MATH1400
Modelling with Differential Equations

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Chapter 1

Introduction

"All models are wrong... but some are useful!"[1]

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1.1 Differential equations: the basics

1.1.1 The derivative

The derivative of a function $y(x)$ at a particular value of x is the slope of the tangent to the curve at the point P, or $(x, y(x))$. Referring to Fig. 1.1, suppose $y(x)$ is a function; then the derivative dy/dx at a particular value of x is given by:

$$\frac{dy}{dx} = \tan \Psi$$

Let Q is a neighbouring point on the curve, then we can take the limit as Q tends to P:

$$\frac{dy}{dx} = \lim_{Q \to P} \frac{QR}{PR} = \lim_{\delta x \to 0} \frac{y(x + \delta x) - y(x)}{\delta x}$$

assuming that the limit exists.

1.1.2 Dependent and independent variables

When considering differential equations, it is important to distinguish between the **dependent** variable(s) and the **independent** variable. For example, time t might be

[1] George Box, a mathematical modeller.

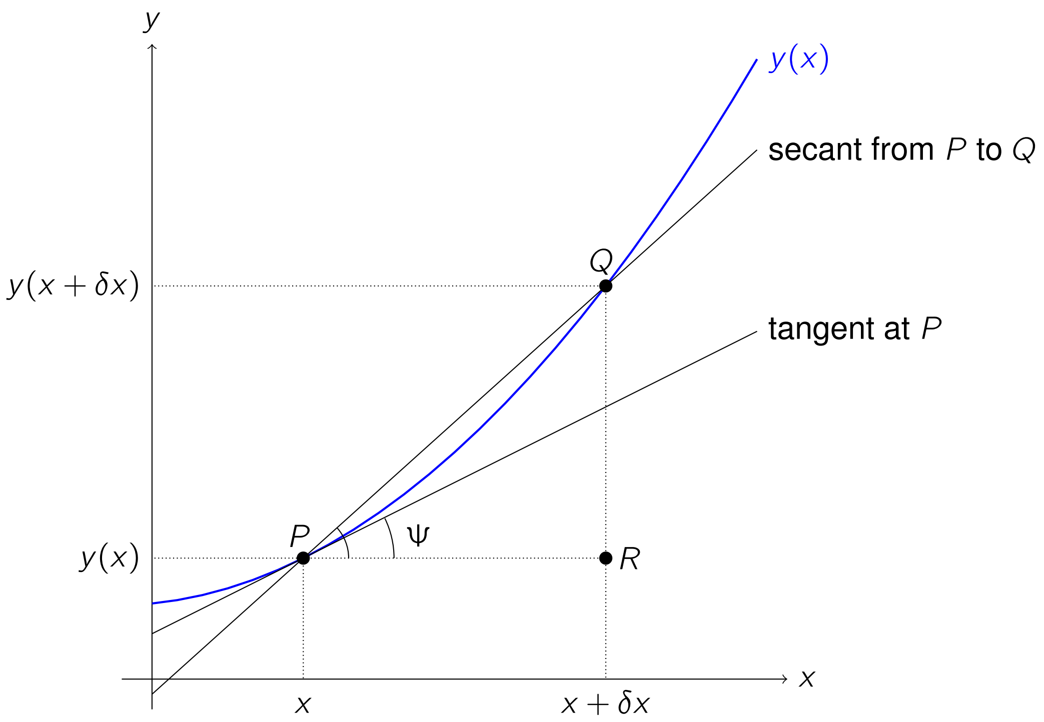


Figure 1.1: Definition of the derivative dy/dx.

the independent variable, and the population $N(t)$ of bacteria on a plate could be the dependent variable: the population depends on time, time does not depend on the population! Thus we would naturally consider dN/dt, rather than dt/dN. Both derivatives make sense mathematically, but dN/dt makes more sense in the context of this problem.

There can be more than one dependent variable, but there can only be one independent variable in an **ordinary differential equation (ODE)**. In situations where there is more than one independent variable, we obtain a partial differential equation (PDE).

Throughout this course, when we write:

$$\frac{dy}{dx},$$

we are thinking of y as the dependent variable and x as the independent variable, but the names x and y are not important. We could equally well think of

$$\frac{dy}{dt} \quad \text{or} \quad \frac{dx}{dt}$$

(and indeed we will do this later on in the course). Then t is the independent variable while y and x are the dependent variables.

1.1.3 Ordinary differential equations

An **ordinary differential equation (ODE)** is an equation that involves the derivatives of a dependent variable $y(x)$ with respect to a *single* independent variable x, e.g. $y' = dy/dx$, $y'' = d^2y/dx^2$. The equation may also involve y itself and some given functions of x. Sometimes, there may also be more than one dependent variable.

Examples of ODEs:

$$
\begin{array}{rl}
\text{(a)} & y' = 4x^7 \\
\text{(b)} & y'' - 7y' + 12y = 5\cos x \\
\text{(c)} & y'' + 2(y')^2 = 75x^3 \\
\text{(d)} & yy''' + 1 = 0 \\
\text{(e)} & (y')^2 + y = 7x \\
\text{(f)} & (y'')^2 + \sqrt{y} = 4
\end{array}
$$

The aim is to **solve** the differential equation, that is, to obtain a relationship between y and x that doesn't involve any derivatives. In some cases, it is possible to work out an **explicit solution**, i.e. $y = f(x)$. In other cases, we may find a relationship between x and y that cannot be solved explicitly for y. For example, $x - y - y^3 = C$ is enough to calculate y for any given value of x. This is called an **implicit** solution.

The majority of the course will mainly be concerned with situations where it is possible to calculate explicit or implicit solutions of ODEs. However, in many cases it is impossible to write down an explicit or implicit solution. Even in these cases, we can get useful information about **qualitative features** of solutions via **phase plane analysis** – this will come at the end of the course [2].

1.1.4 Classification

In order to know whether or not we are in a case where an explicit solution is possible, we need to be able to **classify** the ODE.

Order: The order of an ODE is the largest number of times that the dependent variable is differentiated in the ODE. So the orders of (a)–(f) above are respectively: 1, 2, 2, 3, 1, 2.

Linear: An ODE is linear if it contains no products or powers (other than one) of the **dependent variable** or its derivatives. A linear ODE cannot contain terms like y^2, yy', $\sqrt{y}$, $\cos y$ etc. Powers of the **independent variable** are allowed. So the examples (a) and (b) are linear and the others are nonlinear.

[2] The second year course MATH2391 'Nonlinear differential equations' takes this further.

Autonomous: An ODE is autonomous if there is no explicit mention of the independent variable. So (d) and (f) are autonomous and the others are non-autonomous.

1.1.5 General solutions and particular solutions

We now show that an ODE may, and in general will, have many solutions.

Example 1.1. *Show that the function*

$$y(x) = C_1 e^{3x} + C_2 e^{4x} + \frac{11}{34}\cos x - \frac{7}{34}\sin x$$

is an explicit solution of (b) in section 1.1.3, where C_1 and C_2 are arbitrary constants. This is an example of how to **verify** *a proposed solution to a known ODE.*

Solution: Compute first and second derivatives of the function $y(x)$:

$$y' = 3C_1 e^{3x} + 4C_2 e^{4x} - \frac{11}{34}\sin x - \frac{7}{34}\cos x,$$
$$y'' = 9C_1 e^{3x} + 16C_2 e^{4x} - \frac{11}{34}\cos x + \frac{7}{34}\sin x,$$

and then substitute into the LHS of (b):

$$\begin{aligned} y'' - 7y' + 12y &= (9 - 21 + 12)C_1 e^{3x} + (16 - 28 + 12)C_2 e^{4x} \\ &\quad + \left(-\frac{11}{34} + \frac{49}{34} + \frac{132}{34}\right)\cos x + \left(\frac{7}{34} + \frac{77}{34} - \frac{84}{34}\right)\sin x \\ &= 5\cos x. \end{aligned}$$

So LHS = RHS, and we have **verified** the proposed solution to a known ODE.

Note that we can assign any values to C_1 and C_2, and $y(x)$ will still be a solution of (b). Therefore $y(x)$ represents a family of infinitely many solutions to (b). In general, in order to solve an nth order ODE, we will have to integrate n times, introducing n arbitrary constants C_1, C_2, ..., C_n, so the explicit solution $y(x)$ will depend on n arbitrary constants. We will write the solution as $y(x, C_1, C_2, \ldots, C_n)$ and call it the **general solution** of the nth order ODE. If we assign some definite values to C_1, C_2, ..., C_n, then we obtain a **particular solution** of the ODE. For example, a particular solution of (b) is

$$y(x) = 1.5e^{3x} + 2.4e^{4x} + \frac{11}{34}\cos x - \frac{7}{34}\sin x.$$

Going in the other direction, given $y(x, C_1, C_2, \ldots, C_n)$, we can obtain an nth order ODE that does not contain the constants, as the following examples show.

Example 1.2. *Suppose that $n = 2$, and $y(x, C_1, C_2)$ is given by $y = C_1 e^x + C_2 e^{2x}$. Find an ODE satisfied by y that does not contain the constants C_1 and C_2.*

Solution: We can use the fact that the derivative of a constant is zero as the basis of a method ("isolate and eliminate") to find the ODE. First, multiply the expression for $y(x)$ through by e^{-x} to **isolate** C_1:

$$e^{-x}y = C_1 + C_2e^x.$$

Now, we differentiate with respect to x, which will **eliminate** C_1:

$$\begin{aligned}\frac{d}{dx}\left(e^{-x}y\right) &= \frac{d}{dx}\left(C_1 + C_2e^x\right)\\ -e^{-x}y + e^{-x}y' &= C_2e^x.\end{aligned}$$

Now, we repeat the cycle with C_2. First, **isolate**:

$$-e^{-2x}y + e^{-2x}y' = C_2.$$

Then, **eliminate**:

$$\begin{aligned}\frac{d}{dx}\left(-e^{-2x}y + e^{-2x}y'\right) &= 0\\ 2e^{-2x}y - e^{-2x}y' - 2e^{-2x}y' + e^{-2x}y'' &= 0.\end{aligned}$$

Finally, we re-arrange this expression to obtain the simplest form of the ODE:

$$\begin{aligned}e^{-2x}\left(y'' - 3y' + 2y\right) &= 0\\ y'' - 3y' + 2y &= 0.\end{aligned}$$

Example 1.3. *Suppose that* $n = 1$*, and* $y(x, C)$ *is:*

$$y = \sqrt{x^2 + C}$$

Find an ODE satisfied by y *that does not contain the constant* C*.*

Solution: First, **isolate** the arbitrary constant C by squaring both sides:

$$y^2 = x^2 + C.$$

Then, **eliminate** by taking the derivative with respect to x:

$$2yy' = 2x \implies y' = \frac{x}{y}.$$

If we have some ideas about how the required ODE may look like, a second method to obtain the ODE is by assuming it takes a certain form that contains some unknown coefficients. We then determine these coefficients by substituting the given general solution into the assumed ODE. For example, if we expect the ODE to be second-order and linear, we can assume: $y'' + F(x)y' + G(x)y = H(x)$. More details on this method are given in the lectures and Example Sheet 1.

