Geophysical Fluid Dynamics

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Geophysical fluid dynamics is the study of the dynamics of fluid systems which are

1. shallow or thin, i.e., having a small aspect ratio in which the horizontal length scales is much larger than the vertical length scales,
2. stratified, i.e., the vertical variation in density is important, and
3. rapidly rotating.

1 Equation of motions in rotating frames

We begin by writing the Navier-Stokes equation in a frame of reference which is rotating at angular velocity $\vec{\Omega}$ with respect to an inertial frame. Referring to Fig. 1, if the *position vector* is at rest in the rotating frame, it undergoes circular motion with angular velocity $\vec{\Omega}$ as seen in the inertial frame, therefore,

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_I = \vec{\Omega} \times \vec{r}. \tag{1}$$

We use $(\cdot)_I$ and $(\cdot)_R$ to indicate quantities measured in the inertial and rotating frame respectively. Note that $\vec{r}_I = \vec{r}_R = \vec{r}$. Assume now $\vec{r}$ is moving in the rotating frame, then

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_R + \vec{\Omega} \times \vec{r}, \tag{2a}$$

$$\text{or} \qquad \vec{u}_I = \vec{u}_R + \vec{\Omega} \times \vec{r}. \tag{2b}$$

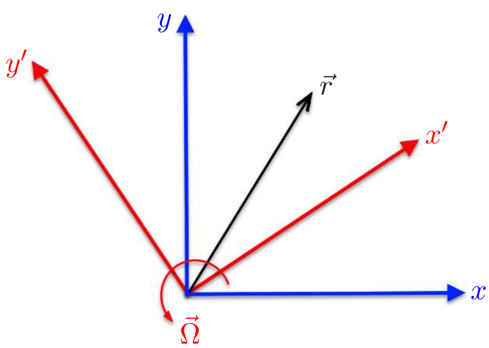

Figure 1: Coordinates (x', y') rotating at angular velocity $\vec{\Omega}$ with respect to inertial coordinates (x, y).

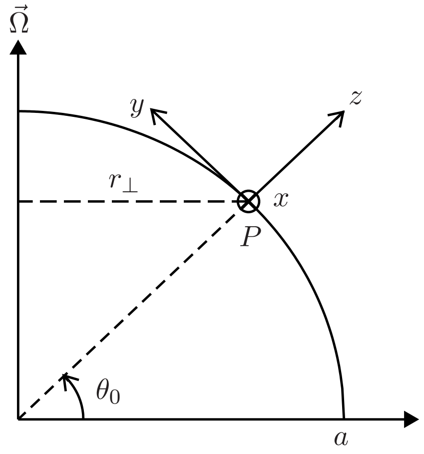

Figure 2: Tangent plane approximation.

Applying Eq. (1) to $\vec{u}_I$, we have

$$\begin{aligned} \left(\frac{\mathrm{d}\vec{u}_I}{\mathrm{d}t}\right)_I &= \left(\frac{\mathrm{d}\vec{u}_I}{\mathrm{d}t}\right)_R + \vec{\Omega} \times \vec{u}_I \\ &= \left(\frac{\mathrm{d}\vec{u}_R}{\mathrm{d}t}\right)_R + \vec{\Omega} \times \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_R + \vec{\Omega} \times (\vec{u}_R + \vec{\Omega} \times \vec{r}) \\ &= \left(\frac{\mathrm{d}\vec{u}_R}{\mathrm{d}t}\right)_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \,. \end{aligned} \tag{3}$$

We shall drop the subscribe "R" with the understanding that from now on all quantities are measured in the rotating frame.

In the geophysical context, the major body force is the conservative gravitational force $\vec{f}_{\text{grav}}$ which can be written in terms of a gravitational potential ϕ,

$$\vec{f}_{\text{grav}} = -\rho \nabla \phi \,. \tag{4}$$

Moreover, molecular viscosity is often negligible. Hence the Navier-Stokes equation in a reference frame attached to the rotating Earth is,

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} + 2\vec{\Omega} \times \vec{u} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\frac{1}{\rho}\nabla p - \nabla\phi \,, \tag{5}$$

where $\vec{\Omega}$ is the angular velocity of the Earth. The second and third terms on the left of Eq. (5) are the Coriolis force and the centrifugal acceleration respectively.

The centrifugal acceleration at a certain latitude θ_0 can be expressed in terms of a potential ϕ_c:

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \nabla \phi_c \tag{6}$$

$$\text{where} \qquad \phi_c = -\frac{1}{2} r_\perp^2 \, \Omega^2 \,, \tag{7}$$

and $r_\perp$ is the perpendicular distance from the Earth's axis of rotation as shown in Fig. 2. Define the *geopotential* $\Phi \equiv \phi + \phi_c$, Eq. (5) becomes,

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi \,. \tag{8}$$

The surfaces of constant Φ are called geopotential surfaces.

The continuity equation written as,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\vec{u} = 0\,, \tag{9}$$

takes the same form in both the inertial and rotating frames. Physically, $\mathrm{D}\rho/\mathrm{D}t$ is the change in density following a fluid particle and thus is independent of reference frames. For the divergence term, we have,

$$\nabla\cdot\vec{u}_I = \nabla\cdot\vec{u}_R + \nabla\cdot(\Omega\times\vec{r}) = \nabla\cdot\vec{u}_R\,. \tag{10}$$

2 Tangent plane approximation

Let us consider a relatively small region near a point P on the Earth's surface at latitude θ_0. Furthermore, if we are interested in phenomena on a somewhat smaller scale such that the curvature of the Earth is less important, we can use a local Cartesian coordinate system (x, y, z) to describe the fluid motions as shown in Fig. 2. Assuming the geopotential surface at P is a perfect sphere. We orient the Cartesian coordinate such that the xy-plane is tangent to the geopotential surface at P. Then $\nabla\Phi$ defines the z-direction:

$$\nabla\Phi = g\hat{k}\,. \tag{11}$$

The symbol g now includes contributions from both the gravitational and centrifugal forces. In this local Cartesian coordinates, $\vec{\Omega} = (0, \Omega_y, \Omega_z)$ and $\vec{u} = (u, v, w)$, therefore the momentum equations are:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega_z v + 2\Omega_y w = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \tag{12a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + 2\Omega_z u = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,, \tag{12b}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} - 2\Omega_y u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,. \tag{12c}$$

The above equations can be simplified if we adopt the *traditional approximation* in which terms involving the horizontal component of the Earth's angular velocity Ω_y are neglected:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \tag{13a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,, \tag{13b}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,, \tag{13c}$$

where we have defined the Coriolis parameter $f = 2\Omega_z$. The traditional approximation has no convincing general justification, it must be justified on a case by case basis.

