MATH1400: Modelling with differential equations (Spring 2021)
Examples 4

Section 1: to be covered in tutorials. The solutions (without method) to these examples are given on the other side of this page, but attempt each one first.

1. Use the Reduction of Order technique to find the general solutions of the following ODEs. For each example, one solution y_1 to the homogeneous problem is given; verify this is the case.

(a) Using $y_1 = e^x$, solve: $(1+x)y'' + (1-2x)y' + (x-2)y = 0$

(b) Using $y_1 = x$, solve: $x^2y'' - xy' + y = x^2$

(c) Using $y_1 = e^x$, solve: $(1+x)y'' - (2x+3)y' + (x+2)y = 6e^x(1+x)^2$

(d) Using $y_1 = e^{-x/2}$, solve: $4y'' + 4y' + y = 8e^{-x/2}$

2. Solve the following second-order linear constant coefficient ODEs:

(a) $y'' + 2y' - 3y = 0$

(b) $y'' + 4y' + 13y = 0$

(c) $4y'' + 4y' + y = 0$

(d) $y'' + y' - 6y = 0$

(e) $y'' - 6y' + 9y = 0$

(f) $y'' + \omega^2 y = 0$

(g) $y'' + 2y' - 8y = \sin x$

(h) $y'' + 2y' - 8y = x^2$

(i) $y'' + 2y' - 8y = e^x$

(j) $y'' + 2y' - 8y = e^{2x}$

(k) $y'' + 6y' + 13y = e^{2x}$

(l) $y'' + y' - 12y = e^{3x}$

(m) $y'' + 2y' + y = e^x$

(n) $y'' - 2y' + y = e^x$

(o) $y'' + y = x^2$

(p) $y'' + 2y' - 3y = x^2 + x$, with $y(0) = y'(0) = 0$.

(q) $y'' + y' - 2y = e^x$, with $y(0) = 0$ and $y'(0) = 1$.

(r) $y'' + 4y = 2x$, with $y(0) = 1$ and $y'(0) = 2$.

(s) $y'' + 9y = \sin 2x$, with $y(0) = 1$ and $y'(0) = 0$.

(t) $y'' + y = \cos x$, with $y(0) = 1$ and $y'(0) = -1$.

3. Solve the following second-order linear constant coefficient ODEs:

(a) $y'' + 2y' + 5y = 17\cos(2x)$ with $y(0) = 0$ and $y'(0) = 0$.

(b) $y'' + 2y' + 5y = 17\cos(2x)e^{-x}$ with $y(0) = 0$ and $y'(0) = 0$.

(c) $y'' + 2y' + 5y = 17e^{-x}$ with $y(0) = 0$ and $y'(0) = 0$.

(d) $y'' + y' - 2y = 3e^x$ with $y(0) = 0$ and $y'(0) = 0$.

(e) $y'' + y' - 2y = 3e^{-x}$ with $y(0) = 0$ and $y'(0) = 0$.

(f) $y'' + y' - 2y = 6\cosh(x)$ with $y(0) = 0$ and $y'(0) = 0$.

(g) $y'' - y = 4\sin x$ with $y(0) = 0$ and $y'(0) = 0$.

(h) $y'' - y = 4\sin 2x$ with $y(0) = 0$ and $y'(0) = 0$.

(i) $y'' + y = 4\sin x$ with $y(0) = 0$ and $y'(0) = 0$.

(j) $y'' + y = 4\sin 2x$ with $y(0) = 0$ and $y'(0) = 0$.

