Problem Set 1, Physics 622 (Fall 2001)
Solutions

1 (a) Bohr-Sommerfeld quantization,

$$\oint p\,dx = nh$$

$$F = -kx \quad \Rightarrow \quad p = \sqrt{2m\left(E - \frac{1}{2}kx^2\right)}$$

Turning points : $x_a = -\sqrt{\frac{2E}{k}}$ and $x_b = \sqrt{\frac{2E}{k}}$ where $p = 0$, thus

$$\begin{aligned} 2\int_{x_a}^{x_b} \sqrt{m(2E_n - kx^2)}\,dx &= nh \\ E_n &= n\hbar\omega \quad (\omega = \sqrt{k/m}) \quad \blacksquare \end{aligned}$$

Period, $T_n = \oint \dfrac{\partial p}{\partial E}\,dx = \dfrac{4}{\omega}\displaystyle\int_0^{x_b} \dfrac{dx}{\sqrt{\frac{2E_n}{k} - x^2}} = \dfrac{2\pi}{\omega} \quad \blacksquare$

Amplitude, $A_n = \sqrt{\dfrac{2E_n}{k}} = \sqrt{\dfrac{2n\hbar\omega}{k}} \quad \blacksquare$

(b) In three dimensions, the system can be considered as three independent one-dimensional oscillators, therefore from (a),

$$\begin{aligned} E_{n_x,n_y,n_z} &= (n_x + n_y + n_z)\hbar\omega \qquad \text{where } n_x, n_y, n_z = 0, 1, 2, \ldots \\ E_N &= N\hbar\omega \qquad (N = n_x + n_y + n_z) \quad \blacksquare \end{aligned}$$

<u>Note</u>: *This problem can also be solved by considering the radial component in spherical coordinates.*
Let $p_r = m\dot{r}$ and L be the angular momentum,

$$\vec{F} = -k\vec{r} \quad \Rightarrow \quad E = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} + \frac{kr^2}{2}$$

L is conserved and quantized, $L = \sqrt{l(l+1)}\hbar$
Bohr-Sommerfeld quantization,

$$\oint p_r\,dr = (n + \frac{3}{4})h$$

$$
\begin{aligned}
2\int_{r_a}^{r_b}\sqrt{2m\left[E_{n,l}-\frac{k}{2}r^2-\frac{\hbar^2 l(l+1)}{2m}\frac{1}{r^2}\right]}\,dr &= (n+\frac{3}{4})h \qquad (p_r(r_a)=p_r(r_b)=0)\\
\frac{\pi}{\omega}[E_{n,l}-\sqrt{l(l+1)}\hbar\omega] &= (n+\frac{3}{4})h\\
E_n &= [2n+\sqrt{l(l+1)}+\frac{3}{2}]\hbar\omega
\end{aligned}
$$

(exact: $E_N = (N+\frac{3}{2})\hbar\omega$ where $N = 2n+l$) ■

2 (a) Wave equation : $\partial_t^2\phi(x,t) = c^2\partial_x^2\phi(x,t)$

let $\phi(x,t) = X(x)T(t)$,

$$
\frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -k^2
$$

$$
X''(x)+k^2X(x)=0 \quad \Rightarrow \quad X(x) = C\sin kx + D\cos kx
$$

$$
\begin{aligned}
\phi(0,t)=0 &\quad\Rightarrow\quad D=0\\
\phi(L,t)=0 &\quad\Rightarrow\quad \sin kL = 0\\
&\quad\Rightarrow\quad k_j=\frac{j\pi}{L} \quad j=1,2,\ldots
\end{aligned}
$$

$$
\therefore \quad X_j(x) = C_j\sin\frac{j\pi}{L}x
$$

$$
\begin{aligned}
\ddot{T}(t)+\omega_j^2T &= 0 \quad (\omega_j = ck_j)\\
\Rightarrow \quad T_j(t) &= A_j\sin\omega_j t + B_j\cos\omega_j t
\end{aligned}
$$

$\therefore$ there is a countable set of solutions,

$$
\phi_j(x,t) = (A_j\sin\omega_j t + B_j\cos\omega_j t)\sin k_j x \quad ■
$$

(b) The first term in the expression of u is the kinetic energy term and the second term is the potential energy term.

$$
\partial_t\phi_j = \omega_j(A\cos\omega_j t - B\sin\omega_j t)\sin k_j x
$$

$$
\partial_x\phi_j = k_j(A\sin\omega_j t + B\cos\omega_j t)\cos k_j x
$$

$$
\therefore \quad u_j(x,t) = \frac{\rho\omega_j^2}{2}[(A^2\cos^2\omega_j t + B^2\sin^2\omega_j t)\sin^2 k_j x + (A^2\sin^2\omega_j t + B^2\cos^2\omega_j t)\cos^2 k_j x] \quad ■
$$

(c) initial conditions:

$$
\begin{aligned}
\phi(x,0) &= \begin{cases} \dfrac{h}{x_\circ}x & x\le x_\circ \\[2ex] \dfrac{h}{L-x_\circ}(L-x) & x > x_\circ \end{cases}\\
\dot{\phi}(x,0) &= 0
\end{aligned}
$$

$$\phi_j(x,t) = \sum_{j=1}^{\infty} (A_j \sin\omega_j t + B_j \cos\omega_j t) \sin k_j x$$

$$\begin{aligned}
\therefore \quad \sum_{j=1}^{\infty} B_j \sin k_j x &= \phi(x,0) \\
B_j &= \frac{2}{L}\int_0^L \phi(x,0) \sin k_j x \, dx \\
&= \frac{2h \sin k_j x_\circ}{k_j^2 x_\circ (L - x_\circ)}
\end{aligned}$$

$$\dot{\phi}_j(x,t) = \sum_{j=1}^{\infty} \omega_j (A_j \cos\omega_j t - B_j \sin\omega_j t) \sin k_j x$$

$$\begin{aligned}
\therefore \quad \sum_{j=1}^{\infty} \omega_j A_j \sin k_j x &= 0 \\
\Rightarrow \quad A_j &= 0
\end{aligned}$$

$$\therefore \quad \phi_j(x,t) = \sum_{j=1}^{\infty} \frac{2h \sin k_j x_\circ}{k_j^2 x_\circ (L - x_\circ)} \sin k_j x \cos\omega_j t$$