In general, if we are given sufficient additional information about y and its derivatives, we can (usually) find the values of the constants, $C_1, C_2, \ldots, C_n$. This additional information can be given in two common ways: initial conditions and boundary conditions.

1.2 Initial conditions and boundary conditions

Initial value problem (IVP): The general solution $y(x, C_1, C_2, \ldots, C_n)$ comprises the multiple solutions of an nth order ODE. Suppose now we are given the values of y, y', y'', ..., $y^{(n-1)}$ at one specific value of $x = x_0$. These n conditions, called the initial conditions (at $x = x_0$), allow us to uniquely determine the values of the n constants. In other words, we pick out the particular solution that satisfies *both* the ODE *and* the initial conditions. This is called an initial value problem.

Boundary value problem (BVP): For an ODE of order n, the n arbitrary constants in the general solution can also be fixed if we are given the boundary conditions, i.e., if we are given n values of y or its derivatives at, at least, two values of x. Finding the particular solution that satisfies both the ODE and the boundary conditions is called the boundary value problem.

Example 1.4. *Solve*

$$y'' - 3y' + 2y = 0$$

with boundary conditions $y(0) = 0$ *and* $y(1) = e - e^2$*.*

Solution: Here $n = 2$ (a second-order ODE) and we are given the value of y at two different values of x, so this is a boundary value problem. We already have a general solution with two arbitrary constants from Example 1.2:

$$y(x) = C_1 e^x + C_2 e^{2x}.$$

Figure 1.2 plots $y(x)$ for several different values of C_1 and C_2. Substitute $x = 0$ and $x = 1$ into this explicit solution, we obtain two simultaneous equations for C_1 and C_2:

$$\begin{aligned} y(0) &= C_1 + C_2 = 0 \\ y(1) &= C_1 e + C_2 e^2 = e - e^2 \end{aligned}$$

The first equation gives us $C_2 = -C_1$, and the second equation gives us $C_1 = 1$, so the solution to our boundary value problem is:

$$y(x) = e^x - e^{2x}.$$

This is shown as the black curve in Fig. 1.2. It is the only particular solution satisfying the given boundary conditions (indicated by the black circles).

Verification: In many of the examples we will examine, it is possible to **verify** that we have the correct solution by checking that our answer indeed satisfies the differential equation and has the right initial or boundary values.

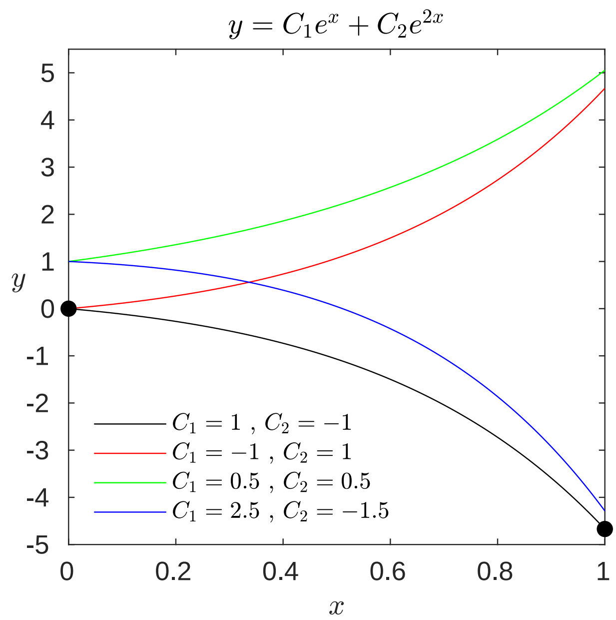


Figure 1.2: Some particular solutions of the ODE in Example 1.4. Only the black curve satisfies the boundary conditions indicated by the black circles.

Check the ODE:

$$
\begin{aligned}
y' &= e^x - 2e^{2x} \\
y'' &= e^x - 4e^{2x} \\
\text{LHS} &= y'' - 3y' + 2y = (1-3+2)e^x + (-4+6-2)e^{2x} = 0 \\
\text{RHS} &= 0 \\
\text{LHS} &= \text{RHS, so the ODE is satisfied} \qquad \checkmark
\end{aligned}
$$

Check the value of y at the "boundary" $x = 0$ and $x = 1$:

$$
\begin{aligned}
y(0) &= e^0 - e^0 = 0 \qquad \checkmark \\
y(1) &= e - e^2 \qquad \checkmark
\end{aligned}
$$

so our solution is correct.

1.3 Mathematical modelling

1.3.1 What is a model?

Mathematical modelling of a system in an application area of interest (e.g., physics, biology, engineering, economics, ...) is the translation of our knowledge or beliefs of that system into the language of mathematics. It typically involves writing down an equation (or set of equations) that describes the behaviour of the system.

The majority of real-world problems are far too complex to model in their entirety. The process of developing a mathematical model therefore involves making a number of simplifications along the way. We accept that all models are limited and, to some extent, wrong; but if the modelling process is conducted with intelligence and care, models are still undeniably useful. They enable us to understand complex phenomena, make predictions about the future and test the impact of changes to a system.

1.3.2 Why differential equations?

Typically, mathematical models can be classified in a number of ways, e.g., empirical vs. mechanistic. Empirical models are often data-driven and use statistical methods to infer relations between different variables in a system; mechanistic models use prior knowledge and understanding of the mechanism that causes change in a system.

In many applications, we are given the value of a quantity **at the present time** (for example, the temperature of coffee in a cup, the number of people infected with a virus, the concentration of carbon dioxide in the atmosphere) and we wish to predict its value in the **future**. To do this, we must know how quickly the quantity is changing. Mathematically, the **rate of change** of this quantity is its **derivative**. If we can write down an equation relating this derivative to some knowledge of the mechanism that causes the change, we can calculate how the temperature changes, how the number of infected people changes, how the concentration of carbon dioxide changes. This gives rise to a **differential equation**—which is a mechanistic model of the quantity of interest.

1.3.3 An example: bacteria growth

Solving a problem using mathematical models often involves three steps: (1) building the model, (2) solving the equations and (3) interpreting the solutions. We illustrate these steps using an example of bacteria growth.

Let $N(t)$ denote the number of bacteria growing on a plate of nutrients at time t. Suppose that we know the **initial value** of N at time $t = t_0$. At this time, the value of N is N_0, or:

$$N(t_0) = N_0. \tag{1.1}$$

We want to know at what time the number of bacteria becomes $10N_0$.

To build a mathematical model for this problem, let us first describe how the number of bacteria changes with time using mathematics. Suppose over the time interval $(t, t + \delta t)$, the number of bacteria increases by an amount δN which we write as $\delta N = k(N, t)\delta t$. Then

$$N(t + \delta t) = N(t) + k(N, t)\delta t.$$

Rearranging:

$$\frac{N(t+\delta t)-N(t)}{\delta t}=k(N,t)$$

In the limit of $\delta t \to 0$, we get the **differential equation**:

$$\frac{dN}{dt}=k(p,t).$$

To make progress, we need some information on the function $k(N,t)$. Suppose by doing experiments, we observe that the rate of change of N is proportional to N. This means that if there are, say, twice as many bacteria, then N will grow twice as rapidly. Mathematically,

$$k(N,t)=\sigma N$$

where σ is a constant we can measure from experiments. Therefore, the mathematical model for our bacteria growth problem is,

$$\frac{dN}{dt}=\sigma N. \tag{1.2}$$

To predict the value of $N(t)$ at any time t, we must now solve the differential equation (1.2) with the initial condition (1.1). We shall learn how to solve (1.2) in the next chapter. For now we claim the solution to this initial value problem is

$$N(t)=N_0 e^{\sigma t}. \tag{1.3}$$

We can easily verify this. First we show that

$$N(0)=N_0 e^0=N_0.$$

Next, calculate dN/dt and verify that it satisfies the differential equation:

$$LHS=\frac{dN}{dt}=N_0\sigma e^{\sigma t},$$

$$RHS=\sigma N=\sigma\left(N_0 e^{\sigma t}\right).$$

So $LHS=RHS$ as required.

Finally from the solution (1.3), we extract the answer to the question we ask at the beginning of this section. Let t_1 be the time $N=10N_0$. Then

$$N(t_1)=N_0 e^{\sigma t_1}=10N_0,$$

which gives

$$t_1=\frac{1}{\sigma}\log 10.$$

So we see that t_1 increases when σ decreases. Does this make sense?

Chapter 2

Solution of first-order ODEs

Contents

2.1 Introduction

A first-order ODE involves the dependent variable y and its first derivative $y' = dy/dx$. There will be a single initial value: $y = y_0$ when $x = x_0$ (often $x_0 = 0$, but not always). Then the general problem takes the form:

$$y' = f(y, x) \qquad \text{with} \quad y(x_0) = y_0, \tag{2.1}$$

where $f(y, x)$ is a given function of y and x.

Some ODEs are easy to solve, for example, if the function $f(x, y)$ depends only on x:

$$y' = f(x),$$

which is solved by $y = \int f(x)\, dx + C$, where C is an arbitrary constant of integration. However, just integrating like this is, in general, not possible.

