Problem Set 5, Physics 622 (Fall 2001)
Solutions

1 Using as a basis the simultaneous eigenstates $|lm\rangle$ of L^2 and L_z, the matrix representation of L^2 and L_z are diagonal, with elements

$$\begin{aligned}(L^2)_{l'm'lm} &= \langle l'm'|L^2|lm\rangle = l(l+1)\hbar^2\delta_{ll'}\delta_{mm'}\\ (L_z)_{l'm'lm} &= \langle l'm'|L_z|lm\rangle = m\hbar^2\delta_{ll'}\delta_{mm'}\end{aligned}$$

Using
$$\begin{aligned}L_+|lm\rangle &= \sqrt{l(l+1)-m(m+1)}\,\hbar|l\ m{+}1\rangle\\ L_-|lm\rangle &= \sqrt{l(l+1)-m(m-1)}\,\hbar|l\ m{-}1\rangle\end{aligned}$$

we can calculate the matrix elements of L_+ and L_-,

$$\begin{aligned}(L_+)_{l'm'lm} &= \sqrt{l(l+1)-m(m+1)}\,\hbar\delta_{l'l}\delta_{m'm+1}\\ (L_-)_{l'm'lm} &= \sqrt{l(l+1)-m(m-1)}\,\hbar\delta_{l'l}\delta_{m'm-1}\end{aligned}$$

Using the relations

$$L_x = \frac{1}{2}(L_+ + L_-)\ ,\ \ L_y = \frac{1}{2i}(L_+ - L_-)$$

we can obtain the matrix elements of L_x and L_y.

(a) $l = 2,\ m = -2,-1,0,1,2$

$$L^2 = 6\hbar^2\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \qquad L_z = \hbar\begin{pmatrix}2&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&-1&0\\0&0&0&0&-2\end{pmatrix}$$

$$L_+ = \hbar\begin{pmatrix}0&2&0&0&0\\0&0&\sqrt{6}&0&0\\0&0&0&\sqrt{6}&0\\0&0&0&0&2\\0&0&0&0&0\end{pmatrix} \qquad L_- = \hbar\begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\0&\sqrt{6}&0&0&0\\0&0&\sqrt{6}&0&0\\0&0&0&2&0\end{pmatrix}$$

$$L_x = \frac{\hbar}{2}\begin{pmatrix}0&2&0&0&0\\2&0&\sqrt{6}&0&0\\0&\sqrt{6}&0&\sqrt{6}&0\\0&0&\sqrt{6}&0&2\\0&0&0&2&0\end{pmatrix} \qquad L_y = \frac{i\hbar}{2}\begin{pmatrix}0&-2&0&0&0\\2&0&-\sqrt{6}&0&0\\0&\sqrt{6}&0&-\sqrt{6}&0\\0&0&\sqrt{6}&0&-2\\0&0&0&2&0\end{pmatrix}$$

∎

(b) $l=\frac{1}{2},\ m=-\frac{1}{2},\frac{1}{2}$

$$\begin{aligned} L^2 &= \frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix} & L_z &= \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} \\ L_+ &= \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix} & L_- &= \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix} \\ L_x &= \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} & L_y &= \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \blacksquare \end{aligned}$$

2 (a) Using Einstein's summation convention,

$$\begin{aligned} L_i &= \epsilon_{ikm} r_k p_m \\ [L_i, r_j] &= \epsilon_{ikm}[r_k p_m, r_j] \\ &= \epsilon_{ikm} r_k [p_m, r_j] \\ &= -i\hbar\epsilon_{ikm} r_k \delta_{jm} \\ &= i\hbar\epsilon_{ijk} r_k \end{aligned}$$

With α, β and γ defined as a cyclic parameter of x, y and z, it is equivalent to write

$$[L_\alpha, r_\beta] = i\hbar r_\gamma \quad \blacksquare$$

Similarly,

$$\begin{aligned} [L_i, p_j] &= \epsilon_{ikm}[r_k p_m, p_j] \\ &= \epsilon_{ikm}[r_k, p_j], p_m \\ &= i\hbar\epsilon_{ikm} p_m \delta_{jk} \\ &= i\hbar\epsilon_{ijm} p_m \\ \Rightarrow \qquad [L_\alpha, p_\beta] &= i\hbar p_\gamma \quad \blacksquare \end{aligned}$$

(b)

$$\begin{aligned} [L_i, V_j V_j] &= [L_i, V_j]V_j + V_j[L_i, V_j] \\ &= i\hbar\epsilon_{ijk} V_k V_j - i\hbar\epsilon_{ijk} V_j V_k \\ &= 0 \quad \blacksquare \end{aligned}$$

