Problem Set 4, Physics 622 (Fall 2001)
Solutions

1 (a) From Eq(2.5.8) in Sakurai, $u_n(x)$ is the energy eigenfunction, the time dependent propagator for the simple harmonic oscillator is given by,

$$
\begin{aligned}
K(x'', t; x', t_0) &= \sum_n e^{-\frac{iE_n(t-t_0)}{\hbar}} u_n^*(x') u_n(x'') \\
&= \sum_n \left[e^{-\frac{iE_n t_0}{\hbar}} u_n(x') \right] \left[e^{-\frac{iE_n t}{\hbar}} u_n(x'') \right]
\end{aligned}
$$

Using Eq(2.5.17) in Sakurai and let $\alpha = \sqrt{\frac{m\omega}{\hbar}}$,

$$
\begin{aligned}
K(x'', t; x', t_0) &= \frac{\alpha}{\sqrt{\pi}} e^{-\frac{\alpha^2}{2}(x''^2+x'^2)} \sum_n \frac{1}{2^n n!} e^{-i\omega(n+\frac{1}{2})(t-t_0)} H_n(\alpha x') H_n(\alpha x'') \\
&= \frac{\alpha}{\sqrt{\pi}} e^{\frac{\alpha^2}{2}(x''^2+x'^2)} e^{-\frac{i\omega}{2}(t-t_0)} \left[e^{-\alpha^2(x''^2+x'^2)} \sum_n \frac{(e^{-i\omega(t-t_0)})^n}{2^n n!} H_n(\alpha x') H_n(\alpha x'') \right] \\
&= \frac{\alpha}{\sqrt{\pi}} e^{\frac{\alpha^2}{2}(x''^2+x'^2)} e^{-\frac{i\omega}{2}(t-t_0)} \frac{1}{\sqrt{1-e^{-i2\omega(t-t_0)}}} \times \\
&\qquad \exp\left[\frac{-\alpha^2(x''^2+x'^2) + 2\alpha^2 x'' x' e^{-i\omega(t-t_0)}}{1-e^{-i2\omega(t-t_0)}} \right]
\end{aligned}
$$

(using Eq(2.5.19) in Sakurai)

$$
= \sqrt{\frac{m\omega}{2\pi i\hbar \sin[\omega(t-t_0)]}} \exp\left\{ \frac{im\omega}{2\hbar} \frac{(x''^2+x'^2)\cos[\omega(t-t_0)] - 2x''x'}{\sin[\omega(t-t_0)]} \right\} \qquad \blacksquare
$$

(b) We try to show that the given $H_n(x)$ indeed satisfies the Hermite differential equation.

$$
\begin{aligned}
H_n(x) &= \frac{(2i)^n}{\pi^{1/2}} \int_{-\infty}^{\infty} dz\, z^n e^{-(z+ix)^2} \\
H_n'(x) &= -2i \frac{(2i)^n}{\pi^{1/2}} \int_{-\infty}^{\infty} dz\, z^n (z+ix) e^{-(z+ix)^2} \\
H_n''(x) &= 2\frac{(2i)^n}{\pi^{1/2}} \int_{-\infty}^{\infty} dz\, z^n [1 - 2(z+ix)^2] e^{-(z+ix)^2}
\end{aligned}
$$

$$
\begin{aligned}
& H_n'' - 2xH_n' + 2nH \\
&= \frac{(2i)^n}{\pi^{1/2}} \int_{-\infty}^{\infty} dz \left\{ z^n [2 - 4(z+ix)^2 + 4xi(z+ix) + 2n] e^{-(z+ix)^2} \right\} \\
&= \frac{(2i)^n}{\pi^{1/2}} \left[2(n+1) \int_{-\infty}^{\infty} dz\, z^n e^{-(z+ix)^2} - 4 \int_{-\infty}^{\infty} dz\, z^{n+1} (z+ix) e^{-(z+ix)^2} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(2i)^n}{\pi^{1/2}}\left\{2(n+1)\int_{-\infty}^{\infty} dz\, z^n e^{-(z+ix)^2} - 2\int_{-\infty}^{\infty} z^{n+1}e^{-(z+ix)^2}d[(z+ix)^2]\right\} \\
&= \frac{(2i)^n}{\pi^{1/2}}\left\{2(n+1)\int_{-\infty}^{\infty} dz\, z^n e^{-(z+ix)^2} + 2\left[z^{n+1}e^{-(z+ix)^2}\right]_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty}(n+1)z^n e^{-(z+ix)^2}dz\right\} \\
&= 0 \quad \blacksquare
\end{aligned}
$$