In general, the Coriolis parameter f is a function of latitude θ: $f(\theta) = 2\Omega\sin\theta$. At a particular latitude θ_0, if we take $f \approx f_0 = 2\Omega\sin\theta_0 =$ constant, we have made what is known as the *f-plane approximation*. When the variation in latitude $\Delta\theta$ is small, referring to Fig. 2, we can account for the change of f with latitude as follow:

$$\begin{aligned} f &\approx 2\Omega \sin(\theta_0 + \Delta\theta) \\ &\approx 2\Omega(\sin\theta_0 + \Delta\theta\cos\theta_0) \\ &\approx 2\Omega\sin\theta_0 + \frac{2\Omega\cos\theta_0}{a} y \qquad (y \approx a\,\Delta\theta) \\ &= f_0 + \beta y\,, \end{aligned} \tag{14}$$

where a is the radius of the Earth. This is called the *β-plane approximation.*

The Earth's atmosphere is well approximated by an ideal gas with constant composition, so an appropriate *equation of state* for the atmosphere is the ideal gas law,

$$p = \rho R T\,, \tag{15}$$

where R is the gas constant for air and T is the temperature. We quote without proof the *thermodynamic energy equation* [1] for an ideal gas which describes the time evolution of T,

$$c_p \frac{\mathrm{D}T}{\mathrm{D}t} - \frac{1}{\rho}\frac{\mathrm{D}p}{\mathrm{D}t} = Q\,. \tag{16}$$

Above, c_p is the specific heat at constant pressure and Q is the diabatic heating rate per unit mass.

The continuity equation Eq. (9), the momentum equations Eq. (13), the equation of state Eq. (15) and the thermodynamic equation Eq. (16) comprise a closed set of equations describing atmospheric flows.

3 Geostrophic and hydrostatic balance

Suppose we are interested in motions associated with large-scale weather systems. These phenomena are dominated by motions in the horizontal directions (we shall see this is a consequence of the shallowness of the atmosphere). Some typical scales for weather systems at mid-latitude are as follow:

horizontal length scale	L	10^6 m (1000 km)
horizontal velocity scale	U	10 ms^{-1}
vertical length scale	H	10^4 m (10 km)
vertical velocity scale	W	10^{-2} ms^{-1}
Coriolis parameter	f_0	10^{-4} s
time scale (advective)	T	10^5 s ($\approx$ 1 day)

We have estimated the Coriolis parameter as $f_0 = 2\Omega\sin\theta \approx 2(2\pi/1\text{day})$. We assume an "advective scaling" for the typical time scale, i.e., $T \sim L/U$.

Using the above values, we now perform a *scale analysis* on the vertical momentum equation, Eq. (13c):

$$\frac{\partial w}{\partial t} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,. \tag{17}$$

$$\frac{W}{T} \qquad \frac{UW}{L} \qquad \frac{W^2}{H}$$

$$10^{-7} \qquad 10^{-7} \qquad 10^{-8} \qquad\qquad 10 \quad (\text{ms}^{-2})$$

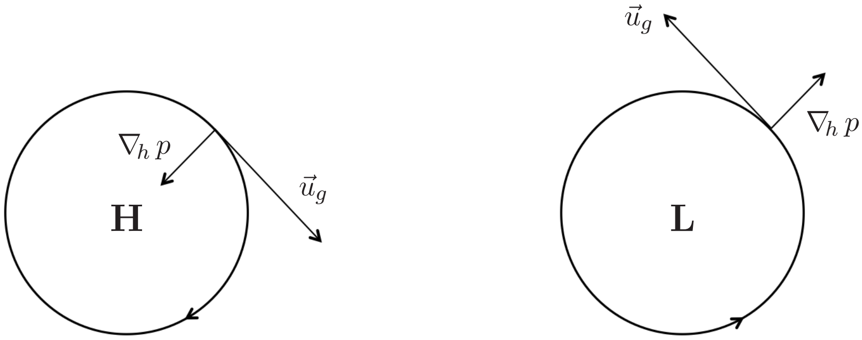

Figure 3: Geostrophic wind is anti-cyclonic around a center of high pressure (left) and cyclonic around a center of low pressure (right) in the Northern Hemisphere ($f > 0$).

Keeping the largest terms, we have the **hydrostatic balance**:

$$\frac{\partial p}{\partial z} \approx -\rho g \,. \tag{18}$$

In other words, we have neglected the vertical acceleration Dw/Dt.

For the x-momentum equation, scale analysis goes as follow,

$$\frac{\partial u}{\partial t} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \,. \tag{19}$$

$$\begin{array}{ccccc} \frac{U}{T} & \frac{U^2}{L} & \frac{WU}{H} & f_0 U & \\ 10^{-4} & 10^{-4} & 10^{-5} & 10^{-3} & (\mathrm{ms}^{-2}) \end{array}$$

Together with a similar scale analysis on the y-momentum equation, we obtain the **geostrophic balance**:

$$fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x} \,, \tag{20a}$$

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \,, \tag{20b}$$

which is a balance between the horizontal pressure gradients and the Coriolis force associated with the horizontal velocities. The velocity $\vec{u}_g = (u_g, v_g)$ that satisfies Eq. (20) exactly is called the geostrophic velocity.

From the geostrophic balance Eq. (20), we see that at a fixed height z_0,

$$(u_g\,\hat{\imath} + v_g\,\hat{\jmath}) \cdot \underbrace{\left(\frac{\partial p}{\partial x}\,\hat{\imath} + \frac{\partial p}{\partial y}\,\hat{\jmath}\right)}_{\nabla_h p} = 0 \,. \tag{21}$$

Recall that $\nabla_h p$ is perpendicular to the isobars: $p(x, y, z_0)$=constant. Therefore, at each fixed height, the geostrophic wind blows along the isobars, i.e., (u_g, v_g) is tangent to the isobars. The direction of the geostrophic wind can also be readily deduced from Eq. (20), as illustrated in Fig. 3 for the case of the Northern Hemisphere ($f > 0$).

4 Validity condition for geostrophic and hydrostatic balance

We first use the mass conservation to obtain a scaling for the vertical velocity w,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \approx 0\,, \tag{22}$$

$$\frac{U}{L} \qquad \frac{U}{L} \qquad \frac{W}{H} \tag{23}$$

$$\Rightarrow \quad W \sim \frac{H}{L} U\,. \tag{24}$$

Hence, we see that the smallness in the vertical velocity is due to the shallowness of the atmosphere $H/L \ll 1$.

To derive a validity condition for the geostrophic balance, we non-dimensionalize the x-momentum equation using the scalings:

$$u' = \frac{u}{U}, \; v' = \frac{v}{U},$$

$$x' = \frac{x}{L}, \; y' = \frac{y}{L}, \; z' = \frac{z}{H},$$

$$f = \frac{f}{f_0} \qquad (f_0 = 2\Omega \sin\theta_0),$$

$$\rho' = \frac{\rho}{\rho_0} \qquad (\rho_0 = \text{mean density}),$$

then we have, $$w' = \frac{w}{HU/L} \qquad \text{(from Eq. (24))}\,,$$

$$t' = \frac{t}{L/U} \qquad \text{(advective scaling)},$$

$$\Delta p' = \frac{\Delta p}{\rho_0 U f_0 L}\,. \tag{25}$$

The scaling for the pressure difference Δp is so chosen in anticipation that rotation, hence f_0, is important. The result is,

$$\underbrace{\frac{\partial u'}{\partial t'} + \vec{u}' \cdot \nabla' \vec{u}'}_{\mathcal{O}(1)} - \frac{1}{\mathrm{Ro}} \underbrace{f' v'}_{\mathcal{O}(1)} = -\frac{1}{\mathrm{Ro}} \frac{1}{\rho'} \frac{\partial p'}{\partial x'}\,, \tag{26}$$

where the **Rossby number** Ro is defined as,

$$\mathrm{Ro} = \frac{U}{f_0 L}\,. \tag{27}$$

The Rossby number estimates the magnitude of horizontal advection relative to the Coriolis term. It can also be interpreted as the ratio of the rotation time scale $1/f_0$ to the advective time scale,

$$\frac{u\,\partial u/\partial x}{f u} \sim \frac{U}{f_0 L} = \frac{1/f_0}{L/U}\,. \tag{28}$$

From Eq. (26), we see that the condition for geostrophic balance to hold is $\mathrm{Ro} \ll 1$, i.e., when rotational effects are important.

