Problem Set 2, Physics 622 (Fall 2001)
Solutions

1 The equation of motion $\langle \hat{\vec{L}} \rangle$ is given by

$$\frac{d\langle \hat{\vec{L}} \rangle}{dt} = \frac{1}{i\hbar}\langle \hat{\vec{L}}, \hat{H} \rangle + \left\langle \frac{\partial \hat{\vec{L}}}{\partial t} \right\rangle$$

where $\hat{H} = \frac{p^2}{2m} + V$ and $\hat{\vec{L}} = \frac{\hbar}{i}\vec{r} \times \vec{\nabla}$.

$$\begin{aligned} [\hat{\vec{L}}, \hat{H}] &= [\hat{\vec{L}}, V] \qquad (\because [\hat{\vec{L}}, \hat{p}] = 0) \\ &= \frac{\hbar}{i}\vec{r} \times (\vec{\nabla} V) \end{aligned}$$

$\hat{\vec{L}}$ is independent of t, thus $\frac{\partial \hat{\vec{L}}}{\partial t} = 0$, so we have

$$\begin{aligned} \frac{d\langle \hat{\vec{L}} \rangle}{\partial t} &= \frac{1}{i\hbar}\langle \frac{\hbar}{i}\vec{r} \times \vec{\nabla} V \rangle \\ &= \langle \vec{r} \times (-\vec{\nabla} V) \rangle \\ &= \langle \hat{\vec{T}} \rangle \qquad (\vec{F} = -\vec{\nabla} V) \quad \blacksquare \end{aligned}$$

2

$$\begin{aligned} |\psi(t=0)\rangle &= |\gamma\rangle \\ |\gamma\rangle &= \sum_j c_j |j\rangle \\ |\psi(t)\rangle &= e^{-\frac{i}{\hbar}\hat{H}t}|\gamma\rangle \qquad (\because \hat{H} \text{ is } t\text{-independent}) \\ &= \sum_j c_j e^{-\frac{i}{\hbar}\epsilon_j t}|j\rangle \quad \blacksquare \\ \langle \gamma | \psi(t) \rangle &= \sum_n c_n^* \sum_j c_j e^{-\frac{i}{\hbar}\epsilon_j t}\langle n | j \rangle \\ &= \sum_j |c_j|^2 e^{-\frac{i}{\hbar}\epsilon_j t} \\ P(B \text{ at time } t \text{ is in } |\gamma\rangle) &= |\langle \gamma | \psi(t) \rangle|^2 \\ &= \sum_{jk} |c_j|^2 |c_k|^2 e^{-\frac{i}{\hbar}(\epsilon_j - \epsilon_k)t} \\ &= \sum_{jk} |c_j|^2 |c_k|^2 \cos\left[\frac{(\epsilon_j - \epsilon_k)}{\hbar}t\right] \quad \blacksquare \end{aligned}$$

3 (a)

$$
\begin{aligned}
\psi(x) &= \begin{cases} Ae^{ik_\circ x} & |x| < a \\ 0 & |x| > a \end{cases} \\
\Rightarrow \Delta x &= 2a
\end{aligned}
$$

$$
\begin{aligned}
\phi(k) &= \int \frac{dx}{2\pi} e^{-ikx} \psi(x) \\
&= \int_{-a}^{a} \frac{dx}{2\pi} e^{-ikx} A e^{ik_\circ x} \\
&= \frac{A}{\pi} \frac{sin[a(k-k_\circ)]}{k-k_\circ} \\
\Rightarrow \Delta k &= \frac{2\pi}{a}
\end{aligned}
$$

Therefore, $\Delta x \Delta k = 4\pi > 1$ ■

(b)

$$
\begin{aligned}
\psi(x) &= Be^{ik_\circ x - b(x-x_\circ)^2} \\
\Rightarrow \Delta x &= \frac{1}{\sqrt{2b}} \qquad \text{(comparing to standardized Gaussian distribution)} \\
\phi(k) &= \int \frac{dx}{2\pi} e^{-ikx} Be^{ik_\circ x - b(x-x_\circ)^2} \\
&= \frac{B}{2\sqrt{\pi b}} e^{-\frac{(k-k_\circ)^2}{4b} - ix_\circ(k-k_\circ)} \\
\Rightarrow \Delta k &= \sqrt{2b}
\end{aligned}
$$

