MATH3476 Numerical Methods
Problem Set 4
Finite-Difference Operators

The following questions are based upon the material covered in Chapter 2 of the notes.

Q1 Using Taylor expansions and "unknown weights" at *integer* mesh points, as per §2.1.1, determine the following.

(a) A *third-order forward-difference* approximation, u_0', to $u'(x)$ at the *left-hand* boundary of a regular mesh with spacing h. Before starting the calculation, think carefully about how many points you will need to use. Deduce that the first term in the truncation error is equal to $\frac{1}{4}h^3u_0^{\mathrm{iv}}$, and state the order of the *next* error term, giving reasons for your answer.

(b) A *second-order forward-difference* approximation, u_0'', to $u''(x)$ at the *left-hand* boundary of a regular mesh with spacing h. How many points will be required in this case? Deduce that the first term in the truncation error is equal to $-\frac{11}{12}h^2u_0^{\mathrm{iv}}$, and state the order of the *next* error term, giving reasons for your answer.

Q2 (a) Prove the following finite-difference operator identities using §2.1.2 in the notes:

(i) $\delta_+\delta_- = \delta_-\delta_+ = \delta_+ - \delta_- = \delta_0^2$;

(ii) $\delta_+ + \delta_- = 2\mu_0\delta_0$;

(iii) $\mu_0^2 = I + \frac{1}{4}\delta_0^2$;

(iv) $\mu_0\delta_0 = \delta$ (see equation (2.31)).

(b) Using equations (2.15) and (2.16) in the notes, show that we may formally write

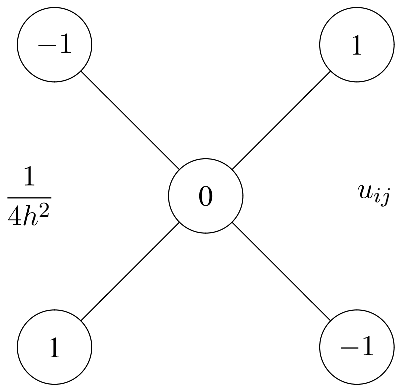
$$E = \sum_{k=0}^{\infty} \delta_-^k \quad \text{and} \quad \delta_+ = \sum_{k=1}^{\infty} \delta_-^k \,.$$

What is strange about these operator relationships?

Q3 Using equations (2.34) and (2.35) in the notes, derive all three results in Example 2.7.

[Hint: you may find it convenient to begin by determining $\delta_0^2u_i$, $\delta_0^4u_i$, $\mu_0\delta_0^2u_i$ and $\mu_0\delta_0^4u_i$.]

Q4 On a *square mesh* with $h = k$, what derivative is approximated by the following molecule (with weights at integer mesh points), and what is the order of the truncation error?



[Hint: The upper-right "atom" corresponds to $E_x E_y u_{ij}$ and the 2-D equivalents of equation (2.19) are $\delta_x = \frac{1}{2}(E_x - E_x^{-1})$ and $\delta_y = \frac{1}{2}(E_y - E_y^{-1})$.]

Q5 *Prove* that the nine-point molecule in Example 2.11 is obtained from the *unique* linear combination of $\frac{2}{3}$ of the "$(5, +)$" and $\frac{1}{3}$ of the "$(5, \times)$" molecules.

Q6 Show that the molecule for the biharmonic operator in Example 2.14 is equivalent to

$$\frac{1}{h^4}\left\{\delta_{0,x}^2 + \delta_{0,y}^2\right\}^2,$$

and so verify that the truncation error is of order $O(h^2)$.