MATH1400: Modelling with differential equations (Spring 2021)
Examples 3

In all cases, clearly specify the ODE that must be solved, along with its initial condition, and use the appropriate method to find the particular solution.

Section 1: to be covered in tutorials.

1. A nuclear breeder reactor produces waste that contains (amongst other things) the isotope ^{239}Pu (Plutonium-239). After 15 years, the initial concentration of ^{239}Pu in the waste has decreased by 0.043%. Find the half-life of the isotope.

2. A body is found at the scene of a crime at 6 a.m. At that time, the temperature of the body was $31°$C. By 8 a.m., the temperature had cooled to $29°$C. The ambient temperature was $20°$C, and normal body temperature is $37°$C. Estimate when the murder occurred.

3. The population of a country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the country. Why is this not a good model for human population?

4. Assume that Lake Erie has a volume V of $480\,\mathrm{km}^3$ of water and that its rate of inflow r_i (from Lake Huron) and outflow r_o (to Lake Ontario) are both $350\,\mathrm{km}^3$ per year. Assume further that the outflow from Lake Erie is perfectly mixed lake water. Suppose that at time $t = 0$ (years), the pollutant concentration of Lake Erie (caused by past industrial pollution) is five times that of Lake Huron, and that the concentration of pollution, c_i, in Lake Huron does not change. Show that we can model the amount of pollution, $x(t)$, in Lake Erie by

$$\frac{dx}{dt} = r_i c_i - \left(\frac{r_o}{V}\right) x.$$

   How long will it take to reduce the pollution concentration in Lake Erie to the level of twice that of Lake Huron?

5. *Extra practice question:* As the salt KNO_3 dissolves in methanol, the number $x(t)$ of grams of the salt in a solution after t seconds satisfies the differential equation

$$\frac{dx}{dt} = 0.8x - 0.004x^2.$$

   (a) What is the maximum amount of the salt that will ever dissolve in the methanol?

   (b) If $x = 50$ when $t = 0$, how long will it take for an additional 50g of salt to dissolve?

6. *Extra practice question:* A particle is acted on by an oscillating force; there is also an air resistance force that opposes the motion. The speed v of the particle is governed by the ODE

$$\frac{dv}{dt} = F \sin t - kv,$$

   where F is proportional to the strength of the oscillating force and k is a constant giving the size of the air resistance. What kind of ODE is this? Suppose that at time $t = 0$, the speed is v_0. Find $v(t)$.

7. *Extra practice question:* Solid cancerous tumours do not grow exponentially in time. As the tumour becomes larger, the doubling time increases. Research has shown that the number of cells $n(t)$ in a tumour at time t satisfies the equation

$$n(t) = n_0 \exp\left(\frac{\lambda}{\alpha}\left(1 - e^{-\alpha t}\right)\right),$$

where n_0, λ and α are constants.

(a) Show that n satisfies the differential equation

$$\frac{dn}{dt} = \lambda e^{-\alpha t} n \quad \text{with} \quad n(0) = n_0.$$

(b) A tumour satisfies the above equation with $\alpha = 0.02$. Originally, when it contained 10^4 cells, the tumour was increasing at a rate of 20% per unit time. What is the limiting number of cells in the tumour?

Section 2: to be handed in

1. A nurse is administering iodine-131 (^{131}I) to a patient with hyperthyroidism. The available medication is in solution form, with a concentration of $100\,\mu\text{g}\,\ell^{-1}$, and the doctor has specified a dose of $50\,\text{MBq}$ (that means 50 million decays per second). Given that ^{131}I has a half-life of $8.1\,\text{days}$, and that 6.02×10^{23} atoms of ^{131}I have a mass of $131\,\text{g}$, calculate the volume of solution, to the nearest $0.01\,\text{m}\ell$, that the nurse must measure out.

2. Suppose that the ambient temperature varies sinusoidally in time. Then Newton's Law of cooling becomes:

$$\frac{d\Theta}{dt} = -k(\Theta - A\sin(\omega t)),$$

with $\Theta(0) = \Theta_0$. Solve this equation, and describe the behaviour of the solution in the limit $t \to \infty$.

3. The population $p(t)$ of the county of Midsomershire obeys the logistic equation:

$$\frac{dp}{dt} = 0.04p\left(1 - \frac{p}{10^6}\right),$$

where t is measured in years.

(a) Modify the equation to take into account the fact that $10\,000$ people move away from Midsomershire each year. Find the general solution $p(t)$.

(b) If the population was 8×10^6 in 1970, find the particular solution. What happens as $t \to \infty$?

4. A lecture theatre contains a volume of air $V = 120{,}000\,\text{m}^3$. It is ventilated by a system that pumps in fresh air at a rate r, and that draws out stale air at the same rate. Assume that the air in the theatre is mixed well. The ventilation system is required to be able to reduce any impurities to a level of 1% of their initial concentration in $30\,\text{min}$. Find the required pumping rate r (in m^3/min). You may leave your answer in terms of logs, or use a calculator.