We will consider three important special cases of first-order ODEs: **separable**, **exact** and **linear**, and show in each case how to find explicit solutions. Note that these three cases are not mutually exclusive, e.g. a linear ODE can be separable. We will also consider some examples that are not separable, exact or linear, but that can be transformed into one of these forms.

2.2 Separable first-order ODEs

A first-order ODE is **separable** if it can be written in the form:

$$\frac{dy}{dx} = a(x)b(y), \tag{2.2}$$

where $a(x)$ and $b(y)$ are functions of x and y respectively. To solve this equation, we follow these steps:
1. **Bring** the $b(y)$ over to the other side:

$$\frac{1}{b(y)}\frac{dy}{dx} = a(x)$$

2. Now **integrate** with respect to x, not forgetting the **constant of integration**:

$$\int \frac{1}{b(y)}\frac{dy}{dx}\,dx = \int a(x)\,dx + C$$

3. Next, **change variables** in the integral for y:

$$\int \frac{1}{b(y)}\,dy = \int a(x)\,dx + C$$

4. If we can **evaluate** these indefinite integrals (depends on the functions a and b), and then re-arrange the answer, we can write down the **general solution** $y(x)$ explicitly.
5. Finally, if we are given the **initial condition** $y(x_0) = y_0$, we can evaluate the constant of integration C, and so find the **particular solution**. (If you don't need the general solution, it is sometimes easier to do step 5 after doing the integrals but before solving for the explicit solution.)
6. And last of all, we can **verify** that the answer satisfies the ODE and the initial value, by working out the LHS and RHS of the original ODE.

Example 2.1. *Consider*

$$\frac{dy}{dx} = \frac{\sin x}{2y} \qquad \textit{with} \quad y\left(\frac{\pi}{2}\right) = -2.$$

Find the explicit solution, and verify that the answer is correct.

Solution: Here,

$$a(x) = \sin x \qquad \text{and} \qquad b(y) = \frac{1}{2y}$$

Proceed as above:
1. Bring the $b(y)$ over to the other side:

$$2y\frac{dy}{dx} = \sin x$$

2. and 3. Now integrate with respect to x, not forgetting the constant of integration, and change variables in the integral for y:

$$\int 2y\frac{dy}{dx}\,dx = \int 2y\,dy = \int \sin x\,dx + C$$

4. We can evaluate these indefinite integrals:

$$y^2 = -\cos x + C$$

and we can re-arrange to find y explicitly by taking the square root of both sides:

$$y = \pm\sqrt{C - \cos x}.$$

This is the general solution.
5. We use $y(\frac{\pi}{2}) = -2$ to evaluate C and determine whether we take the '+' or '−' square root:

$$-2 = \pm\sqrt{C} \qquad \text{so} \quad C = 4,$$

and we take the negative square root. We therefore can write the particular solution:

$$y = -\sqrt{4 - \cos x}.$$

6. Finally, we verify that this is correct. First, check the initial condition

$$y\left(\frac{\pi}{2}\right) = -\sqrt{4 - \cos\frac{\pi}{2}} = -2;$$

then check the ODE by differentiating y with respect to x:

$$y' \;=\; -\frac{1}{2}(4-\cos x)^{-\frac{1}{2}}(\sin x) = \frac{1}{2}\frac{\sin x}{(-\sqrt{4-\cos x})} = \frac{\sin x}{2y}.$$

Thus, we have verified that our solution solves the initial value problem.

Example 2.2. *Find (and verify) the solution of*

$$\frac{dy}{dx} + 4 = y^2 \qquad \textit{with} \quad y(0) = 4.$$

Solution: Separate and integrate (using partial fractions):

$$\begin{aligned}
\int \frac{1}{y^2-4}dy &= \int dx \\
\frac{1}{4}\int \frac{1}{y-2}dy - \frac{1}{4}\int\frac{1}{y+2}dy &= \int dx \\
\frac{1}{4}\log|y-2| - \frac{1}{4}\log|y+2| &= x + C \\
\log\left|\frac{y-2}{y+2}\right| &= 4x + 4C \\
\frac{y-2}{y+2} &= Ae^{4x} \text{ (define a new constant } A = e^{4C}).
\end{aligned}$$

We can apply the initial condition $y(0) = 4$ at this stage:

$$\frac{4-2}{4+2} = Ae^0 \implies A = \frac{2}{6} = \frac{1}{3}.$$

Finally, rearrange for y:

$$y(x) = 2\left(\frac{3+e^{4x}}{3-e^{4x}}\right)$$

Example 2.3. *We can also solve an initial value problem by directly finding the particular solution that satisfies the initial condition without first solving for the general solution. Let's solve Example 2.1 again using this approach.*

Instead of indefinite integrals, we use definite integrals whose lower limits are determined by the given initial condition. The initial condition in Example 2.1 is that $y = -2$ when $x = \frac{\pi}{2}$, so

$$\int_{-2}^{y} (2\hat{y})d\hat{y} = \int_{\pi/2}^{x} \sin\hat{x}\, d\hat{x}.$$

The upper limits are the dependent variable y and the independent variable x of the ODE. Pay attention not to use the same symbol to denote both the dummy integration variable and the upper limit. Integrating,

$$\begin{aligned}\left[\hat{y}^2\right]_{-2}^{y} &= [-\cos\hat{x}]_{\pi/2}^{x} \\ y^2 - 4 &= -\cos x + \cos(\frac{\pi}{2}) \\ y^2 &= 4 - \cos x\end{aligned}$$

Picking the negative square root gives us the same answer as before:

$$y = -\sqrt{4-\cos x}$$

2.2.1 ODEs of homogeneous degree—reduction to separable form

Equations of homogeneous degree (sometimes referred to as 'homogeneous ODEs') are ODEs that can be arranged into the form:

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right) \tag{2.3}$$

that is, the RHS is a function of the combination $\frac{y}{x}$. We can tell whether we can write $f(x,y) = g(\frac{y}{x})$ either by explicitly re-arranging or by showing that $f(tx,ty) = f(x,y)$ for any t.

The differential equation can then be solved by writing

$$v(x) = \frac{y(x)}{x},$$

where $v(x)$ is a new, unknown dependent variable. By the product rule,

$$\frac{dy}{dx} = \frac{d}{dx}(xv) = v + x\frac{dv}{dx}$$

Therefore equation (2.3) becomes

$$\begin{aligned} v + x\frac{dv}{dx} &= g(v) \\ \frac{dv}{dx} &= \frac{g(v) - v}{x}. \end{aligned}$$

This is a separable equation for v and can be solved by the method of Section 2.2. Remember to return to the original dependent variable $y(x)$ at the end of the calculation.

Example 2.4. *Consider*

$$x\frac{dy}{dx} = x + 3y \qquad \textit{with} \quad y(1) = 2.$$

Find the explicit solution, and verify that the answer is correct.

Solution: Divide the equation by x, we get

$$y' = 1 + \frac{3y}{x} = g\left(\frac{y}{x}\right).$$

The RHS is clearly of the form $g(\frac{y}{x})$. If we were less easily able to tell, we could also check by writing $f(x, y) = 1 + \frac{3y}{x}$, and then

$$\begin{aligned} f(tx, ty) &= 1 + \frac{3ty}{tx} \\ &= 1 + \frac{3y}{x} \\ &= f(x, y). \end{aligned}$$

Hence, the equation is homogeneous.

Either way, we find that we can write $y = xv$, and $g(v) = 1 + 3v$. By the product rule, we obtain $y' = v + xv'$, therfore

$$v + xv' = g(v) = 1 + 3v$$

or

$$v' = \frac{1 + 2v}{x}$$

This is separable, so we use the method of section 2.2:
1. Bring the function of the dependent variable to the LHS:

$$\frac{1}{1 + 2v}\frac{dv}{dx} = \frac{1}{x}$$

2. and 3. Now integrate with respect to x, not forgetting the constant of integration, and change variables in the integral for v:

$$\int \frac{1}{1+2v}\frac{dv}{dx}\,dx = \int \frac{1}{1+2v}\,dv = \int \frac{1}{x}\,dx + C$$

4. We can evaluate these indefinite integrals:

$$\frac{1}{2}\log|1+2v| = \log|x| + C$$

and we can re-arrange to find v explicitly by multiplying by 2 and taking exponentials of both sides:

$$\begin{aligned} &\exp(\log|1+2v|) = \exp\left(2\log|x| + 2C\right) \\ \Rightarrow\quad &|1+2v| = x^2 e^{2C}. \end{aligned}$$

Remove the absolute value signs:

$$1+2v = \pm e^{2C}\,x^2$$

If we define $K = \pm e^{2C}$ (a new constant), we can solve for v:

$$v = \frac{1}{2}\left(Kx^2 - 1\right)$$

Recall that $v = y/x$, so we multiply both sides by x to get the general solution:

$$y = \frac{x}{2}\left(Kx^2 - 1\right)$$

5. We use $y(1) = 2$ to evaluate K:

$$2 = \frac{1}{2}(K-1) \qquad \text{so} \quad K = 5$$

and so we write the particular solution:

$$y = \frac{x}{2}\left(5x^2 - 1\right)$$

6. We verify that this is correct. First we check $y(1) = \frac{1}{2}(5-1) = 2$, which is OK, and then we differentiate and substitute into the ODE:

$$\begin{aligned} \text{LHS} &= xy' = x\frac{d}{dx}\frac{1}{2}(5x^3 - x) = \frac{x}{2}\left(15x^2 - 1\right) \\ \text{RHS} &= x + 3y = \frac{x}{2}\left(15x^2 - 1\right) = xy' \end{aligned}$$

so LHS $=$ RHS and hence this is OK too.