For the hydrostatic balance, we first consider the non-rotating case. We employ the same scaling in Eq. (25) except for the pressure difference we now use $\Delta p' = \Delta p'/(\rho_0 U^2)$ instead. The non-dimensionalized vertical momentum equation is then,

$$\frac{H^2}{L^2}\underbrace{\left[\frac{\partial w'}{\partial t'} + \vec{u}' \cdot \nabla' \vec{w}'\right]}_{\mathcal{O}(1)} = -\underbrace{\frac{1}{\rho'}\frac{\partial p'}{\partial z'}}_{\mathcal{O}(1)} - \frac{gH}{U^2}\,. \tag{29}$$

Evidently, for a non-rotating fluid, hydrostatic balance is valid when $(H/L)^2 \ll 1$.

If the fluid is *rapidly* rotating so that geostrophic balance holds, the scaling for w in Eq. (25) should be modified. For simplicity, We take $f = f_0$ and $\rho = \rho_0$ in the following discussion. Recall that $\nabla_h \cdot \vec{u}_g = 0$, this suggests that while it may be true that $\partial u/\partial x \sim \partial v/\partial y \sim U/L$, the sum $\partial u/\partial x + \partial v/\partial y \ll U/L$ due to cancellations between terms. Hence $W \sim (H/L)U$ is an over-estimation. By cross-differentiating the horizontal momentum equations, we deduce

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{f_0}\left[\frac{\partial}{\partial y}\frac{\mathrm{D}u}{\mathrm{D}t} - \frac{\partial}{\partial x}\frac{\mathrm{D}v}{\mathrm{D}t}\right] \sim \frac{1}{f_0}\frac{U^2}{L^2} = \mathrm{Ro}\frac{U}{L}\,. \tag{30}$$

Therefore, an estimation for the vertical velocity that is consistent with the geostrophic balance is,

$$W \sim \mathrm{Ro}\frac{H}{L}U\,. \tag{31}$$

Replacing the scaling for w in Eq. (25) by $w' = w/[\mathrm{Ro}(H/L)U]$, the non-dimensionalized vertical momentum equation takes the from,

$$\mathrm{Ro}^2\frac{H^2}{L^2}\underbrace{\left[\frac{\partial w'}{\partial t'} + \vec{u}' \cdot \nabla' \vec{w}'\right]}_{\mathcal{O}(1)} = -\underbrace{\frac{1}{\rho'}\frac{\partial p'}{\partial z'}}_{\mathcal{O}(1)} - \mathrm{Ro}\,\frac{gH}{U^2}\,. \tag{32}$$

Hence, the validity condition for the hydrostatic balance in low Rossby number flows is $\mathrm{Ro}^2(H/L)^2 \ll 1$. In other words, a rapidly rotating flow is more likely to be in hydrostatic balance.

5 Thermal wind balance

Combining geostrophic and hydrostatic balance, we can derive an expression that relates horizontal temperature gradients to vertical wind shear. We shall also use the ideal gas law Eq. (15) and make the assumption that the vertical temperature variation is negligible, $\partial T/\partial z = 0$ (this assumption can be removed if we use the pressure p instead of height z as our vertical coordinate [2]).

First, we eliminate ρ from Eq. (18) and Eq. (20a):

$$\frac{fv}{T} = R\frac{\partial}{\partial x}\ln p\,, \tag{33}$$

$$-\frac{g}{T} = R\frac{\partial}{\partial z}\ln p\,. \tag{34}$$

Cross-differentiating the above equations and use $\partial T/\partial z = 0$, we have

$$f\frac{\partial v}{\partial z} = \frac{g}{T}\frac{\partial T}{\partial x}\,. \tag{35}$$

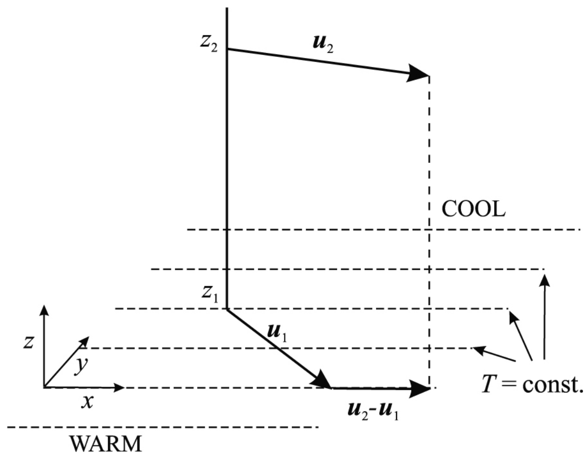

Figure 4: Thermal wind balance (Andrews 2010).

Similarly manipulation using Eq. (18) and Eq. (20b) gives,

$$f\frac{\partial u}{\partial z} = -\frac{g}{T}\frac{\partial T}{\partial y}\,. \tag{36}$$

Eq. (35) and Eq. (36), called the **thermal wind equations**, state that a horizontal temperature gradient is accompanied by a vertical shear of horizontal wind.

Figure 4 illustartes the thermal wind balance using a simple model in which $T = T(y)$. If we take the positive y-direction to be the poleward direction, then $\partial T/\partial y < 0$. On the Northern Hemisphere $f > 0$, assume that we have *cold advection*, i.e., the wind blows across the isotherms from colder to warmer regions. Eq. (36) then implies $\partial u/\partial z > 0$ and hence the geostrophic wind turns cyclonically with height as shown in Fig. 4. This is called *backing*. Similarly, we can readily deduce that *warm advection* is characteried by *veering*, anti-cyclonic rotation with height of the geostrophic wind.

6 Taylor-Proudman theorem

For a fluid of constant density ρ_0 on an f-plane, hydrostatic and geostrophic balance implies that all components of the velocity $\vec{u}$ are independent of height z. This is known as the Taylor-Proudman theorem. The proof goes as follow:

$$\frac{\partial u}{\partial z} = -\frac{1}{f_0\rho_0}\frac{\partial}{\partial y}\frac{\partial p}{\partial z} = -\frac{1}{f_0\rho_0}\frac{\partial}{\partial y}(-\rho_0 g) = 0\,, \tag{37a}$$

$$\frac{\partial v}{\partial z} = \frac{1}{f_0\rho_0}\frac{\partial}{\partial x}\frac{\partial p}{\partial z} = \frac{1}{f_0\rho_0}\frac{\partial}{\partial x}(-\rho_0 g) = 0\,, \tag{37b}$$

$$\because\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ \text{ for geostrophic flows,} \quad \therefore\ \nabla\cdot\vec{u} = 0 \Rightarrow \frac{\partial w}{\partial z} = 0\,. \tag{37c}$$

Furthermore, if there is a solid horizontal boundary in the flow, then $w = 0$ at the boundary which implies $w = 0$ everywhere. These results are also valid for an ideal gas at constant temperature, as can readily be seen from the thermal wind equations. The Taylor-Proudman theorem provides some justifications for approximating large-scale geophysical flows as quasi-two-dimensional.