3 Let m, l and t be the dimension of mass, length and time respectively. Consider the dimension of $\hbar$, c, k_B and T,

$$[\hbar] = \frac{ml^2}{t} \qquad [c] = \frac{l}{t} \qquad [k_B T] = \frac{ml^2}{t^2}$$

Since [energy density] $= \dfrac{m}{lt^2}$, we have

$$[\hbar]^a [c]^b [k_B T]^c = \frac{m}{lt^2}$$

$$\begin{aligned}
\Rightarrow \qquad a + c &= 1 \\
2a + b + 2c &= -1 \\
a + b + 2c &= 2
\end{aligned}$$

Solving the equations, we got $a = -3$, $b = -3$, $c = 4$. Therefore

$$\text{energy density} \sim \frac{k_B^4 T^4}{\hbar^3 c^3} \quad \blacksquare$$

Without $\hbar$, the system of equations becomes,

$$\begin{aligned}
c &= 1 \\
b + 2c &= -1 \\
b + 2c &= 2
\end{aligned}$$

which has no solutions. $\blacksquare$

4 The Klein-Gordon equation is second order in time, therefore both ψ and $\partial_t\psi$ can be given arbitrary initial values. In such case, no conserved probability function P can be constructed which depends on ψ only and not on $\partial_t\psi$, to see this, consider

$$\frac{\partial}{\partial t}\int P(\psi)\,dx = \int \frac{\partial P}{\partial t}\,dx = \int \frac{\partial P}{\partial \psi}\frac{\partial \psi}{\partial t}\,dx\ .$$

If the above expression is to vanish for arbitrary ψ, it is necessary that $\partial_t\psi$ be defined in terms of ψ. But this implies a first order wave equation.

It is possible to set up a conserved P that depends on both ψ and $\partial_t\psi$, e.g.

$$P = \frac{i\hbar}{2mc^2}(\psi^*\partial_t\psi - \psi\partial_t\psi^*)\ .$$

But P is not always positive (try $\psi \sim \exp(-i\omega t)$), so it cannot be interpreted as a position probability density function.

5 (1.5) (a) Write $|a'\rangle = |a_1\rangle, |a''\rangle = |a_2\rangle, \ldots$ and insert two identity operators,

$$\begin{aligned}
|\alpha\rangle\langle\beta| &= \sum_i\sum_j |a_i\rangle\langle a_i|\alpha\rangle\langle\beta|a_j\rangle\langle a_j| \\
&= \sum_i\sum_j |a_i\rangle\langle a_j| \times (\langle a_i|\alpha\rangle\langle a_j|\beta\rangle^*)
\end{aligned}$$

Hence, $|\alpha\rangle\langle\beta| = [\langle a_i|\alpha\rangle\langle a_j|\beta\rangle^*]$, where the expression inside square bracket is the (i,j) matrix element. ■

(b) $|\alpha\rangle = |s_z = \hbar/2\rangle = |+\rangle$, $|\beta\rangle = |s_x = \hbar/2\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$
Hence,

$$\begin{aligned}
|\alpha\rangle\langle\beta| &= \begin{pmatrix} \langle+|\alpha\rangle\langle+|\beta\rangle^* & \langle+|\alpha\rangle\langle-|\beta\rangle^* \\ \langle-|\alpha\rangle\langle+|\beta\rangle^* & \langle-|\alpha\rangle\langle-|\beta\rangle^* \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$ ■

6 (1.6) Since $|i\rangle$ and $|j\rangle$ are eigenkets of A, we have

$$A|i\rangle = a_i|i\rangle \quad A|j\rangle = a_j|j\rangle$$

For $|i\rangle + |j\rangle$ to be an eigenket of A, we must have, for some a,

$$A(|i\rangle + |j\rangle) = a_i|i\rangle + a_j|j\rangle = a(|i\rangle + |j\rangle)$$

$$(a_i - a)|i\rangle + (a_j - a)|j\rangle = 0$$

$$\Rightarrow \quad a_i = a_j = a$$

i.e. $|i\rangle$ and $|j\rangle$ are degenerate.

7 (1.14a) Let λ be the eigenvalues of A.

$$|A - \lambda I| = 0$$
$$\lambda(\lambda^2 - 1) = 0$$
$$\lambda = -1, 0, +1$$

The corresponding eigenvectors are,

$$\frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

There is no degeneracy. ■

8 (1.15) Let $|\alpha\rangle$ be an arbitrary ket. Since $|a', b'\rangle$ form a complete orthonormal set, we have

$$|\alpha\rangle = \sum_{a',b'} c_{a'b'} |a', b'\rangle$$

$$\begin{aligned} [A, B]|\alpha\rangle &= \sum_{a',b'} c_{a'b'}(AB - BA)|a', b'\rangle \\ &= \sum_{a',b'} c_{a'b'}(a'b' - b'a')|a', b'\rangle \\ &= 0 \end{aligned}$$

Since $|\alpha\rangle$ is arbitrary, $[A, B] = 0$. ■

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