Example 2.5. *Find (and verify) the general solution of*

$$x^2\frac{dy}{dx} = y^2 + 3xy + x^2$$

Solution: First, divide through by x^2 to confirm that this is a first-order homogeneous ODE:

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 3\frac{y}{x} + 1 = g(\frac{y}{x}).$$

Then we can write $y = xv$, and $g(v) = v^2 + 3v + 1$. Since $y' = v + xv'$, we get

$$xv' = v^2 + 2v + 1.$$

This is a separable ODE for $v(x)$. Separate and integrate:

$$\begin{aligned} \frac{dv}{dx} &= \frac{(v+1)^2}{x} \\ \int \frac{dv}{(v+1)^2} &= \int \frac{dx}{x} \\ -(v+1)^{-1} &= \log|x| + C \\ v &= -1 - \frac{1}{\log|x| + C}. \end{aligned}$$

Finally, transform back to the original dependent variable $y = xv$:

$$y(x) = -x\left(1 + \frac{1}{C + \log|x|}\right).$$

2.3 Exact first-order ODEs

Suppose the general solution of some first-order ODE can be written in the form

$$H(x, y) = C,$$

where H is a function of two variables whose partial derivatives with respect to x and y are themselves differentiable, and C is a constant. For each value of C, this equation defines a curve in the (x, y) plane. If we think of y as a function of x along these curves, we can write:

$$H(x, y(x)) = C$$

in which $H(x, y(x))$ is considered as a function x. Differentiate the above equation with respect to x using the chain rule, we obtain:

$$\frac{d}{dx}H(x, y(x)) = \frac{\partial}{\partial x}H(x, y) + \frac{\partial}{\partial y}H(x, y)\frac{dy}{dx} = 0. \tag{2.4}$$

Also recall that if $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial y}$ are differentiable, then

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}. \tag{2.5}$$

Now, suppose that we have an ODE of the form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0, \tag{2.6}$$

and the partial derivatives of M and N exist. We ask two questions. First, under what circumstances can we find a function $H(x, y)$ that satisfies

$$\frac{\partial H}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial H}{\partial y} = N(x, y) \tag{2.7}$$

so that (2.6) becomes

$$M(x, y) + N(x, y)\frac{dy}{dx} = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}\frac{dy}{dx} = \frac{dH}{dx}.$$

If such an H exists, it immediately follows that (2.6) reduces to $dH/dx = 0$ and the general solution is $H(x, y) = C$.

From (2.5), we see that the condition on M and N for (2.7) to be true is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If this condition is satisfied, the ODE (2.6) is called **exact**. (This is because $dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy$ is called the total or exact differential of H.)

The second question is, given that the ODE is exact ($\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$), how can the function H be found? From the first equation in (2.7), we integrate with respect to x,

$$H(x, y) = \int M(x, y)\, dx + C_1(y)$$

where $C_1(y)$ is a function of y only. To determine $C_1(y)$, we use the above expression in the second equation of (2.7):

$$N(x, y) = \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)\, dx + \frac{dC_1}{dy}$$

or

$$\frac{dC_1}{dy} = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\, dx.$$

This is an ODE with y as the independent variable which we need to solve to obtain $C_1(y)$. (It is clearly also possible to start from the second equation in (2.7) to get $\int N(x, y)\, dy + C_2(x)$ and so obtain an ODE with the independent variable x.)

Example 2.6. *Find (and verify) the solution of*

$$2xy\frac{dy}{dx} - 2x + y^2 = 0 \quad \textit{with} \quad y(1) = 2.$$

Solution: Here $M(x,y) = -2x + y^2$ and $N(x,y) = 2xy$. We can verify that:

$$\frac{\partial M}{\partial y} = 2y \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 2y,$$

so the ODE is exact. Therefore there is a function $H(x,y)$ that satisfies $\partial H/\partial x = M$ and $\partial H/\partial y = N$. First integrate $\partial H/\partial y = N$ with respect to y,

$$\begin{aligned} H(x,y) &= \int N(x,y)\,dy + C_1(x) \\ &= xy^2 + C_1(x). \end{aligned}$$

Then take the derivative with respect to x and use $\partial H/\partial x = M$

$$\begin{aligned} y^2 + \frac{dC_1}{dx} &= -2x + y^2 \\ \frac{dC_1}{dx} &= -2x \\ C_1(x) &= -x^2 + c \end{aligned}$$

for some constant c. Since $H(x,y) = xy^2 + C_1(x)$, the general solution is

$$H(x,y) = xy^2 - x^2 = C.$$

Putting in the initial value $y(1) = 2$ gives $C = 3$, so the particular solution is

$$xy^2 - x^2 = 3.$$

In this case, this can be written explicitly as

$$y = \sqrt{\frac{x^2+3}{x}},$$

where we have chosen the positive square root to satisfy $y(1) = 2$. To verify that this is correct, we check $y(1) = \sqrt{4} = 2$, then we differentiate

$$y' = \frac{1}{2}\left(\frac{x^2+3}{x}\right)^{-\frac{1}{2}} \frac{(2x)x - (x^2+3)}{x^2} = \frac{1}{2y}\left(\frac{x^2-3}{x^2}\right)$$

and substitute into the ODE:

$$\begin{aligned} 2xy\,y' &= \frac{x^2-3}{x} \\ y^2 - 2x &= \frac{x^2+3}{x} - 2x = \frac{3-x^2}{x} \end{aligned}$$

so $2xy\,y' - 2x + y^2 = 0$ and the ODE is satisfied.

Exercise: attempt the anti-partial differentiation starting from $\int M(x,y)\,dx$.

Note that a separable ODE $y' = a(x)b(y)$ (cf. section 2.2) written in the form

$$\frac{1}{b(y)}\frac{dy}{dx} = a(x)$$

is exact, since $M = -a(x)$ is not a function of y and $N = \frac{1}{b(y)}$ is not a function of x.

2.4 Integrating Factor (IF)

If an ODE is not exact, sometimes multiplying it by a function of x or y (or of both) can make it exact. Such functions are called integrating factors.

Example 2.7. *Find an integrating factor to solve*

$$\sin y + \cos y \frac{dy}{dx} = 0.$$

Solution: The given ODE is not exact because

$$\frac{\partial}{\partial y} \sin y = \cos y \qquad \text{is not equal to} \qquad \frac{\partial}{\partial x} \cos y = 0.$$

An integrating factor to this ODE is e^x. Multiply by e^x to get

$$e^x \sin y + e^x \cos y \frac{dy}{dx} = 0.$$

Let $M(x, y) = e^x \sin y$ and $N(x, y) = e^x \cos y$. We now see that

$$\frac{\partial M}{\partial y} = e^x \cos y \qquad \text{is equal to} \qquad \frac{\partial N}{\partial x} = e^x \cos y$$

so the new equation is exact and there is a function $H(x, y)$ that satisfies $\partial H/\partial x = M$ and $\partial H/\partial y = N$. Obviously the new equation has the same solution as the original equation. We now determine H as follows. From $\frac{\partial H}{\partial x} = M$, we get

$$H(x, y) = \int M(x, y) dx + K(y) = e^x \sin y + K(y)$$
$$\frac{\partial H}{\partial y} = e^x \cos y + \frac{dK(y)}{dy}$$

But we know that $\frac{\partial H}{\partial y} = N$, so

$$\frac{dK(y)}{dy} = 0 \implies K(y) = -C \quad \text{where } C \text{ is an arbitrary constant.}$$

Therefore the general solution is $H(x, y) = 0$ or

$$e^x \sin y = C.$$

There can more than one integrating factor for a given ODE. And in general there is no systematic way to find the integrating factors of an ODE. However for a linear first-order ODE, we shall see in the next section that an integrating factor can be systematically determined.

2.5 Linear first-order ODEs

Linear first-order ODEs are an important class of ODEs of the form:

$$y' + P(x)y = Q(x) \qquad \text{with} \quad y(x_0) = y_0, \tag{2.8}$$

where $P(x)$ and $Q(x)$ are given functions of x. This equation can be solved as follows. Multiplying the ODE (2.8) by $R(x)$ yields:

$$R(x)y' + P(x)R(x)y = Q(x)R(x). \tag{2.9}$$

Suppose that we could choose $R(x)$ so that the left-hand side of this was the derivative of $R(x)y(x)$. Then

$$\begin{aligned} & R(x)y' + P(x)R(x)y = \frac{d}{dx}\left(R(x)y(x)\right) \qquad (2.10)\\ \text{or} \quad & R(x)y' + P(x)R(x)y = R(x)y' + R'(x)y \end{aligned}$$

We see that this can be achieved if

$$\frac{dR}{dx} = RP(x).$$

Recall that $P(x)$ is given to us, so this equation is separable, and can be solved:

$$\begin{aligned} \frac{1}{R}\frac{dR}{dx} &= P \\ \int \frac{1}{R}\,dR &= \int P(x)\,dx \\ \log|R| &= \int P(x)\,dx + C \\ R &= \pm e^C \exp\left(\int P(x)\,dx\right). \end{aligned}$$

Since multiplying R by a constant won't change its ability to convert $y' + P(x)y$ into $(Ry)'$, we can choose the $+$ sign and set e^C to 1, resulting in an expression for R:

$$R(x) = \exp\left(\int P(x)\,dx\right). \tag{2.11}$$

$R(x)$ is an integrating factor of the first-order linear ODE (2.8) with the property (2.10). We now have a method for solving first-order linear ODEs:
1. Arrange the ODE into the form of (2.8):

$$y' + P(x)y = Q(x)$$

and calculate the **integrating factor** $R(x)$:

$$R(x) = \exp\left(\int P(x)\,dx\right).$$

2. Multiply both sides of the ODE by $R(x)$ and use $R' = RP$ to write the LHS as a **perfect derivative**:

$$Ry' + RPy = Ry' + R'y = \frac{d}{dx}(Ry) = RQ$$

3. **Integrate** both sides with respect to x, not forgetting the **constant of integration**:

$$\int \frac{d}{dx}(Ry)\ dx = Ry = \int RQ\,dx + C$$

In principle, we can evaluate the integral on the RHS.
4. **Divide** by $R(x)$ to get the **general solution**:

$$y(x) = \frac{1}{R(x)} \int R(x')Q(x')\,dx' + \frac{C}{R(x)}.$$

Note that we have changed the dummy variable in the integral to x' so that we don't confused it with the independent variable x.
5. Finally, we can use the **initial condition** $y(x_0) = y_0$ to evaluate the constant of integration C, and so find the **particular solution**.
6. Last of all, we can **verify** that the answer satisfies the ODE and the initial value.