7 Inertial oscillations

We now turn to a phenomenon of much smaller length scale (several kilometers) and shorter time scale (< 1 day), the inertial oscillations. In the process, we illustrate the mathematics for studying the evolution of small perturbations to a system: linearization of the equation of motion.

Consider a two-dimensional fluid of constant density on an f-plane, the equations of motions are,

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \tag{38a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,. \tag{38b}$$

Let $\{u_0(x,y,t), v_0(x,y,t), p_0(x,y,t)\}$, called the basic state, be a solution to the equations of motion. Let $\{u_1, v_1, p_1\}$ be small perturbation to the basic state. We now substitute

$$u = u_0 + u_1\,, \quad v = v_0 + v_1\,, \quad p = p_0 + p_1\,, \tag{39}$$

into Eq. (38). Using the fact that $\{u_0, v_0, p_0\}$ is a solution and keeping only terms that are linear in the perturbation $\{u_1, v_1, p_1\}$, after some algebra, we obtain,

$$\frac{\partial u_1}{\partial t} + (\vec{u}_0\cdot\nabla)u_1 + (\vec{u}_1\cdot\nabla)u_0 - fv_1 = -\frac{1}{\rho}\frac{\partial p_1}{\partial x}\,, \tag{40a}$$

$$\frac{\partial v_1}{\partial t} + (\vec{u}_0\cdot\nabla)v_1 + (\vec{u}_1\cdot\nabla)v_0 + fu_1 = -\frac{1}{\rho}\frac{\partial p_1}{\partial y}\,. \tag{40b}$$

Notice that this is a set of *linear* equations for $\{u_1, v_1, p_1\}$.

We now pick the basic state to be a rest state: $u_0 = v_0 = 0$, $p =$ constant. Furthermore, let us look for solution to Eq. (40) in which the perturbation pressure gradient can be ignored, we then have,

$$\frac{\partial u_1}{\partial t} - fv_1 = 0\,, \tag{41a}$$

$$\frac{\partial v_1}{\partial t} + fu_1 = 0\,. \tag{41b}$$

Taking the time derivative of Eq. (41a) and then eliminate v_1 using Eq. (41b) gives,

$$\frac{\partial^2 u_1}{\partial t^2} + f^2 u_1 = 0\,. \tag{42}$$

Therefore, the solution is,

$$u_1 = \mathcal{U}\cos(ft+\theta)\,, \tag{43}$$

$$v_1 = -\mathcal{U}\sin(ft+\theta)\,, \tag{44}$$

where $\mathcal{U}$ and θ are arbitrary constants. This solution is known as inertial oscillation. Recall that the fluid particle trajectories are given by,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_1\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = v_1\,. \tag{45}$$

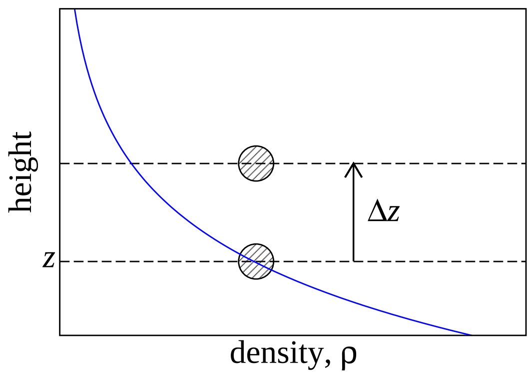

Figure 5: An air parcel is displaced from height z to $z+\Delta z$ with its density conserved.

Taking the initial conditions to be $x(0) = 0$ and $y(0) = 0$, we get,

$$x(t) = \frac{\mathcal{U}}{f}\sin(ft+\theta)\,, \tag{46}$$

$$y(t) = \frac{\mathcal{U}}{f}\cos(ft+\theta)\,, \tag{47}$$

$$\therefore \qquad x^2(t) + y^2(t) = \left(\frac{\mathcal{U}}{f}\right)^2 . \tag{48}$$

So the fluid particles move in circles of radius $\mathcal{U}/f$ at speed $\mathcal{U}$. The motion is anti-cyclonic and the period of the oscillation is $2\pi/f$. For $\mathcal{U} \sim 10$ ms^{-1}, the radius of the oscillation is about 10 km and the period is about 0.5 day which is faster than the advective time scale for large-scale weather systems.

8 Internal gravity waves

In this section, we study a type of atmospheric motion in which the variation in density cannot be ignored: internal gravity waves. These waves are results of the density stratification in the atmosphere. A fluid is said to be *stratified* when its average vertical density gradient is large compared to its horizontal density gradients. We first introduce the idea of buoyancy frequency using the concept of *air parcels*. An air parcel is assumed to be:

1. thermally insulated from its environment, hence it evolves adiabatically, and
2. always at the same pressure as its environment.

The buoyancy frequency. Consider the simple model of an incompressible atmosphere whose density $\rho(z)$ varying with height z only. The system is initially at rest and so in hydrostatic balance. Suppose, as shown in Fig. 5, an air parcel located at z with density $\rho(z)$ is displaced to $z+\Delta z$. We assume the density of the air parcel is conserved as it moves. The difference in the density of the air parcel and its environment at $z+\Delta z$ creates a buoyancy force (per unit volume) equals $\rho(z+\Delta z)g$ according to the Archimedes' principle. Referring to Fig. 5, the equation of motion for the displacement Δz of the air parcel is thus,

$$\rho(z+\Delta z)g - \rho(z)g = \rho(z)\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Delta z\,, \tag{49}$$

$$\text{or} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Delta z + N^2(z)\Delta z = 0\,, \tag{50}$$

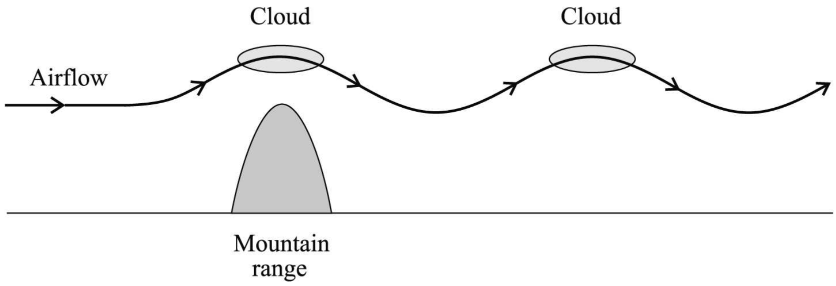

Figure 6: Mountain wave, an example of internal gravity wave in the atmosphere, and its associated parallel bands of cloud (Andrews 2010).

where we have expanded $\rho(z+\Delta z)$ about $\rho(z)$ and the buoyancy frequency $N(z)$ is defined as,

$$N^2(z) \equiv -\frac{g}{\rho(z)}\frac{\partial \rho}{\partial z}\,. \tag{51}$$

So for $\partial\rho/\partial z < 0$, we see that $N^2 > 0$ and the air parcel in general undergoes oscillatory motion. For the special case of constant N, the air parcel undergoes simple harmonic motion with period $2\pi/N$.

