MATH3476 Numerical Methods

Problem Set 2

Norms and Minimax Approximation

The following questions are based upon the material covered in section 1.2.1–1.2.4 in the notes.

Q1 Derive the norms quoted in Example 1.8 in the notes. Sketch the function $u(x) - q_\lambda(x)$ for $x \in [0, 1]$ when $\lambda = 1, \frac{1}{2}, \frac{1}{10}$ and $\frac{1}{50}$ in order to explain why $\|u - q_\lambda\|_\infty$ remains at 1 even as $\lambda \to 0$.

Q2 (a) Write $\|u\|_1 = \int_a^b |u(x)|\,|1|\,dx$ and set $v \equiv 1$ in the Cauchy-Schwarz inequality $|(u, v)| \leq \|u\|_2 \|v\|_2$ to prove the first inequality in (1.35) in the notes. Next, replace the integrand in $\|u\|_2$ by its maximum value to prove the second inequality in (1.35). Hence show that for all functions $u \in \mathcal{C}[a, a + 1]$, we have $\|u\|_1 \leq \|u\|_2 \leq \|u\|_\infty$.

(b) Prove that the norms $\|u - q_\lambda\|_1$, $\|u - q_\lambda\|_2$ and $\|u - q_\lambda\|_\infty$ calculated in Q1 do indeed satisfy the inequalities derived in part (a).

Q3 (i) Using (1.37) in the notes, calculate the Bernstein polynomials $p_1(x)$, $p_2(x)$ and $p_3(x)$ for approximating $u(x) = x^2$ on $[0, 1]$.

(ii) Using your answers to (i), can you *guess* what $p_4(x)$ is? Propose (without proof) a formula for $p_n(x)$ and so verify that it satisfies (1.38) exactly, i.e. that $\|u - p_n\|_\infty = 1/(4n)$.

(iii) Using your proposed formula for $p_n(x)$ derive similar formulae for $\|u - p_n\|_1$ and $\|u - p_n\|_2$ and so show that these, along with $\|u - p_n\|_\infty$, satisfy the inequalities derived in Q2 (a).

Q4 Using the approach of Example 1.10 in the notes, deduce that the *linear* minimax approximation to the function $u(x) = \cos x$ on $[-\frac{1}{2}, 1]$ is $q_1^*(x) \approx 0.8953 - 0.2249x$, and that $\rho_1 \equiv \|u - p_1\|_\infty \approx 0.1301$.

Q5 For the function $u(x) = \ln x$ on $[1, 2]$, compute the linear:

(i) Taylor polynomial, $t_1(x) = t_0 + t_1 x$, by expanding about $x = \frac{3}{2}$;

(ii) Lagrange polynomial, $p_1(x) = p_0 + p_1 x$, and;

(iii) minimax polynomial, $q_1^*(x) = q_0 + q_1 x$.

For each of these linear polynomials $Q(x)$, sketch (on the same axes) the error function $E_1^{(Q)}(x) = u(x) - Q(x)$ and (hence) calculate $\|u - Q\|_\infty$.

Q6 (a) Prove the following result. Let $u \in \mathcal{C}^2[a,b]$ with $u''(x) > 0$ for $x \in [a,b]$. If $q_1^*(x) = q_0 + q_1 x$ is the linear minimax approximation to $u(x)$ on $[a,b]$, then

$$q_1 = \frac{u(b) - u(a)}{b - a} \qquad q_0 = \frac{u(a) + u(c)}{2} - \left(\frac{a + c}{2}\right)\left(\frac{u(b) - u(a)}{b - a}\right)$$

where c is the unique solution of

$$u'(c) = \frac{u(b) - u(a)}{b - a}\,.$$

What is ρ_1? What is the geometrical significance of the formula for $u'(c)$?

(b) Apply the general theory of part (a) to $u(x) = e^x$ on $[-1,1]$ in order to confirm the results of Example 1.10 in the notes.

Quick-check answers and hints.

A2 (b) Hint: if $x, y > 0$, to prove $x \le y$, show that $y^2 - x^2 \ge 0$.

A3 (i) $p_3(x) = x(1 + 2x)/3$.

A4 Hint 1: sketch u and a possible linear approximation $q_1^* \equiv a_0 + a_1 x$ and decide upon the signs of the error $E \equiv u - q_1^*$ at the ends and interior of the interval.

Hint 2: if ξ is the interior point where $|E|$ is maximised, solve for the unknowns in the order a_1, ξ, ρ_1 and a_0.

A5 Hint: the errors in p_1 and q_1^* are translations of each other. Norms are approximately (i) 7.213×10^{-2}, (ii) 5.966×10^{-2} and (iii) 2.983×10^{-2}.

A6 Hint: $c = \xi$ in the notes.