(c) The Hamiltonian of the simple harmonic oscillator can be written as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

where a, $a^\dagger$ are the annihilation and creation operator respectively. Using the commutator relations:

$$
\begin{aligned}
[a, a^\dagger] &= 1 \\
[a, H] &= \hbar\omega a,
\end{aligned}
$$

it can be proved that

$$aH^n = (\hbar\omega + H)^n a.$$

$$
\begin{aligned}
a|\lambda, t\rangle &= a\, e^{-\frac{iHt}{\hbar}}|\lambda\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{it}{\hbar}\right)^n aH^n|\lambda\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{it}{\hbar}\right)^n(\hbar\omega + H)^n a|\lambda\rangle \\
&= \lambda\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{it}{\hbar}\right)^n(\hbar\omega + H)^n|\lambda\rangle \\
&= \lambda e^{-i\omega t}e^{\frac{-iHt}{\hbar}}|\lambda\rangle \\
&= \lambda e^{-i\omega t}|\lambda, t\rangle
\end{aligned}
$$

Therefore, $|\lambda, t\rangle$ is an eigenfunction of a (with eigenvalue $\lambda e^{-i\omega t}$) and thus a coherent state. $\blacksquare$

2 (a)

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \quad , \quad p = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)$$

$$
\begin{aligned}
a|\lambda\rangle &= \lambda|\lambda\rangle \\
\langle\lambda|a|\lambda\rangle &= \lambda \\
\langle\lambda|a^\dagger|\lambda\rangle &= \langle\lambda a|\lambda\rangle = \lambda^*
\end{aligned}
$$

$$
\begin{aligned}
\langle\lambda|x|\lambda\rangle &= \sqrt{\frac{\hbar}{2m\omega}}(\lambda^* + \lambda) \\
\langle\lambda|x^2|\lambda\rangle &= \frac{\hbar}{2m\omega}\langle\lambda|(a^\dagger + a)(a^\dagger + a)|\lambda\rangle = \frac{\hbar}{2m\omega}\langle\lambda|a^{\dagger 2} + a^2 + aa^\dagger + a^\dagger a|\lambda\rangle \\
&= \frac{\hbar}{2m\omega}\langle\lambda|a^{\dagger 2} + a^2 + 2a^\dagger a + 1|\lambda\rangle \\
&= \frac{\hbar}{2m\omega}[(\lambda^* + \lambda)^2 + 1] \\
\langle\lambda|p|\lambda\rangle &= i\sqrt{\frac{m\hbar\omega}{2}}(\lambda^* - \lambda) \\
\langle\lambda|p^2|\lambda\rangle &= -\frac{m\hbar\omega}{2}\langle\lambda|(a^\dagger - a)(a^\dagger - a)|\lambda\rangle = -\frac{m\hbar\omega}{2}\langle\lambda|a^{\dagger 2} + a^2 - aa^\dagger - a^\dagger a|\lambda\rangle \\
&= -\frac{\hbar}{2m\omega}\langle\lambda|a^{\dagger 2} + a^2 - 2a^\dagger a - 1|\lambda\rangle \\
&= -\frac{\hbar}{2m\omega}[(\lambda^* - \lambda)^2 - 1]
\end{aligned}
$$

$$
\Delta x^2 = \langle\lambda|x^2|\lambda\rangle - \langle\lambda|x|\lambda\rangle^2 = \frac{\hbar}{2m\omega}
$$

$$
\Delta p^2 = \langle\lambda|p^2|\lambda\rangle - \langle\lambda|p|\lambda\rangle^2 = \frac{m\hbar\omega}{2}
$$

$$
\Delta x \Delta p = \frac{\hbar}{2} \quad \blacksquare
$$

(b) By definition, a coherent state is an eigenstate of the annihilation operator a, hence

$$
\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)|\lambda\rangle = \lambda\,|\lambda\rangle
$$

In position space representation,

$$
\begin{aligned}
\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}\frac{\hbar}{i}\frac{d}{dx}\right)\psi_\lambda(x) &= \lambda\psi_\lambda(x) \\
\frac{d\psi_\lambda}{dx} &= -\frac{m\omega}{\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\lambda\right)\psi_\lambda \\
\psi_\lambda(x) &= C\,e^{-\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\lambda\right)^2} \quad \blacksquare
\end{aligned}
$$

3(2.14) (a)

$$\begin{aligned}
\langle p'|x|\alpha\rangle &= \int dx' \langle p'|x|x'\rangle\langle x'|\alpha\rangle \\
&= \int dx' x' \langle p'|x'\rangle\langle x'|\alpha\rangle \\
&= \int dx' x' \langle x'|p'\rangle^* \langle x'|\alpha\rangle \\
&= \int dx' x' \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ip'x'}{\hbar}} \langle x'|\alpha\rangle \\
&= i\hbar \frac{\partial}{\partial p'} \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle \\
&= i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle \quad \blacksquare
\end{aligned}$$

(b)