Therefore, $\Delta x \Delta k = 1$ ■

4 For a free particle in one dimension,

$$
-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} = i\hbar \frac{\partial \Psi(x,t)}{\partial t}
$$

and the solution is

$$
\Psi(x,t) = \int dp\, \phi(p) e^{\frac{i}{\hbar}(px-Et)}. \qquad (E = \frac{p^2}{2m})
$$

Given $\Psi(x,0) = \psi(x)$, $\phi(p)$ can be determined as follow,

$$
\begin{aligned}
\psi(x) &= \int dp\, \phi(p) e^{\frac{i}{\hbar}px} \\
\Rightarrow \phi(p) &= \frac{1}{2\pi} \int dx\, \psi(x) e^{-\frac{i}{\hbar}px}
\end{aligned}
$$

The momentum space wavefunction $\Phi(p,t)$ can be obtained by taking the Fourier transform of $\Psi(x,t)$,

$$\Phi(p,t) = \frac{1}{2\pi}\int dx\,\Psi(x,t)e^{-\frac{i}{\hbar}px}$$

or by solving the Schrödinger equation in momentum space,

$$\frac{p^2}{2m}\Phi(p,t) = i\hbar\frac{\partial\Phi(p,t)}{\partial t}$$

with initial condition,

$$\Phi(p,0) = \frac{1}{2\pi}\int dx\,\Psi(x,0)e^{-\frac{i}{\hbar}px} = \phi(p).$$

Therefore,

$$\begin{aligned}\Phi(p,t) &= e^{-\frac{i}{\hbar}\frac{p^2}{2m}t}\Phi(p,0) \\ &= e^{-\frac{i}{\hbar}Et}\phi(p)\end{aligned}$$

(a) From Problem 3(a),

$$\phi(p) = \frac{A\hbar}{\pi}\frac{\sin[\frac{a}{\hbar}(p-p_\circ)]}{p-p_\circ} \qquad (p_\circ = \hbar k_\circ)$$

Therefore,

$$\begin{aligned}\Psi(x,t) &= \frac{A\hbar}{\pi}\int dp\,\frac{\sin[\frac{a}{\hbar}(p-p_\circ)]}{p-p_\circ}e^{\frac{i}{\hbar}(px-\frac{p^2}{2m}t)} \quad \blacksquare \\ \Phi(p,t) &= \frac{A\hbar}{\pi}\frac{\sin[\frac{a}{\hbar}(p-p_\circ)]}{p-p_\circ}e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \quad \blacksquare\end{aligned}$$

(b) From Problem 3(b)

$$\phi(p) = \frac{B}{2\sqrt{\pi b}}e^{-\frac{(p-p_\circ)^2}{4b\hbar^2}-\frac{i}{\hbar}x_\circ(p-p_\circ)}$$

Therefore,

$$\begin{aligned}\Psi(x,t) &= \frac{B}{2\sqrt{\pi b}}\int dp\,e^{-\frac{(p-p_\circ)^2}{4b\hbar^2}-\frac{i}{\hbar}x_\circ(p-p_\circ)}e^{\frac{i}{\hbar}(px-\frac{p^2}{2m}t)} \\ &= \frac{B\hbar}{\sqrt{1+iDt}}e^{\frac{-b(x-x_\circ-\frac{p_\circ}{m}t)^2}{1+iDt}}e^{\frac{i}{\hbar}(p_\circ x-\frac{p^2}{2m}t)} \quad \blacksquare \qquad (D = \frac{2b\hbar}{m}) \\ \Phi(p,t) &= \frac{B}{2\sqrt{\pi b}}e^{-\frac{(p-p_\circ)^2}{4b\hbar^2}-\frac{i}{\hbar}x_\circ(p-p_\circ)}e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \quad \blacksquare\end{aligned}$$