Example 2.8. *Consider*

$$\frac{dy}{dx} - xy = x \qquad \textit{with} \quad y(0) = 5.$$

Find the explicit solution, and verify that the answer is correct.

Solution: Here,

$$P(x) = -x \qquad \text{and} \qquad Q(x) = x$$

Proceed as above:
1. Calculate the integrating factor $R(x)$:

$$R(x) = \exp\left(\int (-x)\,dx\right) = \exp\left(-\frac{x^2}{2}\right) = e^{-\frac{x^2}{2}}$$

2. and 3. Multiply both sides by $R(x)$ and integrate:

$$\int \frac{d}{dx}(R(x)y)\ dx = R(x)y = \int xe^{-\frac{x^2}{2}}\,dx + C = -e^{-\frac{x^2}{2}} + C$$

using the substitution $u = -\frac{x^2}{2}$.
4. Divide by $R(x)$ to get y:

$$y = -1 + Ce^{\frac{x^2}{2}}$$

This is the general solution.
5. Use the initial condition to evaluate C:

$$y(0) = 5 = -1 + Ce^0 = -1 + C, \qquad \Rightarrow \qquad C = 6$$

so the particular solution is

$$y = 6e^{\frac{x^2}{2}} - 1$$

6. Verify the initial condition: $y(0)$ is indeed 5, and verify that the ODE is satisfied:

$$\begin{aligned} y' &= 6xe^{\frac{x^2}{2}} \\ LHS = y' - xy &= 6xe^{\frac{x^2}{2}} - (-x + 6xe^{-\frac{x^2}{2}}) = x = RHS \qquad \text{OK} \end{aligned}$$

Example 2.9. *Find (and verify) the general solution of*

$$(x+1)y' + y = x(x+1)$$

Solution: Divide by $(x+1)$, so the ODE

$$y' + \frac{y}{x+1} = x$$

is first-order linear with $P(x) = \frac{1}{x+1}$ and $Q(x) = x$. The integrating factor is:

$$R(x) = \exp\left(\int P(x)\,dx\right) = \exp\left(\int \frac{1}{x+1}\,dx\right) = \exp\left(\log|x+1|\right) = x+1.$$

Multiply by R and integrate:

$$\begin{aligned} \frac{d}{dx}\left((x+1)y\right) &= x(x+1) \\ (x+1)y &= \int (x^2 + x)\,dx + C \\ (x+1)y &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + C. \end{aligned}$$

Therefore:

$$y(x) = \frac{\frac{1}{3}x^3 + \frac{1}{2}x^2 + C}{x+1}.$$

2.5.1 Bernoulli's equation

Jacob Bernoulli (1654–1705) was one of the many prominent mathematicians in the Bernoulli family. Bernoulli's equation is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{2.12}$$

where $n \neq 0, 1$. If $n = 0$ the equation is first-order linear and if $n = 1$, we can simply write $(P(x) - Q(x))y$, and we get a first-order linear ODE.

This can be solved by writing

$$z(x) = y^{1-n},$$

where $z(x)$ is a new dependent variable. Then

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx} \tag{2.13}$$

Now divide (2.12) by y^n to get

$$\begin{aligned} y^{-n}\frac{dy}{dx} + P(x)y^{1-n} &= Q(x) \\ \frac{1}{1-n}\frac{dz}{dx} + P(x)z &= Q(x) \\ \frac{dz}{dx} + (1-n)P(x)z &= (1-n)Q(x) \end{aligned}$$

This is a first-order linear ODE for z and can be solved by the method of section 2.5. Remember to return to the original dependent variable $y(x)$ at the end of the calculation.

Example 2.10. *Consider*

$$\frac{dy}{dx} + 4y = e^x y^3$$

Find the general solution, and verify that the answer is correct.

Solution: Here $n = 3$, $P(x) = 4$ and $Q(x) = e^x$. Write $z(x) = y^{1-n} = y^{-2}$ and differentiate,

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

Divide the ODE by y^3 to get

$$\begin{aligned} y^{-3}\frac{dy}{dx} + 4y^{-2} &= e^x \\ -\frac{1}{2}\frac{dz}{dx} + 4z &= e^x \\ \frac{dz}{dx} - 8z &= -2e^x \end{aligned}$$

This is a first-order linear ODE for z, so we use the method of section 2.5.

1. We calculate the integrating factor $R(x)$:

$$R(x) = \exp\left(\int (-8)\,dx\right) = e^{-8x}$$

2. and 3. Multiply both sides by $R(x)$ and integrate:

$$e^{-8x}\frac{dz}{dx} - e^{-8x}8z = -2e^{x}e^{-8x}$$
$$\frac{d}{dx}(ze^{-8x}) = -2e^{-7x}$$
$$ze^{-8x} = \int(-2e^{-7x})dx + C = \frac{2}{7}e^{-7x} + C$$

4. Divide by $R(x)$ to get z:

$$z = \frac{2}{7}e^{x} + Ce^{8x}$$

Remember that $z = y^{-2}$, so $y = \pm z^{-1/2}$:

$$y(x) = \frac{\pm 1}{\sqrt{\frac{2}{7}e^{x} + Ce^{8x}}}$$

This is the general solution.
6. Verify that the ODE is satisfied (we'll only do the $+$ case here):

$$y' = -\frac{\frac{2}{7}e^{x} + 8Ce^{8x}}{2\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{3/2}}$$

Then

$$\begin{aligned}
LHS = y' + 4y &= -\frac{\frac{2}{7}e^{x} + 8Ce^{8x}}{2\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{3/2}} + \frac{4}{\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{1/2}} \\
&= \frac{-\frac{1}{7}e^{x} - 4Ce^{8x} + \frac{8}{7}e^{x} + 4Ce^{8x}}{\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{3/2}} \\
&= \frac{e^{x}}{\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{3/2}} = e^{x}y^{3} = RHS \qquad \text{OK}
\end{aligned}$$

Example 2.11. *For $x > 0$, find the general solution of*

$$\frac{dy}{dx} + \frac{y}{x} = 2x^{3}y^{4}$$

Solution: Let $z = y^{-3}$, hence $z' = -3y^{-4}y'$. Then multiply the ODE by y^{-4} and change variables:

$$y^{-4}\frac{dy}{dx} + \frac{1}{x}y^{-3} = 2x^{3}$$
$$\frac{dz}{dx} - 3x^{-1}z = -6x^{3}$$

The integrating factor for this new ODE is $R(x) = \exp(\int(-3x^{-1})dx)$, we take $R = x^{-3}$. Multiplying by R then gives,

$$\begin{aligned} x^{-3}\frac{dz}{dx} - 3x^{-4}z &= -6 \\ \frac{d}{dx}\left(x^{-3}z\right) &= -6 \\ x^{-3}z &= -6x + C \end{aligned}$$

Reverting back to y and rearranging, we get:

$$y(x) = \left(-6x^4 + Cx^3\right)^{-1/3}$$

Chapter 3

Applications of first-order ODEs

Contents

In this chapter, we apply the theory and solution methods of first-order ODEs from chapter 2 to a number of application areas. For each modelling problem, **the goal is to formulate the ODE** that must be solved, along with its initial condition, and use an appropriate method to find the particular solution. The 'maths' in this chapter is no more difficult than what we have already seen in chapter 2. However, before getting to the maths, we have to learn how to formulate the problem from a description (in words) of a certain situation. So, while the maths might be no more difficult, the tricky part here is coming up with the correct ODE in the first place. Throughout this chapter, we consider the situation carefully and translate the words describing this situation into an ODE—this is a key part of the process of mathematical modelling.

3.1 Radioactivity and Carbon dating

3.1.1 Decay and half-life

Atoms of a given radioactive isotope have a fixed percentage chance of decaying in any given time period. Provided a sufficiently large number of atoms are considered as a mass, the number of decays that occur in a fixed time period is simply proportional to the number of atoms and the length of time.

Let $N(t)$ be the number of atoms at time t and consider this number a (short) time later $N(t + \delta t)$. Assume we know that there are N_0 atoms at a particular time

t_0. In the time interval between t and $t + \delta t$, the number of atoms decreases by an amount proportional to (i) the number itself at that time (i.e., $N(t)$) and (ii) the length of the time interval (i.e., δt). Mathematically, this translates to:

$$N(t + \delta t) = N(t) - kN(t)\delta t$$

where k is a **positive** constant and we take the negative sign since the number is decreasing. Rearranging:

$$\frac{N(t + \delta t) - N(t)}{\delta t} = -kN$$

and taking the limit of small δt (see chapter 1):

$$\frac{dN}{dt} = -kN \qquad \text{with initial condition } N(t_0) = N_0. \tag{3.1}$$

This is a separable ODE. We can solve this by separating and integrating:

$$\begin{aligned} \int \frac{1}{N}\frac{dN}{dt}\,dt = \int \frac{1}{N}\,dx \quad &= \quad -\int k\,dt + C \\ \log|N| \quad &= \quad -kt + C \\ |N| \quad &= \quad e^C e^{-kt} \end{aligned}$$