An example of atmospheric internal gravity waves is the *mountain waves* (or lee waves) that formed when a stratified airflow passes over a mountain range as illustrated in Fig. 6. These wave motions are generated downstream of the mountain and may be made visible by *wave clouds* that form when water vapor condensed at the crests of the waves.

Our model. To develop a mathematical model for internal gravity waves in the atmosphere, we shall:

1. focus on motions of relatively small horizontal scale so that the Earth's rotation can be ignored: $f = 0$,

2. consider the atmosphere as incompressible, and

3. assume the atmosphere is in hydrostatic balance.

For the density stratification, it is convenient to decompose, without approximation, the exact density ρ into a mean vertical profile $\tilde{\rho}(z)$ and a residual fluctuating part $\rho'(x,y,z,t)$. Sometimes, we may also write $\tilde{\rho}(z) = \rho_0 + \bar{\rho}(z)$ where the constant ρ_0 is a typical value for the density of the atmosphere, hence,

$$\rho(x,y,z,t) = \underbrace{\rho_0 + \bar{\rho}(z)}_{\tilde{\rho}(z)} + \rho'(x,y,z,t)\,. \tag{52}$$

It is also customary to divide the pressure p into two parts,

$$p(x,y,z,t) = \tilde{p}(z) + p'(x,y,z,t)\,, \tag{53}$$

where we require $\tilde{p}$ and $\tilde{\rho}$ to satisfy the hydrostatic balance,

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}z} = -\tilde{\rho}\,g\,. \tag{54}$$

The Boussinesq approximation. First, let us consider the continuity equation. Recall that for an incompressible fluid, $\mathrm{D}\rho/\mathrm{D}t = 0$. Using the decomposition Eq. (52), we have the following equation for ρ',

$$\frac{\mathrm{D}\rho'}{\mathrm{D}t} - \frac{\rho_0}{g} N^2(z)\, w = 0\,, \tag{55}$$

where the buoyancy frequency $N(z)$ here is defined as,

$$N^2(z) = -\frac{g}{\rho_0}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}\,. \tag{56}$$

Furthermore, the continuity equation becomes,

$$\nabla \cdot \vec{u} = 0\,. \tag{57}$$

We now turn to the momentum equations. By the hydrostatic assumption, the vertical momentum equation becomes,

$$\frac{\partial p'}{\partial z} = -\rho' g\,. \tag{58}$$

In our current model, the horizontal momentum equations are,

$$(\rho_0 + \bar{\rho} + \rho')\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}\,, \tag{59a}$$

$$(\rho_0 + \bar{\rho} + \rho')\frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}\,. \tag{59b}$$

In the atmosphere, the deviations of ρ and p from a hydrostatically balanced state are usually very small, hence $\rho' \ll \tilde{\rho}$. On the other hand, $\bar{\rho}(z)$ is not necessarily much smaller than ρ_0 (this is in contrast to the case of the ocean in which $\bar{\rho}, \rho' \ll \rho_0$). However, in our present discussion, we nonetheless assume $\bar{\rho} \ll \rho_0$. If the scale height is sufficiently large, this might be a good approximation (see Ref. [3] for further discussions). Hence Eq. (59) reduces to,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}\,, \tag{60}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}\,. \tag{61}$$

Equations (55), (57), (58), (60) and (61) constitute our model for studying internal gravity waves in the atmosphere. Note that we have retained the full density in the continuity equation and the vertical momentum equation but only kept its constant part ρ_0 in the horizontal momentum equations. This is known as the Boussinesq approximation. Loosely speaking, we have ignored density variations *except* where they are coupled with gravity g.

Linearized equations and wave solutions. Consider the following basic state that satisfies exactly the five equations in our model:

$$\vec{u}^{(0)} = 0, \quad \rho^{(0)} = \rho_0 + \bar{\rho}(z), \quad p^{(0)} = \bar{p}(z)\,, \tag{62}$$

$$\text{such that} \quad \frac{\partial p^{(0)}}{\partial z} = -\rho^{(0)} g\,. \tag{63}$$

Let $\{\vec{u}^{(1)}, \rho^{(1)}, p^{(1)}\}$ be small perturbations to the basic state. Substitute $\{\vec{u}^{(0)} + \vec{u}^{(1)}, \rho^{(0)} + \rho^{(1)}, p^{(0)} + p^{(1)}\}$ into Eqs. (55), (57), (58), (60), (61) and keep only terms that are linear in the perturbations. After dropping the superscripts (with the understanding that the variables now refer to the perturbation variables), we get

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}\,, \tag{64a}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}\,, \tag{64b}$$

$$\frac{\partial p}{\partial z} = -\rho\, g\,, \tag{64c}$$

$$\nabla \cdot \vec{u} = 0\,, \tag{64d}$$

$$\frac{\partial \rho}{\partial t} - \frac{\rho_0}{g} N^2 w = 0\,. \tag{64e}$$

For simplicity, we shall take N to be constant and seek solutions that does not depend on y. Equation (64) is a set of linear partial different equations with *constant coefficients*. Therefore we can find plane wave solutions of the form,

$$(u, v, w, p, \rho) = \mathrm{Re}\left\{(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\rho})\, e^{i(kx+mz-\omega t)}\right\}\,, \tag{65}$$

where $(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\rho}), k, m, \omega$ are to be determined. Substitution of Eq. (65) into Eq. (64) gives $\hat{v} = 0$ and the following set of algebraic equations,

$$\begin{pmatrix} -i\omega & 0 & ik/\rho_0 & 0 \\ 0 & 0 & im & g \\ k & m & 0 & 0 \\ 0 & \rho_0 N^2/g & 0 & i\omega \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{w} \\ \hat{p} \\ \hat{\rho} \end{pmatrix} = 0\,. \tag{66}$$

For non-trivial solutions to exist, the determinant of the 4×4 matrix on the left of Eq. (66) must equal zero. This gives the dispersion relation,

$$\omega^2 = \frac{N^2 k^2}{m^2}\,, \tag{67}$$

which relates the wavenumber (k, m) to the angular frequency ω of the plane waves. We can now solve, say, $\hat{u}, \hat{w}$ and $\hat{\rho}$ in terms of $\hat{p}$. We arbitrarily take $\hat{p}$ to be real and the solution is,

$$u = \frac{k\hat{p}}{\rho_0 \omega} \cos(kx + mz - \omega t)\,, \tag{68a}$$

$$v = 0\,, \tag{68b}$$

$$w = -\frac{k^2 \hat{p}}{\rho_0 \omega m} \cos(kx + mz - \omega t)\,, \tag{68c}$$

$$p = \hat{p} \cos(kx + mz - \omega t)\,, \tag{68d}$$

$$\rho = \frac{m\hat{p}}{g} \sin(kx + mz - \omega t)\,. \tag{68e}$$

The solution Eq. (68) is illustrated in Fig. 7 for the case $k > 0$, $m < 0$ and $\omega > 0$, the solid lines are the line of constant phase