5(1.11) Let $|1\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|2\rangle = \begin{bmatrix}0\\1\end{bmatrix}$, then H can be written as

$$H = \begin{bmatrix}H_{11} & H_{12}\\ H_{12} & H_{22}\end{bmatrix}.$$

The eigenvalues λ are determined from the secular equation $\det(H - \lambda I) = 0$. The eigenkets $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are obtained from

$$\begin{aligned} (H_{11} - \lambda)x_1 + H_{12}x_2 &= 0 \\ H_{12}x_1 + (H_{22} - \lambda)x_1 &= 0. \end{aligned}$$

This leads to

$$\begin{aligned} \lambda_1 &= \frac{H_{11} + H_{22}}{2} + \Delta \\ \lambda_1 &= \frac{H_{11} + H_{22}}{2} - \Delta \end{aligned} \qquad \text{where} \quad \Delta = \sqrt{\frac{(H_{11} - H_{22})^2}{4} + H_{12}^2} \quad \blacksquare$$

Let $H_0 = H_{11} - \lambda_1$, then $H_{22} - \lambda_2 = -(H_{11} - \lambda_1) = -H_0$, the eigenkets are

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{H_0^2 + H_{12}^2}} \begin{bmatrix} -H_{12} \\ H_0 \end{bmatrix} \\ X_2 &= \frac{1}{\sqrt{H_0^2 + H_{12}^2}} \begin{bmatrix} H_0 \\ H_{12} \end{bmatrix} \end{aligned} \quad \blacksquare$$

6(1.17) $[A_1, H] = 0 \Rightarrow \exists\ \{|a_1 n\rangle\}$ *s.t.* $A_1|a_1 n\rangle = a_1|a_1 n\rangle,\ H|a_1 n\rangle = n|a_1 n\rangle$
$[A_2, H] = 0 \Rightarrow \exists\ \{|a_2 n\rangle\}$ *s.t.* $A_2|a_2 n\rangle = a_2|a_2 n\rangle,\ H|a_2 n\rangle = n|a_2 n\rangle$
We are given that A_1 and A_2 do not commute, *i.e.* $[A_1, A_2]|\psi\rangle \neq 0$ for some $|\psi\rangle$. Assume the energy eigenkets are non-degenerate, we can write

$$|a_1 n\rangle = |a_2 n\rangle = |a_1 a_2 n\rangle,$$

then

$$\begin{aligned} [A_1, A_2]|a_1 a_2 n\rangle &= (a_1 a_2 - a_2 a_1)|a_1 a_2 n\rangle \\ &= 0 \end{aligned}$$

which is possible only if

(1) $[A_1, A_2] = 0$ which is given untrue, or

(2) $[A_1, A_2]|a_1 a_2 n\rangle = 0$ for this particular state $|a_1 a_2 n\rangle$ regardless of any assumptions. One such case is when $a_1 = a_2 = 0$, then $[A_1, A_2]|a_1 a_2 n\rangle = 0$ no matter $|a_1 a_2 n\rangle$ is degenerate or not, therefore no conclusion can be drawn. (e.g. for a spherically symmetric potential, we have $[L_x, H] = [L_z, H] = 0$ and $[L_x, L_z] \neq 0$ and yet the S-states for which, $L_x|n, l{=}0, m_l{=}0\rangle = L_z|n, l{=}0, m_l{=}0\rangle = 0$, are non-degenerate.)