Use the fact that $N > 0$ and set $A = e^C$:

$$N = Ae^{-kt}$$

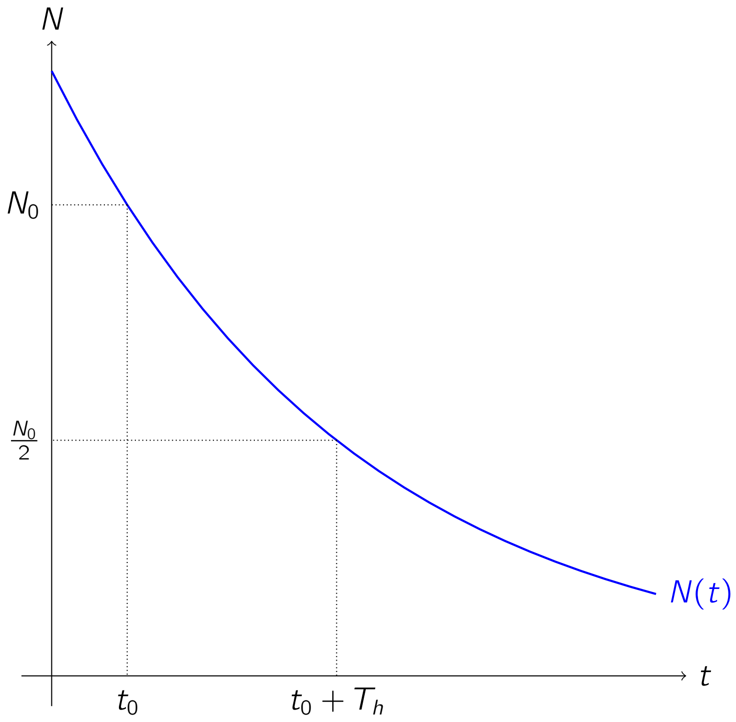
We can now use the initial condition $N(t_0) = N_0$ to find A:

$$N_0 = Ae^{-kt_0} \implies A = N_0 e^{kt_0}$$

which results in the particular solution for $N(t)$:

$$N = N_0 e^{-k(t-t_0)}. \tag{3.2}$$

Note that the symbol N for the dependent variable is our choice; we could also choose to denote the number of atoms by, for example, x and the problem we must solve is the same. For this problem (and many others), the independent variable is time. For modelling problems, we try to keep a sensible naming convention, so if we are discussing rates of change with respect to time, we almost always name the independent variable as t. However, there is nothing wrong mathematically with choosing another symbol for time!



What does this solution (3.2) look like? The blue line in this figure plots $N(t)$ for a general initial condition $N(t_0) = N_0$; it is an example of exponential decay where k is the rate of decay.

Half-life is the time required for a quantity to reduce to half of its initial value. In radioactivity, the **half-life**, which we will denote T_h, is the time after which half of the isotope has decayed. Initially, at time t_0, there are N_0 atoms. So, we are interested in the time T_h in the future at which there are $\frac{1}{2}N_0$ atoms. Mathematically, this means $N(t_0 + T_h) = \frac{1}{2}N_0$. Using the solution (3.2) at time $t_0 + T_h$:

$$N(t_0 + T_h) = \frac{1}{2}N_0 \qquad \Longrightarrow \qquad N_0 e^{-k(t_0+T_h-t_0)} = N_0 e^{-kT_h} = \frac{1}{2}N_0.$$

Rearranging, we find an expression for the half-life in terms of the rate of decay k:

$$T_h = \frac{1}{k}\log 2.$$

3.1.2 Carbon dating

There are three isotopes of Carbon: ^{12}C, ^{13}C and ^{14}C. Almost all Carbon is made up of the first two (^{12}C and ^{13}C) because ^{14}C is radioactive: it decays with a half-life of 5730 years to form ^{14}N, a Nitrogen isotope. Although it decays quite quickly, it is constantly being produced in the upper atmosphere by the action of cosmic rays. The equilibrium level of ^{14}C is about 1 part per trillion (10^{12}).

When an organism dies, it ceases to absorb Carbon from the environment, so the amount of ^{14}C it contains will decrease as this decays radioactively. The time since the death of the organism can be estimated by measuring how much ^{14}C remains.

For ^{14}C, the half-life is $T_h = 5730$ years, so $k = \frac{1}{T_h}\log 2 = 0.000121/$year.

Example 3.1. *A fossilised bone is found to contain* 0.1% *of its original* ^{14}C*. Find the age of the fossil.*

Solution: Let t be today's date, and suppose the fossil is T years old, so it was fossilised at time $t_0 = t - T$. At this time it had x_0 ^{14}C, and now it has $0.001x_0$. So the number of atoms x at time t is

$$x(t) = 0.001x_0 = x_0 e^{-k(t-t_0)} = x_0 e^{-kT},$$

which we can solve for T:

$$T = -\frac{1}{k}\log 0.001 = \frac{1}{k}\log 1000 \approx 57100.$$

Therefore the fossil is approximately 57100 years old.

3.2 Newton's law of cooling

Sir Isaac Newton FRS: 1643–1727, English physicist, mathematician, astronomer, natural philosopher, alchemist and theologian.

Newton's law of cooling states that if an object is hotter than the ambient temperature, then the rate of change of the object's temperature is proportional to the temperature difference between the object and its surrounding. Mathematically, we write:

$$\frac{d\Theta}{dt} = -k(\Theta - A), \qquad \text{with } \Theta(t_0) = \Theta_0 \tag{3.3}$$

where $\Theta(t)$ is the object's temperature, A is the ambient temperature (a constant[1]), t is time and k is a **positive** constant. This is a first-order linear (separable) ODE:

$$\frac{d\Theta}{dt} + k\Theta = kA$$

so the integrating factor is $R(t) = \exp\left(\int k\,dt\right) = \exp(kt)$. If we multiply (3.3) by this integrating factor, we get:

$$\frac{d}{dt}\left(\Theta e^{kt}\right) = kAe^{kt}$$

Integrating this, using the initial condition and rearranging results in:

$$\Theta(t) = A + (\Theta_0 - A)e^{-k(t-t_0)}$$

[1]Note that, in more complicated examples, the ambient temperature may be time-dependent $A = A(t)$.

Exercise: Verify that this expression is correct.

Note that as $t \to \infty$, we have $\Theta(t) \to A$, i.e., the temperature decays exponentially to the ambient temperature.

Example 3.2. *A horse shoe is heated to* $100°C$ *and then placed in a room to cool. The temperature of the room is* $10°C$*. After 20 minutes, the temperature of the bar is* $50°C$*. It is safe to handle at* $30°C$*; how long must we wait to handle it?*

Solution: We consider time in minutes with $t_0 = 0$ the time at which the metal bar is placed in the room. We use the particular solution to the cooling problem with $\Theta(t_0 = 0) = \Theta_0 = 100$, $A = 10$:

$$\Theta(t) = 10 + 90e^{-kt},$$

where k is the (unknown) decay constant. Next we "translate" the remainder of the question into equation form:

- *"After 20 minutes, the temperature of the bar is 50° C"* translates to $\Theta(20) = 50$.
- *"It is safe to handle at 30° C; how long must we wait to handle it?"* This is the crux of the problem: find T such that $\Theta(T) = 30$.

We use the first point to determine k:

$$\begin{aligned}\Theta(20) = 10 + 90e^{-20k} = 50 \implies e^{-20k} = \frac{40}{90} &\implies -20k = \log\frac{4}{9} \\ &\implies k = 0.0405...\end{aligned}$$

Given this k, we can find T such that $\Theta(T) = 30$:

$$\begin{aligned}\Theta(T) = 10 + 90e^{-kT} = 30 \implies e^{-kT} = \frac{20}{90} &\implies -kT = \log\frac{2}{9} \\ &\implies T \approx 37\text{mins}.\end{aligned}$$

So, we must wait approximately 37 minutes to handle the metal bar.

3.3 Population growth models

Let $p(t)$ be the population of a country at time t. The rate of change of population in a country is equal to the rate at which people enter the country (e.g., births, immigration) minus the rate at which people leave the country (e.g, deaths, emigration). Hence we can write:

$$\frac{dp}{dt} = B(p, t) - D(p, t) + M(p, t),$$

where
$B(p, t)$ represents births,
$D(p, t)$ represents deaths,
$M(p, t)$ represents net migration **into** the country (hence the **plus** sign).
The processes B, D and M may depend on the population p itself and the time t.

3.3.1 The Malthusian model

Thomas Robert Malthus FRS: 1766–1834, English clergyman, political economist and demographer.

Malthus (1798) suggested a simple model that has no migration, so $M = 0$. For the birth and death rates, he assumed they are proportional to the population p:

$$B(p, t) = bp(t), \qquad \text{and} \qquad D(p, t) = dp(t),$$

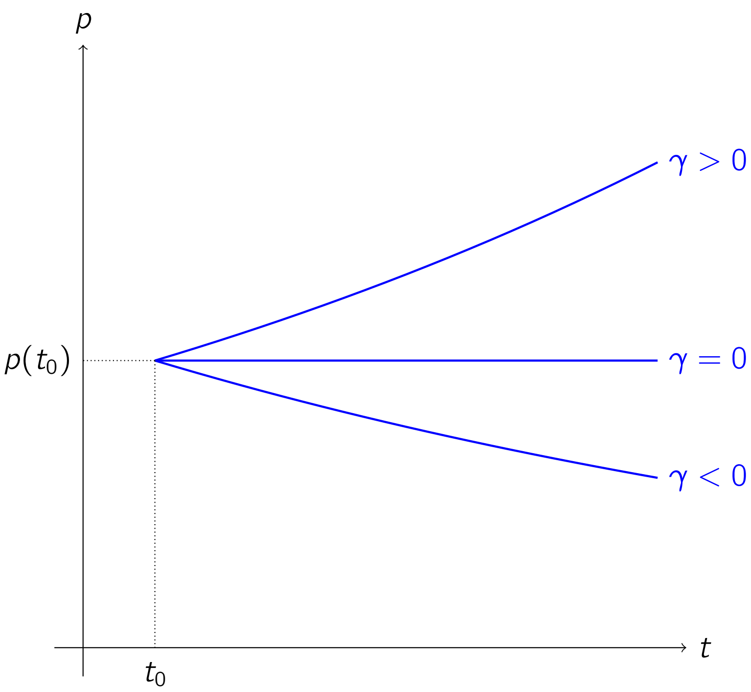
where b and d are positive constants, so

$$\frac{dp}{dt} = (b - d)p = \gamma p \tag{3.4}$$

where $\gamma = b - d$ is a constant called the **growth rate**. Note that depending on the values of b and d, γ can be positive, zero or negative. (3.4) is a separable ODE which we can solve with the initial condition $p(t_0) = p_0$:

$$p(t) = p(t_0)e^{\gamma(t-t_0)}.$$

What does this solution look like? It clearly depends on γ: the plot below shows three possible solutions $p(t)$ depending on the sign of γ. Its sign determines whether the population will increase (more births than deaths) or decrease (more deaths than births). If the birth and death rate are the same (so $\gamma = 0$), then the population is steady, i.e., it does not change in time. As a model it is quite limited: it predicts that the population will increase without bound if $\gamma > 0$ and the population will die out if $\gamma < 0$.