$$\theta(x, z, t) \equiv kx + mz - \omega t = \text{constant}\,. \tag{69}$$

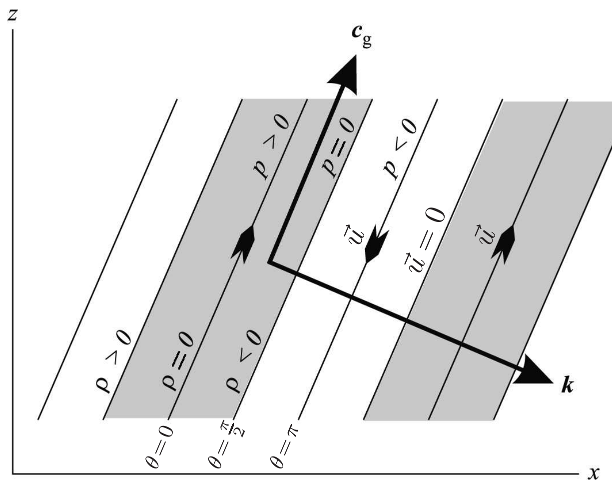

Figure 7: Instantaneous velocity $\vec{u}$, pressure p and density ρ fields of a plane internal gravity wave with $k > 0$, $m < 0$ and $\omega > 0$, see Eq. (68). The solid lines are line of constant phase θ. Fluid particles are moving upward in the shaded area and downward in the unshaded area (adopted from Andrews 2010).

For the particular signs we have chosen, the group velocity is

$$\vec{c}_g \equiv \left(\frac{\partial \omega}{\partial k} \, , 0 \, , \frac{\partial \omega}{\partial m} \right) = \left(-\frac{N}{m} \, , 0 \, , \frac{Nk}{m^2} \right) \, , \tag{70}$$

and the following features can be deduced from the solution Eq. (68).

1. The wave pattern, or the lines of constant phase, propagates obliquely downward in the direction of the wave vector $\vec{k} = (k, 0, m)$.

2. The velocity vector $\vec{u} = (u, 0, w)$ is parallel to the lines of constant phase.

3. Fluid particles oscillate up and down along straight lines parallel to the lines of constant phase.

4. The group velocity is perpendicular to the wave vector since $\vec{c}_g \cdot \vec{k} = 0$. Furthermore, $\vec{c}_g$ is pointing obliquely upward.

Hence, as the wave pattern propagating obliquely downward, fluid particles oscillates up and down along tilted straight lines and energy is being pumped obliquely upward.

9 Rotating shallow-water system

In the previous section, we study internal gravity waves using a non-rotating system with density straitification. Now we introduce the rotating shallow-water system in which

1. the fluid density ρ_0 is constant,

2. the Coriolis parameter varies with latitude: $f(y) = f_0 + \beta y$, and

3. the fluid is in hydrostatic balance (due the shallowness of the fluid layer).

Later, we shall use this system to study Rossby waves.

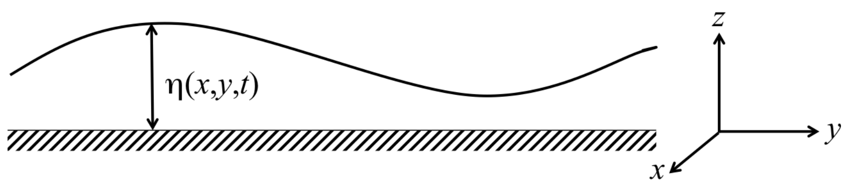

Figure 8: Shallow-water system

We consider the case with a flat solid bottom. The thickness of the fluid layer is $\eta(x, y, t)$. Referring to the coordinate system shown in Fig. 8, the equations of motion are:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}\,, \tag{71a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}\,, \tag{71b}$$

$$\frac{\partial p}{\partial z} = -\rho_0\, g\,, \tag{71c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,. \tag{71d}$$

There are four unknowns (u, v, w, p) which in general are functions of time and the three-dimensional space (x, y, z, t). We now rewrite the system in terms of the dependent variables $\{(u, v, \eta), w\}$ where (u, v, η) depend only on (x, y, t) and constitute a closed two-dimensional system.

We start by integrating the hydrostatic equation Eq. (71c) in z,

$$p(x, y, z, t) = \rho_0 g[\eta(x, y, t) - z]\,. \tag{72}$$

We see that $\partial p/\partial x$ and $\partial p/\partial y$, which appear on the right of Eqs. (71a) and (71b), do not depend on z. Therefore we can consistently *choose* to consider cases where (u, v) are independent of z, i.e., we assume the flow is columnar (for example, if we pick an initial condition of (u, v) that does not depend on z, then (u, v) at subsequent time will be independent of z). Define the notations,

$$\vec{u}_h = (u, v)\,, \tag{73}$$

$$\frac{\mathrm{D}_h}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\,, \tag{74}$$

and using Eq. (72), the horizontal momentum equations become,

$$\frac{\mathrm{D}_h u}{\mathrm{D}t} - fv = -g\frac{\partial \eta}{\partial x}\,, \tag{75}$$

$$\frac{\mathrm{D}_h v}{\mathrm{D}t} + fu = -g\frac{\partial \eta}{\partial y}\,. \tag{76}$$

We then integrate the continuity equation Eq. (71d) from the bottom $z = 0$ to the surface $z = \eta$ and use the kinematic boundary condition,

$$w(x, y, \eta, t) = \frac{\mathrm{D}_h \eta}{\mathrm{D}t}\,, \tag{77}$$

to obtain the following equation for η,

$$\frac{\partial \eta}{\partial t} + \nabla_h \cdot (\eta\, \vec{u}_h) = 0\,. \tag{78}$$

Equations (75), (76) and (78), called the *shallow-water equations*, constitute a closed system in the three dependent variables $\{u(x,y,t), v(x,y,t), \eta(x,y,t)\}$. The vertical velocity can be obtained from the horizontal velocity by integrating Eq. (71d) in z,

$$w(x,y,z,t) = -z\,(\nabla_h \cdot \vec{u}_h)\,. \tag{79}$$

We see that w is linear in z.

Shallow-water potential vorticity. The potential vorticity is an important quantity in geophysical fluid dynamics. Here, we consider a version of the potential vorticity for the shallow-water system. For a two-dimensional system, the vorticity has only the z-component given by,

$$\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,. \tag{80}$$

Subtract the y-derivative of Eq. (71a) from the x-derivative of Eq. (71b), we get

$$\frac{\mathrm{D}_h}{\mathrm{D}t}(\zeta + f) + (\zeta + f)(\nabla_h \cdot \vec{u}_h) = 0\,. \tag{81}$$

We have used $\mathrm{D}_h f/\mathrm{D}t = v\partial f/\partial y$. From Eq. (78),

$$\nabla_h \cdot \vec{u}_h = -\frac{1}{\eta}\frac{\mathrm{D}_h \eta}{\mathrm{D}t}\,. \tag{82}$$

Finally, eliminating $\nabla_h \cdot \vec{u}_h$ from Eqs. (81) and (82), we get the *potential vorticity equation*,

$$\frac{\mathrm{D}_h}{\mathrm{D}t}\left(\frac{\zeta + f}{\eta}\right) = 0\,. \tag{83}$$

Above, the quantity in the bracket is the potential vorticity (PV) for the shallow-water system,

$$\mathrm{PV} = \frac{\zeta + f}{\eta}\,. \tag{84}$$

According to Eq. (83), the potential vorticity is conserved following the motion of the vertical fluid columns.

