MATH3476 Numerical Methods

Problem Set 3

Near-Minimax Approximations using Chebyshev Polynomials

The following questions are based upon the material covered in §§1.2.5–1.2.10 in the notes.

Q1 If $f(x)$ is square-integrable on $[a,b]$ then, by (1.33) and (1.35), we have $(f,f) = \|f\|_{w,2}^2$. Use this information to show that the definition of Φ given immediately before (1.57) is equivalent to $\Phi = (C_n - u, C_n - u)$, and hence use (1.55) and (1.56) to prove (1.57).

Q2 Let $u(x) = \cos^{-1} x$ for $x \in [-1,1]$, taking the principal branch with $u \in [0,\pi]$.

(a) Show using (1.59) in the notes that the Fourier coefficients c_k, $k = 0(1)n$, are here given explicitly by

$$c_k = \begin{cases} \pi & k = 0 \\ -\dfrac{4}{k^2\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases} .$$

(b) Using the results of part (i), show that the *cubic* Chebyshev least-squares approximation to $u(x)$ is

$$C_3(x) = \frac{\pi}{2} - \frac{8}{3\pi}x - \frac{16}{9\pi}x^3 \approx 1.570796 - 0.848826x - 0.565884x^3 .$$

(c) Plot (using a graphics calculator or computer) the error function $E_3(x) = u(x) - C_3(x)$. At how many points in $[-1,1]$ does this error function vanish? Using this information and (1.67) in the notes, *estimate* $\|u - C_3\|_\infty$. Now *calculate* $\|u - C_3\|_\infty$ and compare it with your estimate. Explain the cause of the discrepancy.

Q3 (a) By splitting the $\displaystyle\int_0^\pi$ integral in (1.59) into $\displaystyle\int_0^{\pi/2} + \int_{\pi/2}^{\pi}$, derive formula (1.60) in the notes.

(b) Let $u(x) = \sin(\pi x/2)$ for $x \in [-1,1]$.

(i) Which Fourier coefficients c_k will vanish, and why? Using $N = 4$ in the trapezoidal-rule applied to the appropriate formula for evaluating the non-zero c_k, show that the Fourier coefficients in this case are given by

$$c_k \approx \frac{1}{2}\sum_{i=0}^{4}{}'' \sin\left(\frac{\pi}{2}\cos\frac{i\pi}{8}\right)\cos\frac{ki\pi}{8} ,$$

where the double prime on the sum means that the first and last terms are to be halved.

(ii) Using the formula in (i), calculate the Fourier coefficients c_k, $k = 0(1)3$, and so show that the *cubic* least-squares approximation to $u(x)$ is

$$C_3(x) \approx 1.547863506x - 0.5522871056x^3 .$$

(iii) Which Fourier coefficient c_k estimates the error $\|u - C_3\|_\infty$? Calculate it using the same trapezoidal rule as in (ii). Estimate $\|u - C_3\|_\infty$ (using Python or otherwise) from the polynomial $C_3(x)$ and compare your answers. Why is there no discrepancy here, as there was in Q2 (iii)?

Q4 (a) Using the approach in §1.2.8 in the notes, show that the *quadratic* Chebyshev interpolation polynomial, $I_2(x)$, for the function $u(x) = \sqrt{4+x}$ on $x \in [-1, 1]$ is

$$I_2(x) \approx 2 + 0.251496x - 0.0158597x^2 .$$

(b) Estimate $\|u - I_2\|_\infty$; first, using (1.71), then directly from $u(x) - I_2(x)$. What is the relevance of the information $c_3 \approx 5.056 \times 10^{-4}$?

(c) Although $I_n(x)$ is a Lagrange polynomial, why can the divided-difference formula (1.17) not be used to construct $I_3(x)$ as $I_3(x) = I_2(x) + u[x_0, x_1, x_2, x_3]\Psi_2(x)$?

Q5 (a) Using the approach in §1.2.9 in the notes, show that the *quadratic* forced-oscillation approximating polynomial, $F_2(x)$, for the function $u(x) = \sqrt{4+x}$ on $x \in [-1, 1]$ is

$$F_2(x) \approx 2.0000796 + 0.251503x - 0.0160202x^2 ,$$

and so use (1.82) to determine the associated forced-error coefficient ϕ_2.

(b) Estimate $\|u - F_2\|_\infty$ directly from $u(x) - F_2(x)$. How does the value of ϕ_2 compare with these estimates and with the Fourier coefficient $c_3 \approx 5.056 \times 10^{-4}$?

Q6 Let $u(t) = \sqrt{t}$ on $[1, 3]$. Using the transformation $t = \frac{1}{2}[(b + a) + (b - a)x]$, write u as a function of x on $[-1, 1]$, and so find the *linear* forced-oscillation approximating polynomial, $F_1(x)$, on $[-1, 1]$. Hence show that the corresponding polynomial on $[1, 3]$ is

$$F_1(t) \approx 0.658069 + 0.366025t .$$

Compute the forced-error coefficient $|\phi_1|$ and compare it with $c_2 \approx -2.407 \times 10^{-2}$ and $\|u - F_1\|_\infty \approx 2.494 \times 10^{-2}$

Quick-check answers and hints.

A1 Hint: note that $\left\{ \sum_{k=0}^{n}{}' c_k\, T_k \right\}^2$ should be written as

$$\left\{ \sum_{k=0}^{n}{}' c_k\, T_k \right\} \left\{ \sum_{\ell=0}^{n}{}' c_\ell\, T_\ell \right\} = \sum_{k=0}^{n}{}' \sum_{\ell=0}^{n}{}' c_k c_\ell T_k T_\ell\,,$$

i.e. two different dummy suffices are required to retain the cross-products in the square.

A2 (iii) $|c_5| \approx 5.093 \times 10^{-2}$, $\|u - C_3\|_\infty = |u(\pm 1) - C_3(\pm 1)|$. Hint: make an accurate sketch of $u(x)$ and so think why does (1.65) not hold as $x \to \pm 1$?

A3 (a) (i) Hint: make the substitution $\theta = \pi - \phi$ in the second integral. (b) (ii) Accurate values are $c_1 \approx 1.133648177$, $c_3 \approx -0.1380717764$ and $c_5 \approx 0.004490714244$.

A4 (ii) $-5.938 \times 10^{-4} \le u(x) - I_2(x) \le 5.422 \times 10^{-4}$.

A5 (i) $\phi_2 \approx 5.056 \times 10^{-4}$. (ii) $-5.098 \times 10^{-4} \le u(x) - F_2(x) \le 5.110 \times 10^{-4}$.

A6 $\phi_1 \approx -2.409 \times 10^{-2}$.