Therefore, except for the situation mentioned in (2), the energy eigenkets must be degenerate. $\blacksquare$

7(1.18) (a) Notice that for any complex number λ,

$$(\langle\alpha| + \lambda^*\langle\beta|) \cdot (|\alpha\rangle + \lambda|\beta\rangle) \geq 0$$

Set $\lambda = -\dfrac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$, we have

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2 \quad \blacksquare$$

(b)

$$\begin{aligned}
\Delta A|\alpha\rangle &= \lambda\Delta B|\alpha\rangle \\
\Rightarrow \langle\alpha|\Delta A &= \lambda^*\langle\alpha|\Delta B = -\lambda\langle\alpha|\Delta B \qquad \text{(since } \lambda \text{ is purly imaginary)} \\
\langle(\Delta A)^2\rangle &= \lambda\langle\Delta A\Delta B\rangle \\
\langle(\Delta B)^2\rangle &= -\lambda^{-1}\langle\Delta A\Delta B\rangle \\
\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle &= -(\langle\Delta A\Delta B\rangle)^2 \\
&= -\left(\frac{\langle[A,B]\rangle + \langle\{\Delta A, \Delta B\}\rangle}{2}\right)^2 \\
&= -\frac{1}{4}(\langle[A,B]\rangle)^2 \qquad (\langle\{\Delta A, \Delta B\}\rangle = 0) \\
&= \frac{1}{4}|\langle[A,B]\rangle|^2 \qquad (\langle[A,B]\rangle \text{is purly imagnary}) \quad \blacksquare
\end{aligned}$$

(c)

$$\begin{aligned}
\psi(x') = \langle x'|\alpha\rangle &= \frac{1}{(2\pi d^2)^{1/4}} e^{\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}} \\
\hat{p}\psi(x') = \frac{\hbar}{i}\frac{\partial\psi}{\partial x'} &= \frac{\hbar}{i(2\pi d^2)^{1/4}}\left[\frac{i\langle p\rangle}{\hbar} - \frac{2(x'-\langle x\rangle)}{4d^2}\right] e^{\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}} \\
&= \left[\langle p\rangle - \frac{\hbar(x'-\langle x\rangle)}{2d^2 i}\right]\psi(x') \\
(x'-\langle x\rangle)\psi(x') &= -i\frac{2d^2}{\hbar}(\hat{p} - \langle p\rangle)\psi(x') \\
\therefore \langle x'|\Delta x|\alpha\rangle &= \lambda\langle x'|\Delta p|\alpha\rangle \qquad \text{where } \lambda = -i\frac{2d^2}{\hbar} \text{ is purly imaginary.} \quad \blacksquare
\end{aligned}$$

8(1.28) (a) $[x, F(p_x)]_{classical} = \dfrac{\partial x}{\partial x}\dfrac{\partial F}{\partial p_x} - \dfrac{\partial x}{\partial p_x}\dfrac{\partial F}{\partial x} = \dfrac{\partial F(p_x)}{\partial p_x}$ $\blacksquare$

(b) Using the commutator relation $[A, BC] = B[A, C] + [A, B]C$, it can be proved that $[x, p_x^n] = ni\hbar p_x^{n-1}$.

$$
\begin{aligned}
[x, e^{\frac{ip_x a}{\hbar}}] &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{ia}{\hbar}\right)^n [x, p_x^n] \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{ia}{\hbar}\right)^n ni\hbar p_x^{n-1} \\
&= -a \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\frac{ia}{\hbar}\right)^{n-1} p_x^{n-1} \\
&= -a \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{ia}{\hbar}\right)^n p_x^n \\
&= -ae^{\frac{iap}{\hbar}} \quad \blacksquare
\end{aligned}
$$

(c)

$$
\begin{aligned}
xe^{\frac{ip_x a}{\hbar}} |x'\rangle &= \left(e^{\frac{ip_x a}{\hbar}} x + [x, e^{\frac{ip_x a}{\hbar}}]\right) |x'\rangle \\
&= \left(e^{\frac{ip_x a}{\hbar}} x' - ae^{\frac{ip_x a}{\hbar}}\right) |x'\rangle \\
&= (x' - a)e^{\frac{ip_x a}{\hbar}} |x'\rangle
\end{aligned}
$$

Therefore, $e^{\frac{ip_x a}{\hbar}} |x'\rangle$ is an eigenstate of x with eigenvalue $x' - a$ $\blacksquare$

– END –