Example 3.3. *In 1770, the population of Great Britain was estimated to be 6.4 million. By 1790, the population had grown to 8 million. Estimate γ, and predict the population in the year 2012.*

Solution: Take $t_0 = 1770$, and $p(t_0) = 6.4 \times 10^6$. We know that $p(1790) = 8 \times 10^6$. So

$$p(1790) = 8 \times 10^6 = p(t_0)e^{(1790-t_0)\gamma} = 6.4 \times 10^6 \times e^{20\gamma}.$$

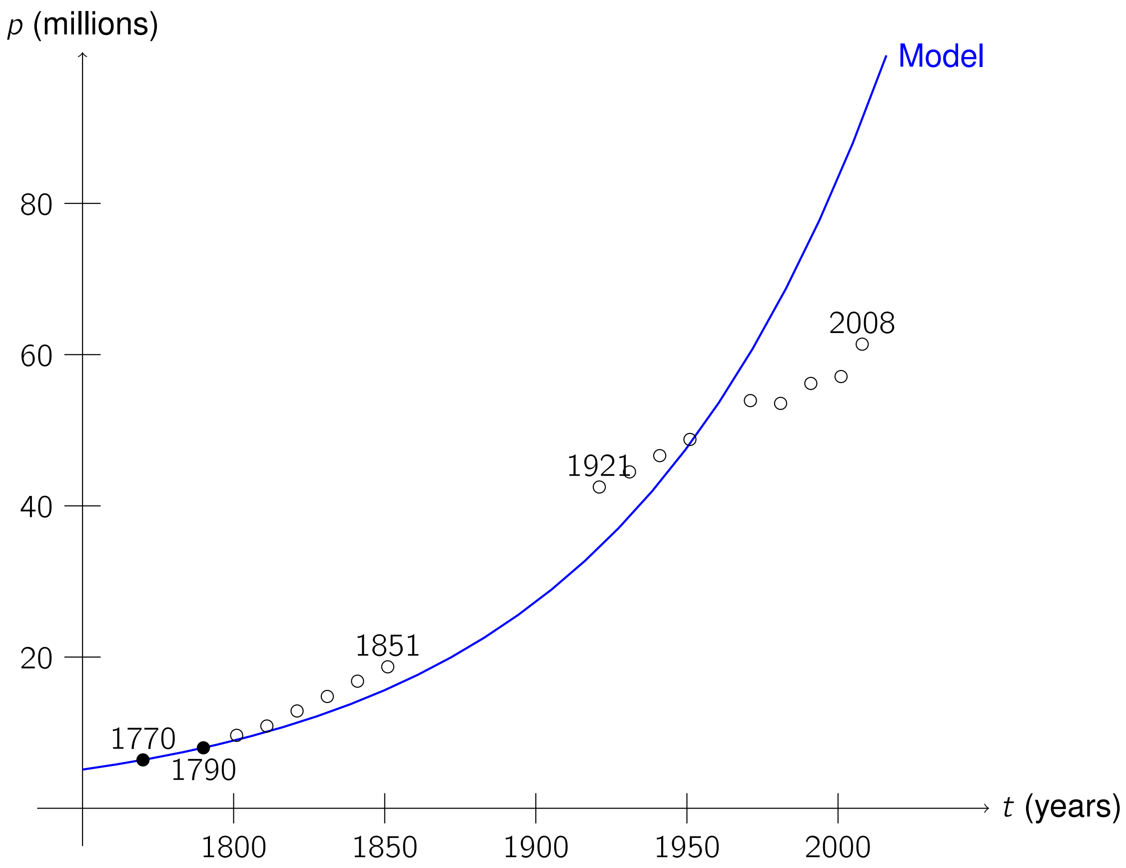
Therefore

$$\gamma = \frac{1}{20} \log\left(\frac{8 \times 10^6}{6.4 \times 10^6}\right) = 0.0112\,\text{year}^{-1}$$

Now we can get $p(2012)$:

$$p(2012) = p(t_0)e^{(2012-t_0)\gamma} = 6.4 \times 10^6 \times e^{242 \times 0.0112} = 96 \times 10^6$$

In fact, the current population of Great Britain is about 61×10^6 so the estimate is approximately 50% too large. This is not surprising, since we have not included effects like immigration, birth and death rates that change with time, changes in agriculture that allow more food production etc.



3.3.2 The Logistic model

Pierre François Verhulst: 1804–1849, Belgian mathematician.

One problem with the Malthusian model is that it predicts either the population grows without bound ($\gamma > 0$) or that it decays to extinction ($\gamma < 0$), which is fairly unrealistic. Of course, populations cannot grow without bound. As the population becomes too large, there can be competition for food, resources or space which prevent the population from growing further. Such self-limiting processes can be modelled by letting the growth rate γ depend on p.

Suppose that there is a limiting population $p_\infty > 0$ such that

$$\begin{aligned} \text{the population } p \text{ grows } (\gamma > 0) &\quad \text{when} \quad p < p_\infty, \\ \text{the population } p \text{ decays } (\gamma < 0) &\quad \text{when} \quad p > p_\infty, \\ \text{the population } p \text{ stops changing } (\gamma = 0) &\quad \text{when} \quad p = p_\infty. \end{aligned}$$

The simplest model of this would have γ depending linearly on p:

$$\gamma = \mu\left(1 - \frac{p}{p_\infty}\right)$$

where the positive constant μ is the growth rate in the limit of very small population ($p \to 0$). Using this growth rate in $\frac{dp}{dt} = \gamma p$ results in the **logistic equation**:

$$\frac{dp}{dt} = \mu p\left(1 - \frac{p}{p_\infty}\right) \qquad \text{with } p(t_0) = p_0. \tag{3.5}$$

Comparing to (3.4), there is now an extra *quadratic* (p^2) term on the right which limits the growth of p. This population model was first written down by Verhulst (1838) and is a successful model of yeast, bacteria or fruit flies (in a controlled environment), but still too simple for more realistic situations.

Nonetheless, we can solve the nonlinear separable ODE (3.5) (it is also a Bernoulli equation):

$$\begin{aligned} \int \frac{1}{p\left(1 - \frac{p}{p_\infty}\right)}\, dp &= \int \mu\, dt + C \\ \int \frac{p_\infty}{p(p_\infty - p)}\, dp &= \int \mu\, dt + C \\ \int \left(\frac{1}{p} + \frac{1}{p_\infty - p}\right) dp &= \mu t + C \\ \log|p| - \log|p_\infty - p| &= \mu t + C \\ \frac{p}{p_\infty - p} &= A e^{\mu t} \end{aligned}$$

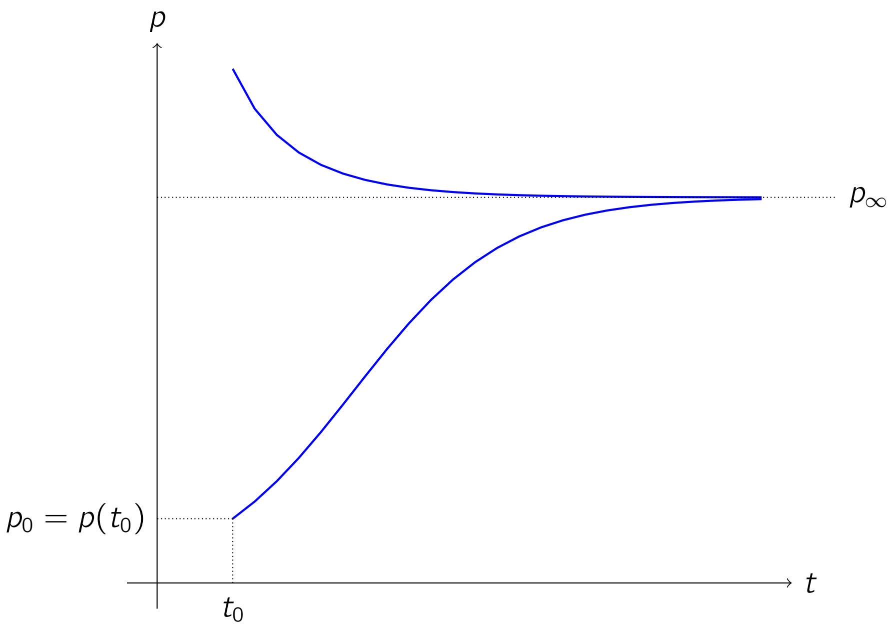
where $A = \pm e^C$ and we have used partial fractions to do the integral. We can now use the initial condition $p(t_0) = p_0$ to find A:

$$A = \frac{p_0}{p_\infty - p_0} e^{-\mu t_0} \qquad \text{so} \qquad \frac{p}{p_\infty - p} = \frac{p_0}{p_\infty - p_0} e^{\mu(t - t_0)}$$

Now rearrange to find $p(t)$:

$$
\begin{aligned}
p &= \frac{p_0}{p_\infty - p_0}(p_\infty - p)e^{\mu(t-t_0)} \\
(p_\infty - p_0)p + p_0 p e^{\mu(t-t_0)} &= p_0 p_\infty e^{\mu(t-t_0)} \\
p[(p_\infty - p_0) + p_0 e^{\mu(t-t_0)}] &= p_0 p_\infty e^{\mu(t-t_0)} \\
p(t) &= \frac{p_\infty p_0 e^{\mu(t-t_0)}}{(p_\infty - p_0) + p_0 e^{\mu(t-t_0)}} \\
p(t) &= \frac{p_\infty p_0}{(p_\infty - p_0)e^{-\mu(t-t_0)} + p_0}
\end{aligned}
$$

Exercise: Verify that this expression has $p(t_0) = p_0$ and that it satisfies the logistic equation (3.5). Verify also that as $t \to \infty$, we have $p(t) \to p_\infty$.