Physical interpretation. Consider a cylindrical fluid column of radius R and height η rotating at angular velocity ω about the axis of the cylinder. The moment of inertia I of the fluid cylinder is

$$I = \frac{\rho_0}{2}\pi R^4 \eta\,. \tag{85}$$

Since pressure exerts no torque about the axis of rotation, therefore *in an inertial frame*, we must have the conservation of (1) total angular momentum $\ell = I\omega$ and (2) total mass $M = \rho_0 \pi R^2 \eta$. This implies, after eliminating R from ℓ and M,

$$\frac{\ell}{M^2} \sim \frac{\omega}{\eta} = \text{constant}\,. \tag{86}$$

Recall that the vorticity is twice the local angular velocity, therefore,

$$2\omega = \hat{k} \cdot (\nabla \times \vec{u}_I) = \hat{k} \cdot \nabla \times (\vec{u}_R + \Omega \times \vec{r}) = \zeta + f \,. \tag{87}$$

Using Eq. (87) in Eq. (86) gives PV=constant. Hence, the conservation of potential vorticity is a result of the conservation of angular momentum and the conservation of mass. For example, if the fluid column is being stretched, η increases. Since mass is conserved, R must decrease and so I decreases. The conservation of angular momentum then implies ω increases and the net result is ω/η remains constant.

Shallow-water quasi-geostrophic equations. Recall that when the Rossby number is small, rotational effects dominates nonlinear advection. In the limit Ro $\to 0$, we have the geostrophic balance Eq. (20). The geostrophic balance is not a dynamical equation in the sense that it contains no time derivatives and hence, does not allow us to predict the time dependence of the system.

We now discuss the quasi-geostrophic (QG) approximation *for the shallow-water model* and derive a single equation with one dependent variable that would allow us to obtain the time evolution of the geostrophic velocities (u_g, v_g). We make the following three assumptions:

1. the motion is nearly geostrophic, i.e. Ro $\ll 1$,
2. fractional changes in f are small, i.e., $\beta L/f_0 \ll 1$ where L is a typical horizontal scale,
3. fractional changes in the fluid thickness are small, so if H_0 is the mean thickness and

$$\eta'(x, y, t) = \eta(x, y, t) - H_0 \,, \tag{88}$$

then we require $|\eta'|/H_0 \ll 1$.

Since we are interested in nearly geostrophic flows, we write the variables in the shallow-water system as the sum of a large geostrophic part and a small ageostrophic part,

$$u = u_g + u_a, \quad v = v_g + v_a, \quad \eta = \eta_g + \eta_a \,, \tag{89}$$

where the geostrophic velocities (u_g, v_g) satisfy,

$$f_0 u_g = -g\frac{\partial \eta_g}{\partial y} \,, \tag{90a}$$

$$f_0 v_g = g\frac{\partial \eta_g}{\partial x} \,. \tag{90b}$$

Our goal is to predict how the geostrophic component of the flow changes with time. Notice that we have used f_0 instead of the full $f(y)$ in Eq. (90). This is consistent with the assumption that the geostrophic component is the lowest order approximation to the system. Since $\nabla \cdot \vec{u}_g = 0$, we can define a stream function ψ_g such that

$$\eta_g = \frac{f_0}{g}\psi_g \,, \tag{91a}$$

$$(u_g, v_g) = \left(-\frac{\partial \psi_g}{\partial y}, \frac{\partial \psi_g}{\partial x}\right) \,, \tag{91b}$$

$$\zeta_g \equiv \frac{\partial v_g}{\partial y} - \frac{\partial u_g}{\partial x} = \nabla^2 \psi_g \,. \tag{91c}$$

To derive the QG equation, the general idea is to replace (u, v) by (u_g, v_g) in the PV conservation equation Eq. (83). Keeping only first-order small terms, we have the following approximation for the PV,

$$\zeta + f \approx \zeta_g + f_0 + \beta y = f_0 \left(1 + \frac{\beta y}{f_0} + \frac{\zeta_g}{f_0}\right) , \tag{92}$$

$$\eta \approx H_0 \left(1 + \frac{\eta'}{H_0}\right) , \tag{93}$$

$$\begin{aligned} \Rightarrow \quad \text{PV} &\approx \frac{f_0}{H_0}\left(1 + \frac{\beta y}{f_0} + \frac{\zeta_g}{f_0}\right)\left(1 - \frac{\eta'}{H_0}\right) \\ &\approx \frac{1}{H_0}\underbrace{\left(\nabla^2\psi_g + f - \frac{f_0^2}{gH_0}\psi_g\right)}_{q} + \frac{f_0}{H_0} . \end{aligned} \tag{94}$$

Note that $\zeta_g/f_0 \sim U/(f_0 L) \ll 1$ where U is a typical horizontal velocity scale. q is called the quasi-geostrophic potential vorticity (QGPV). Next we approximate the operator $\mathrm{D}_h/\mathrm{D}t$ as follow,

$$\begin{aligned} \frac{\mathrm{D}_h}{\mathrm{D}t} &\approx \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y} , \\ &= \frac{\partial}{\partial t} + J(\psi_g\,, \cdot\,) , \end{aligned} \tag{95}$$

where the Jacobian J is defined as,

$$J(A, B) \equiv \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x} . \tag{96}$$

Finally, substituting Eq. (94) and Eq. (95) into Eq. (83), we have the *shallow-water quasi-geostrophic equation*,

$$\frac{\partial q}{\partial t} + J(\psi_g, q) = 0 , \tag{97a}$$

$$\nabla^2\psi_g + f - \frac{1}{L_d^2}\psi_g = q , \tag{97b}$$

where $L_d \equiv \sqrt{gH_0}/f_0$ is the deformation radius. L_d is an important parameter that often appears in stratified rotating systems.

Eq. (97) contains a single dependent variable ψ_g. Once ψ_g is known, (u_g, v_g) and η_g can be determined using Eq. (91). Hence under the quasi-geostrophic approximations, Eq. (97) describes the whole dynamics of the system. Given $q(n)$ [or equivalently $\psi_g(n)$] at the n-th time step, we obtain $q(n + 1)$ by time-stepping Eq. (97a) forward and then determine $\psi_g(n + 1)$ from $q(n + 1)$ using Eq. (97b).

Rossby waves in shallow-water systems. The variation of the Coriolis parameter with latitude and the conservation of potential vorticity lead to the formation of Rossby waves. We now study Rossby waves in the quasi-geostrophic shallow-water model Eq. (97).

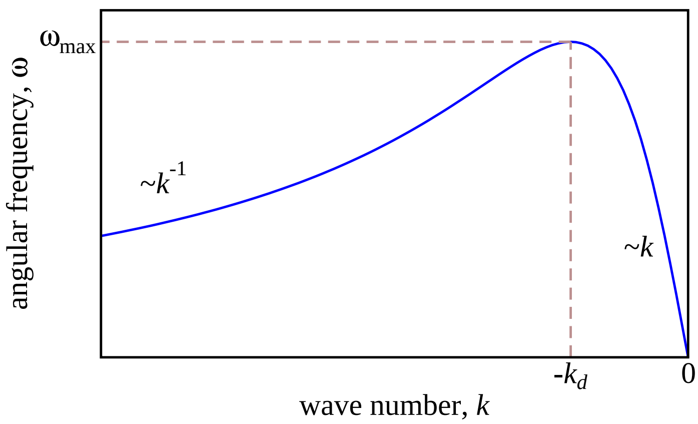

Figure 9: Dispersion relation for Rossby waves in a quasi-geostrophic shallow-water system.

