phi_{xx} - \phi_{yy}) \qquad \blacksquare
\end{aligned}
$$

In the $|m\rangle$ basis, we have

$$
H_{\text{int}} = \begin{pmatrix} 3A & 2\sqrt{3}B & 0 & 0 \\ 2\sqrt{3}B & -3A & 0 & 0 \\ 0 & 0 & -3A & 2\sqrt{3}B \\ 0 & 0 & 2\sqrt{3}B & 3A \end{pmatrix}\hbar^2
$$

Diagonalizing the above matrix gives the energy eigenvalues and eigenkets:

$$
E_1 = E_3 = \sqrt{9A^2 + 12B^2}\hbar^2
$$
$$
E_2 = E_4 = -\sqrt{9A^2 + 12B^2}\hbar^2
$$

$$
\begin{aligned}
|E_{1,2}\rangle &= 2\sqrt{3}B|\frac{3}{2},\frac{3}{2}\rangle + (E_{1,2} - 3A)|\frac{3}{2},-\frac{1}{2}\rangle \\
|E_{3,4}\rangle &= 2\sqrt{3}B|\frac{3}{2},-\frac{3}{2}\rangle + (E_{3,4} + 3A)|\frac{3}{2},\frac{1}{2}\rangle
\end{aligned}
$$

Therefore, there exists a two-fold degeneracy. $\blacksquare$

7 Let $\langle jj; m_1m_2|JM\rangle$ be the Clebsch-Gordon coefficients with $J = 0, 1, \ldots, 2j$ and $M = m_1 + m_2$. Then we have,

$$
\begin{aligned}
|JM\rangle &= \sum_{m_1}\langle jj; m_1m_2|JM\rangle Y_{jm_1}Y_{jm_2} \\
|JM\rangle' &= \sum_{m_2}\langle jj; m_2m_1|JM\rangle Y_{jm_2}Y_{jm_1}
\end{aligned}
$$

Applying the recursion relation Eq(3.7.49) in Sakurai to the present case,

$$\begin{aligned}
& \sqrt{(J-M)(J+M+1)}\langle jj; m_1 m_2 | J, M+1\rangle \\
= \; & \sqrt{(j-m_1+1)(j+m_1)}\langle jj; m_1-1 m_2|JM\rangle + \sqrt{(j-m_2+1)(j+m_2)}\langle jj; m_1 m_2-1|JM\rangle
\end{aligned}$$

and put $M = J$,

$$\langle jj; m_1 - 1 m_2 | JJ\rangle = -\sqrt{\frac{(j-m_2+1)(j+m_2)}{(j-m_1+1)(j+m_1)}}\langle jj; m_1 m_2 - 1|JJ\rangle$$

By convention, $\langle jj; jJ-j|JJ\rangle$ is real and positive, therefore from the above equation, we can conclude that

$$\begin{aligned}
\text{sign of } \langle jj; m_1, J-m_1|JJ\rangle &= (-1)^{j-m_1} \\
\text{sign of } \langle jj; J-m_1, m_1|JJ\rangle &= (-1)^{j+m_1-J}
\end{aligned}$$

Hence, if we invert m_1, m_2, the sign of $|JJ\rangle$ is multiplied by the factor

$$\frac{(-1)^{j+m_1-J}}{(-1)^{j-m_1}} = (-1)^{2m_1-J} = (-1)^{2j+J}$$

since $(-1)^{2J} = 1$ for J is an integer, $(-1)^{2m_1} = (-1)^{2(j-n)} = (-1)^{2j}$ for n is an integer. The same is true for $|JM\rangle$, which are constructed from $|JJ\rangle$ by the action of J_- which does not change the symmetry. Therefore

$$\begin{aligned}
\langle jj; m_2 m_1|JM\rangle &= (-1)^{2j+J}\langle jj; m_1 m_2|JM\rangle \\
\Rightarrow \qquad |JM\rangle' &= (-1)^{2j+J}|JM\rangle \quad \blacksquare
\end{aligned}$$

8 Given $W(x,p,t) = \int_{-\infty}^{\infty} dy \langle x + y/2|\psi(t)\rangle\langle\psi(t)|x - y/2\rangle e^{ipy/h}$, we then perform the integrations,

$$\begin{aligned}
\frac{1}{2\pi\hbar}\int dp\, W(x,p,t) &= \int_{-\infty}^{\infty} dy\langle x+y/2|\psi(t)\rangle\langle\psi(t)|x-y/2\rangle\frac{1}{2\pi\hbar}\int dp\, e^{ipy/h} \\
&= \int_{-\infty}^{\infty} dy\langle x+y/2|\psi(t)\rangle\langle\psi(t)|x-y/2\rangle\delta(y) \\
&= \langle x|\psi(t)\rangle\langle\psi(t)|x\rangle \\
&= |\langle x|\psi(t)\rangle|^2
\end{aligned}$$

which is the probability density in x. $\blacksquare$

$$
\begin{aligned}
\frac{1}{2\pi\hbar}\int dx\, W(x,p,t) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp_1 dp_2 dy dx \langle x+y/2|p_1\rangle\langle p_1|\psi(t)\rangle\langle\psi(t)|p_2\rangle\langle p_2|x-y/2\rangle e^{ipy/h} \\
&= \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{\infty} dp_1 dp_2 dy dx\, e^{\frac{ip_1}{\hbar}(x+\frac{y}{2})}\langle p_1|\psi(t)\rangle\langle\psi(t)|p_2\rangle e^{-\frac{ip_2}{\hbar}(x-\frac{y}{2})} e^{ipy/h} \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp_1 dp_2 dy\, \delta(p_1-p_2)\langle p_1|\psi(t)\rangle\langle\psi(t)|p_2\rangle e^{\frac{i}{\hbar}(p_1+p_2)\frac{y}{2}} e^{ipy/h} \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp_1 dy\, \langle p_1|\psi(t)\rangle\langle\psi(t)|p_1\rangle e^{\frac{i}{\hbar}(p-p_1)y} \\
&= \int_{-\infty}^{\infty} dp_1\, \langle p_1|\psi(t)\rangle\langle\psi(t)|p_1\rangle\delta(p-p_1) \\
&= |\langle p|\psi(t)\rangle|^2
\end{aligned}
$$

which is the probability density in p. ■

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