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}|\alpha\rangle &= \frac{p^2}{2m}|\alpha\rangle + \frac{m\omega^2}{2}x^2|\alpha\rangle \\
i\hbar\frac{\partial}{\partial t}\langle p'|\alpha\rangle &= \frac{\langle p'|p^2|\alpha\rangle}{2m} + \frac{m\omega^2}{2}\langle p'|x^2|\alpha\rangle \\
&= \frac{p'^2}{2m}\langle p'|\alpha\rangle + \frac{m\omega^2}{2} i\hbar\frac{\partial}{\partial p'}\langle p'|x|\alpha\rangle \\
&= \frac{p'^2}{2m}\langle p'|\alpha\rangle + \frac{m\omega^2}{2}\left(i\hbar\frac{\partial}{\partial p'}\right)\left(i\hbar\frac{\partial}{\partial p'}\right)\langle p'|\alpha\rangle \\
\Rightarrow \qquad i\hbar\frac{\partial}{\partial t}|\alpha\rangle &= \left[\frac{p^2}{2m} - \frac{m\hbar^2\omega^2}{2}\frac{\partial^2}{\partial p'^2}\right]\langle p'|\alpha\rangle \quad \blacksquare
\end{aligned}$$

From the symmetry between x and p in the coordinate and momentum space Schrödinger equation, we can deduce that the energy eigenfunctions in momentum space should be of the form

$$e^{-\frac{p^2}{2p_\circ^2}} H_n\left(\frac{p}{p_\circ}\right)$$

where $p_\circ = \sqrt{m\hbar\omega}$. $\blacksquare$

4(2.15) From Eq(2.3.45a) in Sakurai,

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$

$$\begin{aligned}
C(t) &= \langle 0|x(t)x(0)|0\rangle \\
&= \langle 0|x^2(0)|0\rangle\cos\omega t + \langle 0|p(0)x(0)|0\rangle\frac{\sin\omega t}{m\omega}
\end{aligned}$$

$$
\begin{aligned}
\langle 0|p(0)x(0)|0\rangle &= \frac{i\hbar}{2}\langle 0|(a^\dagger - a)(a^\dagger + a)|0\rangle \\
&= -\frac{i\hbar}{2}\langle 0|aa^\dagger|0\rangle \\
&= -\frac{i\hbar}{2} \\
\langle 0|x^2(0)|0\rangle &= \int dx' \langle 0|x^2(0)|x'\rangle\langle x'|0\rangle \\
&= \int dx' x'^2 |\langle x'|0\rangle|^2 \\
&= \frac{\alpha}{\sqrt{\pi}}\int dx' x'^2 e^{-\alpha^2 x'^2} \qquad (\alpha = \sqrt{\tfrac{m\omega}{\hbar}}) \\
&= \frac{\alpha}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2\alpha^3} \\
&= \frac{\hbar}{2m\omega}
\end{aligned}
$$

Gathering the above results, we have

$$
C(t) = \frac{\hbar}{2m\omega}e^{-i\omega t} \quad \blacksquare
$$

5(2.17)

$$
\begin{aligned}
\langle 0|e^{ikx}|0\rangle &= \int dx' \langle 0|e^{ikx}|x'\rangle\langle x'|0\rangle \\
&= \int dx' \langle 0|e^{ikx'}|x'\rangle\langle x'|0\rangle \\
&= \int dx' e^{ikx'}|\langle x'|0\rangle|^2 \\
&= \frac{\alpha}{\sqrt{\pi}}\int dx' e^{ikx'}e^{-\alpha^2 x'^2} \qquad (\alpha = \sqrt{\tfrac{m\omega}{\hbar}}) \\
&= \frac{\alpha}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\alpha}e^{-\frac{k^2}{4\alpha^2}} \\
&= e^{-\frac{k^2\hbar}{4m\omega}}
\end{aligned}
$$

From Eq(2.3.34) in Sakurai, $\langle 0|x^2|0\rangle = \dfrac{\hbar}{2m\omega}$, therefore

$$
\langle 0|e^{ikx}|0\rangle = \exp\left[-\frac{k^2}{2}\langle 0|x^2|0\rangle\right] \quad \blacksquare
$$

6(2.20) (a) For $x > 0$, $\psi(x)$ obeys the same differential equation as the simple harmonic oscillator, but only those solutions $\psi_n(x=0) = 0$ are acceptable in the present problem. Noticing that $H_n(0) \neq 0$ for even n, the eigenfunctions for the given potential are those of the simple harmonic oscillator with odd n, and the ground state energy is therefore,

$$E = \frac{3}{2}\hbar\omega \quad \blacksquare$$

(b) The ground state wavefunction vanishes for $x < 0$ and is the same as the first excited state wavefunction of the simple harmonic oscillator for $x \geq 0$, thus

$$\psi_0(x) = \begin{cases} 0 & x < 0 \\ 2\sqrt{\frac{\alpha^3}{\sqrt{\pi}}}\, x\, e^{-\frac{\alpha^2 x^2}{2}} & x \geq 0 \end{cases} \qquad (\alpha = \sqrt{\tfrac{m\omega}{\hbar}})$$

$$\begin{aligned} \langle x^2 \rangle &= \frac{4\alpha^3}{\sqrt{\pi}} \int_0^\infty dx\, x^4\, e^{-\alpha^2 x^2} \\ &= \frac{4\alpha^3}{\sqrt{\pi}} \frac{3\sqrt{\pi}}{8\alpha^5} \\ &= \frac{3\hbar}{2m\omega} \quad \blacksquare \end{aligned}$$

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