Section 2: to be handed in

1. State (proof not required) whether the following pair of functions are linearly independent on the given interval.

(a) $x, x+1 \quad (0 < x < 1)$

(b) $1, e^{2x} \quad (x < 0)$

(c) $\sin x + \cos x, 2\sin(x + \frac{\pi}{4}) \quad$ (all x)

(d) $e^{kx}, xe^{kx} \quad$ (all x)

(e) $|x|x, 2x^2 \quad (0 < x < 1)$

2. Which of the following, $\sin x$ or $\cos x$, is a solution of $y'' - 2(\cot x)y' + (1 + 2\cot^2 x)y = 0$? Hence, use the Reduction of Order technique to find the general solution of

$$y'' - 2(\cot x)y' + (1 + 2\cot^2 x)y = \sin x\,.$$

3. Solve the following boundary value problem:

$$y'' - 2y' + 2y = 0, \quad y(0) = -3, \quad y(\frac{\pi}{2}) = 0$$

4. Solve the following initial value problem:

$$y'' - 2y' + y = x + e^x, \quad y(0) = 3, \quad y'(0) = 4$$

Solutions to section 1.

1. (a) $y = (C_1(1+x)^{-2} + C_2)e^x$

(b) $y = x^2 + C_2x + C_1x\log|x|$

(c) $y = (2x^3 + 3x^2 + C_1(1+x)^2 + C_2)e^x$

(d) $y = (x^2 + C_1x + C_2)e^{-x/2}$

2. (a) $y = C_1e^x + C_2e^{-3x}$

(b) $y = (C_1\cos(3x) + C_2\sin(3x))e^{-2x}$

(c) $y = (C_1 + C_2x)e^{-x/2}$

(d) $y = C_1e^{2x} + C_2e^{-3x}$

(e) $y = (C_1 + C_2x)e^{3x}$

(f) $y = C_1\cos(\omega x) + C_2\sin(\omega x)$

(g) $y = C_1e^{2x} + C_2e^{-4x} - \frac{2}{85}\cos x - \frac{9}{85}\sin x$

(h) $y = C_1e^{2x} + C_2e^{-4x} - \frac{1}{8}x^2 - \frac{1}{16}x - \frac{3}{64}$

(i) $y = C_1e^{2x} + C_2e^{-4x} - \frac{1}{5}e^x$

(j) $y = C_1e^{2x} + C_2e^{-4x} + \frac{1}{6}xe^x$

(k) $y = (C_1\cos 2x + C_2\sin 2x)e^{-3x} + \frac{1}{29}e^{2x}$

(l) $y = C_1e^{3x} + C_2e^{-4x} + \frac{1}{7}xe^{3x}$

(m) $y = (C_1 + C_2x)e^{-x} + \frac{1}{4}e^x$

(n) $y = (C_1 + C_2x)e^x + \frac{1}{2}x^2e^x$

(o) $y = C_1\cos x + C_2\sin x + x^2 - 2$

(p) $y = \frac{3}{4}e^x - \frac{1}{108}e^{-3x} - \frac{1}{3}x^2 - \frac{7}{9}x - \frac{20}{27}$

(q) $y = \frac{2}{9}e^x - \frac{2}{9}e^{-2x} + \frac{1}{3}xe^x$

(r) $y = \cos 2x + \frac{3}{4}\sin 2x + \frac{1}{2}x$

(s) $y = \cos 3x - \frac{2}{15}\sin 3x + \frac{1}{5}\sin 2x$

(t) $y = \cos x - \sin x + \frac{1}{2}x\sin x$

3. (a) $y = (-\cos 2x - \frac{9}{2}\sin 2x)e^{-x} + \cos 2x + 4\sin 2x$

(b) $y = \left(\frac{17}{4}x\sin 2x\right)e^{-x}$

(c) $y = \frac{17}{4}(1 - \cos 2x)e^{-x}$

(d) $y = xe^x - \frac{1}{3}e^x + \frac{1}{3}e^{-2x}$

(e) $y = -\frac{3}{2}e^{-x} + \frac{1}{2}e^x + e^{-2x}$

(f) $y = xe^x - \frac{3}{2}e^{-x} + \frac{1}{6}e^x + \frac{4}{3}e^{-2x}$

(g) $y = e^x - e^{-x} - 2\sin x$

(h) $y = \frac{4}{5}e^x - \frac{4}{5}e^{-x} - \frac{4}{5}\sin 2x$

(i) $y = 2\sin x - 2x\cos x$

(j) $y = \frac{8}{3}\sin x - \frac{4}{3}\sin 2x$