3 The easiest way to prove Eq.(3) is by using the Addition Theorem of spherical harmonics:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)$$

where P_l is the Legendre polynomial of order l and

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

When $\theta = \theta'$ and $\phi = \phi'$, $\cos\gamma = 1$ which implies $P_l(\cos\gamma) = 1$, hence

$$\sum_{m=-l}^{l} |Y_{lm}(\theta,\phi)|^2 = \frac{2l+1}{4\pi} \quad \blacksquare$$

<u>Note</u>: Proof of the Addition Theorem can be found in :
Classical Electrodynamics 2ed., J.D.Jackson, Sec. 3.6
Mathematical Methods for Physicists 4ed., G.B.Arfken & H.J.Weber, Sec. 12.8
Quantum Mechanics 3ed., E.Merzbacher, Sec. 11.4

4(3.15) (a) Rewrite $\psi(\vec{x})$ in spherical coordinates,

$$\psi(\vec{x}) = r(\sin\theta\cos\phi + \sin\theta\sin\phi + 3\cos\theta) f(r)$$

Notice that

$$\begin{aligned} Y_{1,0} &= \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_{1,1} &= -\sqrt{\frac{3}{8\pi}}\sin\theta(\cos\phi + i\sin\phi) \\ Y_{1,-1} &= \sqrt{\frac{3}{8\pi}}\sin\theta(\cos\phi - i\sin\phi) \end{aligned}$$

We can write $\psi(\vec{x})$ in terms of the spherical harmonics,

$$\psi(\vec{x}) = rf(r)\sqrt{\frac{4\pi}{3}}\left(3\,Y_{1,0} - \frac{1+i}{\sqrt{2}}\,Y_{1,1} + \frac{1-i}{\sqrt{2}}\,Y_{1,-1}\right)$$

Since $L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$, it is obvious that $\psi(\vec{x})$ is an eigenfunction of L^2 with $l = 1$. $\blacksquare$

(b) Notice that the given $\psi(\vec{x})$ is not normalized, probabilities for the particle to be found in various m_l states are:

$$\begin{aligned} m_l &= -1 &: \frac{1}{11} \\ m_l &= 0 &: \frac{9}{11} \\ m_l &= 1 &: \frac{1}{11} \quad \blacksquare \end{aligned}$$

(c) Since $\psi(\vec{x})$ is an energy eigenfunction with eigenvalue E, we have in spherical coordinates,

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{L^2}{2mr^2}+V(r)\right]\psi(\vec{x})=E\psi(\vec{x})$$

With $\psi(\vec{x})$ given in (a) and $L^2Y_{lm}=l(l+1)\hbar^2Y_{lm}$, we get the radial equation

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{\hbar^2}{mr^2}+V(r)\right]rf(r)=Erf(r)$$

Performing the differentiation and after some rearrangements, the potential is given by

$$V(r)=\frac{\hbar^2}{2mrf(r)}[rf''(r)+4f'(r)]+E \quad \blacksquare$$

5(3.16) Expressing L_x and L_y in terms of L_+ and L_-, we have

$$\begin{aligned}
\langle L_x\rangle &= \frac{1}{2}\langle lm|(L_+ + L_-)|lm\rangle \\
&= \frac{1}{2}\left(\langle lm|L_+|lm\rangle+\langle lm|L_-|lm\rangle\right) \\
&= \frac{1}{2}\left(c_{m+1}\langle lm|l\,m+1\rangle+c_{m-1}\langle lm|l\,m-1\rangle\right) \quad (c_{m+1}, c_{m-1} \text{ are complex constants}) \\
&= 0 \quad \blacksquare \qquad (\because \langle l'm'|lm\rangle=\delta_{ll'}\delta_{mm'})
\end{aligned}$$

$$\begin{aligned}
\langle L_y\rangle &= \frac{1}{2i}\langle lm|(L_+ - L_-)|lm\rangle \\
&= 0 \quad \blacksquare
\end{aligned}$$

Next, we try to show $\langle L_x^2\rangle=\langle L_y^2\rangle$,

$$\begin{aligned}
\langle L_+L_+\rangle &= 0 \\
\Rightarrow \qquad \langle L_x^2\rangle-\langle L_y^2\rangle+i\langle L_xL_y+L_yL_x\rangle &= 0
\end{aligned}$$

Since L_x^2, L_y^2 and $L_xL_y+L_yL_x$ are hermitian, their expectation values are real, thus we conclude that $\langle L_x^2\rangle=\langle L_y^2\rangle$. Then

$$\begin{aligned}
L^2 &= L_x^2+L_y^2+L_z^2 \\
\langle L^2\rangle &= \langle L_x^2\rangle+\langle L_y^2\rangle+\langle L_z^2\rangle \\
\therefore \quad \langle L_x^2\rangle=\langle L_y^2\rangle &= \frac{\langle L^2\rangle-\langle L_z^2\rangle}{2} \\
&= \frac{l(l+1)\hbar^2-m^2\hbar^2}{2} \quad \blacksquare
\end{aligned}$$

