MATH3476 Numerical Methods

Problem Set 1

Polynomial Interpolation

The following questions are based upon the material in sections 1.1.1 to 1.1.3 of the notes.

Q1 (i) With $u(x) = \cos x$ for $x \in [0, \frac{\pi}{2}]$, obtain the Lagrange interpolating polynomial (1.1) with regularly spaced nodes and $n = 2$ and so show that

$$p_2(\tfrac{\pi}{5}) = \tfrac{3}{25}(1 + 4\sqrt{2}),$$

and compute the error E_2 at the interpolation point $x = \frac{\pi}{5}$. Evaluate $\sum_{i=0}^{n} \ell_i(x)$ and comment on your answer.

(ii) Obtain the Lagrange polynomial when $n = 3$ and so show that

$$p_3(\tfrac{\pi}{5}) = \tfrac{3}{50} + \tfrac{54}{125}\sqrt{3},$$

and compute the error E_3 at the interpolation point $x = \frac{\pi}{5}$. Evaluate $\sum_{i=0}^{n} \ell_i(x)$ and comment on your answer.

Q2 Obtain the Lagrange interpolating polynomial $p_2(x)$ for a function $u(x)$ given at regularly spaced nodes $x_0 = -h$, $x_1 = 0$ and $x_2 = h$. Hence, using the error formula (1.9) in the notes, proceed along the lines of Example 1.3 to deduce that

$$u'(x) = \frac{(2x-h)u(x_0) - 4xu(x_1) + (2x+h)u(x_2)}{2h^2} + \frac{3x^2 - h^2}{6}u'''(\xi),$$

where $x, \xi \in [-h, h]$, and so write down approximations for $u'(x_0)$, $u'(x_1)$ and $u'(x_2)$, including the error terms. Where is the error minimised and maximised?

Q3 Let $p_2^{(0,2)}(x)$ and $p_2^{(1,3)}(x)$ denote quadratic Lagrange polynomials that interpolate the data at $\{(x_0, u_0),\ (x_1, u_1),\ (x_2, u_2)\}$ and $\{(x_1, u_1),\ (x_2, u_2),\ (x_3, u_3)\}$ respectively. Show that

$$P \equiv \frac{(x_3 - x)p_2^{(0,2)}(x) + (x - x_0)p_2^{(1,3)}(x)}{x_3 - x_0}$$

is the *cubic* Lagrange polynomial $p_3(x)$ interpolating the data $\{(x_0, u_0), (x_1, u_1), (x_2, u_2), (x_3, u_3)\}$.

Q4 Using "property (i)" of divided differences and the fundamental definition of a derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

deduce that

$$\frac{d}{dx}u[x_0, x_1, \ldots, x_n, x] = u[x_0, x_1, \ldots, x_n, x, x].$$

This formula is used to compute interpolation errors when divided differences are used for numerical differentiation.

Q5 Use equations (1.16) to (1.18) in the notes to show that the divided-difference interpolating polynomial $p_2(x)$ for the function $u(x) = \cos x$ with nodes at $x_i = 0.2\,i$, $i = 0, 1, 2\ldots$ is

$$p_2(x) \approx 1 - 0.001986706\,x - 0.488402022\,x^2\,,$$

and use this to compute approximations $p_2(0.1)$ and $p_2(0.3)$ to $\cos 0.1$ and $\cos 0.3$ respectively.

Without evaluating $p_3(x)$ explicitly, determine the correction term to be added to $p_2(x)$ and so compute the refined approximations $p_3(0.1)$ and $p_3(0.3)$. Hence determine the errors E_2 and E_3 at both $x = 0.1$ and $x = 0.3$ and comment on your results.

Take the Python script `3476_1.4.py` from the course homepage, modify it and use it to calculate $p_4(0.1)$ and $E_4(0.1)$. Equation (1.19) in the notes will be useful here.

Quick-check answers to numerical questions

A1 (i) $E_2(\frac{\pi}{5}) \approx 1.02 \times 10^{-2}$ and $E_3(\frac{\pi}{5}) \approx 7.71 \times 10^{-4}$.

A5 $E_2(0.1) \approx 8.69 \times 10^{-5}$, $E_2(0.3) \approx -1.11 \times 10^{-4}$, $E_3(0.1) \approx -6.02 \times 10^{-5}$ and $E_3(0.3) \approx 3.57 \times 10^{-5}$.