MATH3476 Numerical Methods
Problem Set 5
Numerical Linear Algebra

The following questions are based upon the material covered in Chapter 3 of the notes.

Q1 Use LU-factorisation to show that the solution of the tridiagonal system of equations

$$\begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

is $\boldsymbol{x} = \left(\frac{1}{6}, \frac{1}{3}, -\frac{5}{18}, \frac{5}{18}\right)^T$. Retain the use of exact fractions throughout your calculations.

Q2 Consider the system of equations

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

(i) Determine H_J, H_{GS} and H_ω.

(ii) Evaluate $\rho(H_J)$ and $\rho(H_{GS})$.

(iii) Show that the eigenvalues μ of H_ω satisfy the quadratic equation

$$\mu^2 - \left(\frac{\omega^2}{4} - 2\omega + 2\right)\mu + (\omega - 1)^2 = 0,$$

and, along the lines of argument in §3.2.5, determine ω^* and $\rho(H_{\omega^*})$. Confirm this value of ω^* using (3.21).

(iv) Using the spectral radius of each matrix, evaluate the *theoretical* number of iterations, N_J, N_{GS} and N_{ω^*},which ensure that $\left\|\boldsymbol{e}^{(N)}\right\| \leq \frac{1}{10}\left\|\boldsymbol{e}^{(0)}\right\|$, i.e. to reduce the initial error by a factor of 10.

(v) If now $\boldsymbol{x}^{(0)} = \boldsymbol{0}$, determine the exact solution and evaluate $\left\|\boldsymbol{e}^{(0)}\right\|_2$. Hence calculate the *actual* value of N required to reduce $\left\|\boldsymbol{e}^{(0)}\right\|_2$ by a factor of 10 for each of the three iterative schemes (using ω^* in SOR).

Q3 Example 3.2 in the notes demonstrates that Gauss-Seidel is not always superior to Jacobi. This is also the case for the system of equations with coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -3 \end{pmatrix}.$$

(i) Determine both H_J and H_{GS} in terms of an ILU splitting of A.

(ii) Determine the eigenvalues of H_{GS}, and so its spectral radius $\rho(H_{GS})$. What can you say about the convergence of Gauss-Seidel?

(iii) Show that the *characteristic equation* for the eigenvalues of H_J is

$$C(\lambda) \equiv \lambda^3 + \frac{1}{3}\lambda + \frac{2}{3} = 0.$$

(iv) By considering the turning points and sign properties of $C(\lambda)$, deduce that there is only one real root (λ_1) and two complex (conjugate) roots (λ_2 and $\bar{\lambda}_2$) of $C(\lambda)$. Using Python, or whatever other method you choose, determine the approximate value of the real root.

(v) By comparing the above characteristic equation with the expanded form of $C(\lambda) \equiv (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\bar{\lambda}_2) = 0$, deduce the *magnitude* of the complex roots. Hence show that Jacobi converges in this case, even though you have not yet derived the exact form of all the eigenvalues.

(vi) Use the coefficient of λ^2 in $C(\lambda)$ to deduce both the real and imaginary parts of λ_2.

(vii) Hence plot two diagrams showing the eigenvalues and the Gerschgorin row and column disks of H_J.

Q4 By expressing the n-dimensional initial error vector $\boldsymbol{e}^{(0)}$ in equation (3.17) as a linear combination of n linearly independent eigenvectors of H, deduce the necessary condition for convergence, i.e. $\rho(H) < 1$.

Q5 (a) For the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix},$$

evaluate $\|A\|_F$, $\|A\|_1$, $\|A\|_2$ and $\|A\|_\infty$ using the norm definitions of §3.1.1.

(b) Consider the general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(i) Show that

$$\|A\|_2^2 = \frac{1}{2}\left\{\|A\|_F^2 + \left(\|A\|_F^4 - 4|A|^2\right)^{\frac{1}{2}}\right\},$$

where $|A|$ is the determinant of A.

(ii) Show that

$$\|A\|_F^2 \geq 2|A|$$

for *any* 2×2 matrix, and hence that $\|A\|_2$ is real.

(iii) Hence deduce that

$$\frac{1}{\sqrt{2}}\|A\|_F \leq \|A\|_2 \leq \|A\|_F.$$

Q6 A function satisfies the Poisson equation $\Delta u = -2$ in the square region $-1 < x, y < 1$, and is zero everywhere on the boundary of the square.

(i) Use the (5,+) molecule on a square mesh with $h = k = 1$ to approximate the value of $u(0,0)$.

(ii) On a square mesh with $h = k = \frac{1}{2}$, use symmetry arguments to show that there are just *three* points (of the nine possible) at which u takes different values. Using the (5,+) molecule, establish three equations in the three unknown values of u and show that they have solution 0.5625, 0.34375 and 0.4375.

(iii) Solve the problem with $h = k = 1$ using the 9-point *Mehrstellenverfahren* scheme in §2.2.2.

(iv) Solve the problem with $h = k = \frac{1}{2}$ using the 9-point *Mehrstellenverfahren* scheme.

(v) Solve (3.43) in the notes for p. Given that the *exact* answers to 5 decimal places are 0.58937, 0.36229 and 0.45868, use the above numerical results at $(0,0)$ to determine the error order p for both the 5-point and 9-point schemes.