MATH1400: Modelling with differential equations (Spring 2021)
Examples 2

Note: In all cases, use the appropriate method to find the particular solution in the case when an initial condition is given, or the general solution in the case when an initial condition is not given. If possible, write the solution y as an explicit function of x.

Section 1: to be covered in tutorials. The solutions to some of these examples are given on the other side of this page.

1. Solve the following **first-order linear** ODEs.

(a) $y' = \frac{y}{x} + 1$.

(b) $y' - xy = x$.

(c) $y' + y\cos x = \exp(-\sin x)$.

(d) $y' + y\tan x = \cos^2 x$.

(e) $y' - y\tan x = 1$, with $y(\pi/4) = 3$.

(f) $xy' - 2y = x^2$, with $y(1) = 1$.

2. Verify that the following ODEs are of **homogeneous degree**. Solve the ODEs.

(a) $y' = \frac{y}{x} + 1$.

(b) $x\frac{dy}{dx} = 3x + 4y$.

(c) $(x+y)\frac{dy}{dx} = x - y$.

(d) $x\frac{dy}{dx} = y + 2\sqrt{xy}$.

(e) $2xy\frac{dy}{dx} = x^2 + 2y^2$.

(f) $x(x+y)\frac{dy}{dx} = y(x-y)$.

(g) $xy\frac{dy}{dx} = y^2 + x\sqrt{4x^2 + y^2}$.

3. Verify that the following ODEs are of the form of **Bernoulli's** equation, or can be put in to that form. Solve the ODEs.

(a) $\frac{dy}{dx} + \frac{1}{x}y = x^2y^2$.

(b) $\frac{dy}{dx} - y = \sqrt{y}$.

(c) $x^2\frac{dy}{dx} + 2xy = 5y^4$.

(d) $y^2\frac{dy}{dx} + 3xy^3 = 6x$.

(e) $\frac{dy}{dx} = y + y^3$.

(f) $y^2\left(x\frac{dy}{dx} + y\right) = (1 + x^3)^{\frac{1}{2}}$.

4. Verify that the following ODEs are **exact**. Solve the ODEs.

(a) $x^2(x + 2y^3)y' + xy(3x + y^3) = 0$.

(b) $xy'\sin y - \cos y = 2x$.

5. *Extra question:* what is the Integrating Factor for the following **first-order linear** ODE

$$y' + ay = h(x),$$

where a is a constant with $a \neq 0$? Write down the general solution of the ODE, and note that it comes in two parts: one with an integral involving $h(x)$, and the other with an arbitrary constant. (Later on we will learn that the first part is called the **Particular Integral** and the second part is called the **Complementary Function**.) Solve the ODE in the case that $h(x)$ is:
(a) $h(x) = 1$; (b) $h(x) = x$; (c) $h(x) = x^2$; (d) $h(x) = e^{kx}$ with $k \neq -a$; (e) $h(x) = e^{-ax}$; (f) $h(x) = xe^{-ax}$; (g) $h(x) = \sin x$.

Section 2: to be handed in.
Note that up to 3 marks (out of 20) are awarded for **clarity and presentation**. To gain these marks, show your working, structure your answers clearly and logically, and give appropriate explanations (using sentences as well as equations).

1. Solve the following **first-order linear** ODE:
$$y' + \frac{2}{x}y = 4x.$$

2. Verify that the following ODE is of **homogeneous degree** and solve the initial value problem:
$$x^2y' = y(x+y), \quad \text{with} \quad y(1) = 1.$$

3. Verify that the following ODE is of the form of **Bernoulli's** equation and find its general solution:
$$x^3\frac{y'}{y} - x^2 = -y^3\cos x.$$

4. Verify that the following ODE is **exact**. Solve the ODE.
$$(xe^{xy} - x^2\cos y + 2)y' + ye^{xy} - 2x\sin y = 0.$$

5. Use an **appropriate solution technique** to solve the following first-order ODEs.
   (a) $x\cos(y)\, y' + \sin y = 0.$
   (b) $y' + 2y\tan x = \sin x$, with $y(\frac{\pi}{3}) = 0.$
   (c) $yy' + xy^2 = \exp(-x^2).$

Selected solutions to section 1.

1. First-order linear: (a) $y = x(\log|x| + C)$; (b) $y = Ce^{x^2/2} - 1$; (c) $y = (x+C)e^{-\sin x}$; (d) $y = (C + \sin x)\cos x$; (e) $y = (\sin x + \sqrt{2})\sec x$; (f) $y = x^2(1 + \log|x|)$.

2. Homogeneous: (b) $y = -x + Cx^4$; (c) $y^2 + 2xy - x^2 = C$; (d) $y = x(\log|x| + C)^2$; (e) $y = \pm x\sqrt{\log|x| + C}$; (f) $C = \log|xy| - \frac{x}{y}$; (g) $y = x\sqrt{(\log|x| + C)^2 - 4}$.

3. Bernoulli: (a) $y = \frac{1}{Cx - \frac{1}{2}x^3}$; (b) $y = (Ce^{x/2} - 1)^2$; (c) $y = (\frac{15}{7x} + Cx^6)^{-1/3}$; (e) $y = (-1 + Ce^{-2x})^{-1/2}$.

4. Exact: (a) $H(x,y) = C = 2x^3y + x^2y^4$; (b) $H(x,y) = C = x^2 + x\cos y$.