Consider the following basic state which satisfies Eq. (97),

$$\psi_g^{(0)} = \frac{g}{f_0} H_0 \tag{98}$$

$$\Rightarrow \quad q^{(0)} = f_0 + \beta y - \frac{\psi_g^{(0)}}{L_d^2}\,. \tag{99}$$

This corresponds to a fluid at rest $\vec{u}^{(0)} = 0$ with uniform thickness $\eta^{(0)} = H_0$. Let $\psi_g^{(1)}$ be a small perturbation to the streamfunction: $\psi_g = \psi_g^{(0)} + \psi_g^{(1)}$. From Eq. (97b), the corresponding perturbation in q is

$$q^{(1)} = q - q^{(0)} = \nabla^2 \psi_g^{(1)} - \frac{\psi_g^{(1)}}{L_d^2}\,. \tag{100}$$

Linearize the shallow-water QG equation Eq. (97a) about the basic state, we have

$$\frac{\partial q^{(1)}}{\partial t} + J(\psi_g^{(1)}, q^{(0)}) + J(\psi_g^{(0)}, q^{(1)}) = 0 \tag{101}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\nabla^2 \psi_g^{(1)} - \frac{1}{L_d^2}\frac{\partial \psi_g^{(1)}}{\partial t} + \beta\,\frac{\partial \psi_g^{(1)}}{\partial x} = 0\,. \tag{102}$$

Eq. (102) is a linear equation with constant coefficients, hence substitute plane wave solution of the form,

$$\psi_g^{(1)} = \mathrm{Re}\,\hat{\psi}\, e^{i(kx+ly-\omega t)}\,, \tag{103}$$

where $\hat{\psi}$ is a constant, into Eq. (102), we obtain the dispersion relation for Rossby waves,

$$\omega = \frac{-\beta k}{K^2 + k_d^2}\,, \tag{104}$$

where $K^2 \equiv k^2 + l^2$ and $k_d \equiv 1/L_d$, see Fig. 9. Several features should be noted.

1. Rossby waves exist only when $\beta \neq 0$.
2. From Eq. (104), the x-component of the phase speed ω/k is always negative. Therefore Rossby waves have a *westward* phase speed at every wave number. Also note that by definition $\omega > 0$, hence $k < 0$.

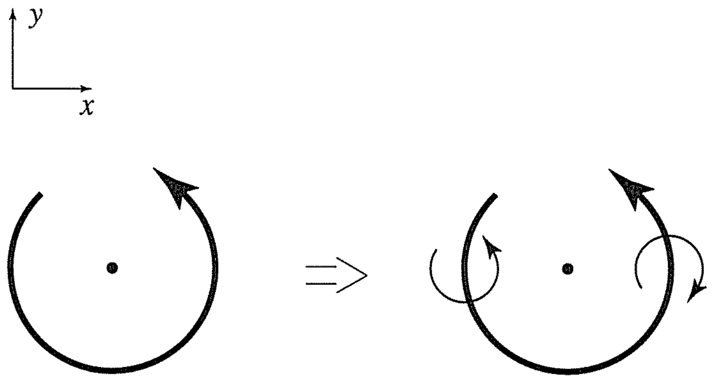

Figure 10: Physical mechanism of short Rossby waves (Salmon 1998)

3. It is straight forward to verify from Eq. (104) that the maximum frequency $\omega_{\max}$ occurs at $k = -k_d$ and $l = 0$. Furthermore,

$$\omega_{\max} = \frac{\beta}{2k_d}\,. \tag{105}$$

Using $g \approx 10\ \mathrm{ms}^{-2}$, $H_0 \approx 20$ km, $f_0 \approx 10^{-4}\ \mathrm{s}^{-1}$ and $\beta \approx 10^{-11}\ \mathrm{m}^{-1}\mathrm{s}^{-1}$, we have the maximum period of Rossby waves equals $2\pi/\omega_{\max} \approx 3$ days. Since this period is longer than the Earth's rotation period, Rossby waves are nearly geostrophic, i.e. the effect of the Earth's rotation is important. This is of course not surprising since our analysis is done under the QG approximations.

Physical mechanism. To understand the physical origin of Rossby waves, we look at two limiting cases.

(i) Short waves ($K \gg k_d$). In this limit, the linearized QG equation Eq. (102) becomes,

$$\frac{\partial}{\partial t}\nabla^2\psi_g^{(1)} + \beta\,\frac{\partial \psi_g^{(1)}}{\partial x} \approx 0\,. \tag{106}$$

Recall that we derive the QG equation and its linearized counterpart from the PV conservation equation Eq. (83). In the short-wavelength limit, Eq. (83) reduces to,

$$\frac{\mathrm{D}}{\mathrm{D}t}(\zeta + f) \approx 0\,. \tag{107}$$

Suppose a perturbation in the form of a positive vortex is introduced into the vorticity field ζ, see Fig. 10. Since $f(y)$ increases with y, the northward ($+y$) flow on the east side of the vortex causes a decreases in ζ as required by the conservation of $\zeta + f$ in Eq. (107). Similar argument implies ζ increases on the west side of the vortex. Hence the perturbation propagates westward.

(ii) Long waves ($K \ll k_d$). In the long-wavelength limit, the linearized QG equation is

$$-\frac{1}{L_d^2}\frac{\partial \psi_g^{(1)}}{\partial t} + \beta\,\frac{\partial \psi_g^{(1)}}{\partial x} \approx 0\,, \tag{108}$$

corresponding to the approximate PV conservation:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{f}{\eta}\right) \approx 0\,. \tag{109}$$

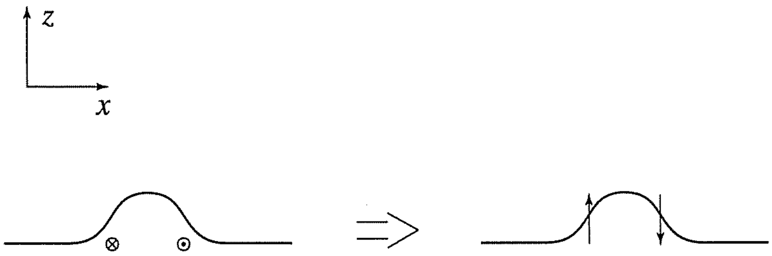

Figure 11: Physical mechanism of long Rossby waves (Salmon 1998)

Now consider a perturbation in the thickness η as shown in Fig. 11. Since the flow is nearly goestrophic, from Eq. (90), the fluid near the hump flows in the directions indicated in Fig. 11. Hence from Eq. (109), η on the west side of the hump increases due to the northward flow; the converse occurs on the east side of the hump. As a result, the perturbation moves westward.

References

[1] D. G. Andrews, *An Introduction to Atmospheric Physics* 2nd ed., Cambridge University Press, 2010.

[2] R. Salmon, *Lectures on Geophysical Fluid Dynamics*, Oxford University Press, 1998.

[3] G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics*, Cambridge University Press, 2006.