6(3.17) (a)

$$\begin{aligned} L_- Y_{\frac{1}{2},\frac{1}{2}}(\theta,\phi) &\sim \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)e^{\frac{i\phi}{2}}\sqrt{\sin\theta} \\ &\sim e^{-\frac{i\phi}{2}}\frac{\cos\theta}{\sqrt{\sin\theta}} \end{aligned}$$

(b)

$$\begin{aligned} L_z Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) &= -\frac{\hbar}{2}Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) \\ -i\hbar\frac{\partial}{\partial\phi}Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) &= -\frac{\hbar}{2}Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) \\ \Rightarrow \qquad Y_{\frac{1}{2},-\frac{1}{2}} &\sim f(\theta)e^{\frac{-i\phi}{2}} \end{aligned}$$

$$\begin{aligned} L_- Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) &= 0 \\ \left(-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) &= 0 \\ \Rightarrow \qquad \frac{1}{\cot\theta}\frac{1}{f}\frac{df}{d\theta} &= \frac{1}{2} \\ f(\theta) &\sim \sqrt{\sin\theta} \\ \Rightarrow \qquad Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) &\sim e^{-\frac{i\phi}{2}}\sqrt{\sin\theta} \end{aligned}$$

We see that the results in (a) and (b) are contradictory to each other. ∎

7 In position representation,

$$\begin{aligned} L_z &= -i\hbar\frac{\partial}{\partial\phi} \\ L^2 &= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\phi^2}\right] \\ L_\pm &= \hbar e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right] \end{aligned}$$

and the $Y_{lm}(\theta,\phi)$'s satisfy the follow relations,

$$\begin{aligned} L_z Y_{lm}(\theta,\phi) &= m\hbar Y_{lm}(\theta,\phi) \\ L^2 Y_{lm}(\theta,\phi) &= l(l+1)\hbar^2 Y_{lm}(\theta,\phi) \\ Y_{l\,m\pm 1} &\sim \frac{1}{\hbar}L_\pm Y_{lm}(\theta,\phi) \end{aligned}$$

For $l=3$, $m=0$,

$$\begin{aligned} L_z Y_{3,0}(\theta,\phi) &= 0 \\ \Rightarrow \qquad Y_{3,0}(\theta,\phi) &= Y_{3,0}(\theta) \end{aligned}$$

$$
\begin{aligned}
L^2 Y_{3,0}(\theta,\phi) &= 3(3+1)\hbar^2 Y_{3,0}(\theta,\phi) \\
\Rightarrow -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] Y_{3,0}(\theta,\phi) &= 12\hbar^2 Y_{3,0}(\theta,\phi) \\
(1-w^2)\frac{d^2Y_{3,0}}{dw^2} - 2w\frac{dY_{3,0}}{dw} - 12\hbar^2 Y_{3,0} &= 0 \qquad (w=\cos\theta)
\end{aligned}
$$

which is the Legendre's equation,

$$\therefore \quad Y_{3,0} \sim P_3 \sim 5\cos^3\theta - 3\cos\theta$$

$$
\begin{aligned}
Y_{3,\pm1} &\sim e^{\pm i\phi}\frac{dY_{3,0}}{d\theta} \\
&\sim \sin\theta(5\cos^2\theta - 1)e^{\pm i\phi} \\
Y_{3,\pm2} &\sim e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right] Y_{3,\pm1} \\
&\sim \sin^2\theta\cos\theta e^{\pm 2i\phi} \\
Y_{3,\pm3} &\sim e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right] Y_{3,\pm2} \\
&\sim \sin^3\theta e^{\pm 3i\phi}
\end{aligned}
$$

Putting back in the normalization constants and adopting a commonly used phase convention, we have:

$$
\begin{aligned}
Y_{3,0}(\theta,\phi) &= \sqrt{\frac{7}{16\pi}}(5\cos^3\theta - 3\cos\theta) \\
Y_{3,\pm1}(\theta,\phi) &= \mp\sqrt{\frac{21}{64\pi}}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi} \\
Y_{3,\pm2}(\theta,\phi) &= \sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta e^{\pm 2i\phi} \\
Y_{3,\pm3}(\theta,\phi) &= \mp\sqrt{\frac{35}{64\pi}}\sin^3\theta e^{\pm 3i\phi} \qquad \blacksquare
\end{aligned}
$$

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