The graph above illustrates that if $0 < p_0 < p_\infty$, the population grows, and saturates at p_∞, but if $p_0 > p_\infty$, the population decays down to p_∞ (assuming that $p_\infty > 0$). What happens if $p_0 = 0$?

3.4 Mixing problems

In mixing problems, we typically consider a substance that is dissolved in a fluid, and that the fluid is entering and leaving some enclosed volume V which may or may not be constant. Here, fluid can mean a liquid or a gas, so we could be dealing with salt (substance) dissolved in water (fluid) in a container or tank (enclosed volume), or a pollutant (substance) mixed in air (fluid) of a room (enclosed volume). Furthermore, the rates of entering and leaving may be different.

We are interested in how the amount $x(t)$, or the concentration $C(t)$, of the substance in the volume of fluid changes over time. By definition, concentration is the amount of substance per unit volume:

$$C(t) = \frac{x(t)}{V(t)}.$$

It follows that the amount $x(t)$ (or mass) of substance at time t equals the concentration $C(t)$ multiplied by the volume $V(t)$: $x(t) = C(t)V(t)$.

Our goal is to find an ODE for $x(t)$ (or $C(t)$). To do so, we use conservation of mass and make some simplifying modelling assumptions. The conservation of mass, in this context, means that the rate of change of $x(t)$ is equal to the rate of inflow of $x(t)$ minus the rate of outflow of $x(t)$:

$$\begin{pmatrix}\text{Rate of change}\\ \text{of } x(t)\end{pmatrix} = \begin{pmatrix}\text{Rate of inflow}\\ \text{of } x(t)\end{pmatrix} - \begin{pmatrix}\text{Rate of outflow}\\ \text{of } x(t)\end{pmatrix}.$$

This is a similar idea to the population models: rate of input minus rate of output gives that net rate of change of a quantity. The key modelling assumption is that mixing is infinitely fast so that the concentration is uniform in the fluid. This is seldom the case in real life, but making this assumption allows us to make some progress[2].

Suppose we have a container full of a salt solution of a certain concentration $C(t)$ and $x(t)$ is the amount (in kg) of salt dissolved in the liquid in the container at any time t. Let's say we pump in a solution with a different concentration C_{in} at a certain rate r_{in}, **mix well** and extract the mixture at a different rate r_{out}. The flow rate r_{in} and r_{out} are, respectively, the volume of liquid entering and leaving the container per unit time. We want to know the amount $x(t)$, or concentration $C(t)$, in the container at any time t.

To solve this, we consider the law of conservation of mass:

$$\begin{pmatrix}\text{Rate of change}\\ \text{of } x(t)\end{pmatrix} = \begin{pmatrix}\text{Rate at which } x(t)\\ \text{enters the container}\end{pmatrix} - \begin{pmatrix}\text{Rate at which } x(t)\\ \text{leaves the container}\end{pmatrix}$$

where

- rate of change of $x(t)$ is $\dfrac{dx}{dt}$;
- rate at which $x(t)$ enters the container is equal to (flow rate of liquid entering, r_{in}) $\times$ (concentration of substance in liquid entering, C_{in});
- rate at which $x(t)$ leaves the container is equal to (flow rate of liquid leaving, r_{out}) $\times$ (concentration of substance in liquid leaving, C_{out}).

[2] If we allowed concentration to vary with location, the problem becomes much more difficult and requires *partial differential equations*.

This gives the basic ODE of the mixing problem:

$$\frac{dx}{dt} = C_{in} r_{in} - C_{out} r_{out}.$$

Three of the four quantities on the right of the above equation, namely r_{in}, C_{in} and r_{out} are given while C_{out} is unknown. But C_{out} is just the concentration in the container, so we set $C_{out} = C(t)$ and use $x(t) = C(t)V$ (assuming volume is fixed) to obtain:

$$\frac{dx}{dt} = C_{in} r_{in} - \left(\frac{r_{out}}{V}\right) x(t).$$

If we want to solve for concentration $C(t)$, we divide the above equation by V to get:

$$\frac{dC}{dt} = \frac{C_{in} r_{in}}{V} - \left(\frac{r_{out}}{V}\right) C(t). \tag{3.6}$$

Example 3.4. *A* $1000\,\ell$ *tank of water initially contains* $10\,\mathrm{kg}$ *of dissolved salt. A pipe brings a salt solution (concentration* $0.005\,\mathrm{kg}\,\ell^{-1}$*) into the tank at a rate of* $2\,\ell\,\mathrm{s}^{-1}$*, and a second pipe carries away the excess solution at the same rate.*

(a) *Calculate* $C(t)$*, the concentration of salt in the tank, assuming that the tank is well mixed*[3]*.*

(b) *After how many minutes has the concentration of salt decreased 25% of its initial value?*

(c) *What happens as* t *tends to infinity?*

Solution (a): Let $V = 1000\,\ell$ be the volume of water in the tank (this is constant) and let $x(t)$ be the mass of salt dissolved in the water, so $x(0) = 10\,\mathrm{kg}$. Let $C(t) = x/V$ be the concentration of salt in the tank, assuming that the salt is well mixed, so the initial concentration is $C(0) = 0.01\,\mathrm{kg}\,\ell^{-1}$.

The rate of inflow of salt (in $\mathrm{kg\,s^{-1}}$) is $C_{in} r_{in}$, where $C_{in} = 0.005\,\mathrm{kg}\,\ell^{-1}$ is the concentration of salt in the inflow, and $r_{in} = 2\,\ell\,\mathrm{s}^{-1}$ is the rate of inflow.

The rate of outflow of salt (in kg/s) is $C_{out} r_{out}$, where C_{out} is the concentration of salt in the outflow which is the same as the concentration of salt in the tank in this case, so $C_{out} = C(t)$. We also know the rate of outflow equals the rate of inflow, so $r_{out} = r_{in} = 2\,\ell\,\mathrm{s}^{-1}$.

From the law of conservation of mass, or (3.6), we get the equation for $C(t)$

$$\frac{dC}{dt} + \frac{r_{in}}{V} C = \frac{C_{in} r_{in}}{V}$$

This is a first-order linear ODE; its integrating factor is $\exp\left(\frac{r_{in}}{V} t\right)$ and the solution in this case is

$$C(t) = C_{in} + K \exp\left(\frac{-r_{in}}{V} t\right),$$

[3] What does well-mixed mean? This is a modelling assumption. It just means that mixing happens so quickly that the concentration is always uniform over the whole fluid.

where K is a constant. Putting in the initial condition and the numbers results in:

$$\begin{aligned} C(t) &= C_{in} + (C(0) - C_{in}) \exp\left(\frac{-r_{in}}{V}t\right) \\ &= 0.005 + 0.005e^{-0.002t} \end{aligned}$$

where C is in $\mathrm{kg}\,\ell^{-1}$ and t is in seconds. (**Exercise: work through the details and verify this yourself.**)

Solution (b): We want the the time T such that $C(T) = (1-0.25)C(0) = 0.75C(0) = 0.0075$, and use the solution from (a):

$$\begin{aligned} &0.0075 = 0.005 + 0.005e^{-0.002T} \\ \implies &\frac{1}{2} = e^{-0.002T}. \end{aligned}$$

So $\log(2) = 0.002T$ and $T = 346.57...$s. Therefore it takes approximately 5mins 47secs for the concentration to decrease by 25% of its original value.

Solution (c): What happens as t tends to infinity? Since $\exp\left(\frac{-r_{in}}{V}t\right) \to 0$ as $t \to \infty$, we see from the solution in (a) that

$$C \to C_{in} = 0.005\mathrm{kg}\,\ell^{-1}.$$

So, for large time t, the concentration of the solution in the container is close to the concentration of the inflowing solution $C_{in} = 0.005\mathrm{kg}\,\ell^{-1}$. For example, after an hour ($t = 3600$s), the concentration is $0.0050037...\mathrm{kg}\,\ell^{-1}$.

Remark: the volume was constant in this example; it is possible to incorporate a time-dependent volume $V = V(t)$ in the ODE for $C(t)$:

$$\frac{dC}{dt} + \frac{r_{in}}{V(t)}C = \frac{C_{in}r_{in}}{V(t)}$$

If the expression $V(t)$ is known (or can be worked out from the problem), then we can solve this ODE with integrating factor:

$$R(t) = \exp\left(\int \frac{r_{in}}{V(t)}dt\right).$$

Exercise (variable volume): A water tank of volume $1000\,\ell$ is initially *half-full* of fresh water. A pipe brings a salt solution of concentration $0.005\,\mathrm{kg}\,\ell^{-1}$ into the tank at a rate of $5\,\ell\,\mathrm{s}^{-1}$, and a second pipe carries away the overflow at the same rate. Calculate $C(t)$, the concentration of salt in the tank, assuming that the liquid is well mixed.

Answer: $C(t) = \begin{cases} \dfrac{0.025t}{500+5t} & 0 \le t < 100 \\ 0.005 - 0.0025\exp[-0.005(t-100)] & t \ge 100. \end{cases}$

3.5 Economics and Finance

3.5.1 First-order model of supply and demand

Let $P(t)$ be the price of a product, and let Q_S and Q_D be the quantity supplied and quantity demanded for the product. Suppose that

1. the demand for a product decreases as the price goes up,
2. the supply of a product increases as the price goes up, and
3. the rate of change of price of the product is positive (the price goes up) when demand exceeds supply.

If we model these processes by some linear functions, the variation of the product price with time can be described by a linear first-order differential equation.

Example 3.5. *Let us model Q_D and Q_S by linear functions in P as follow:*

$$Q_D = a - bP \qquad \textit{and} \qquad Q_S = f + gP,$$

where a, b, e and f are positive constants. Suppose also that

$$\frac{dP}{dt} = h(Q_D - Q_S), \qquad \textit{with} \qquad P(0) = P_0$$

and h is a positive constant. Set up, solve and interpret the ODE for $P(t)$.

Solution: We firstly substitute the quantity equations into the ODE:

$$\begin{aligned} \frac{dP}{dt} &= h(a - bP - f - gP) \\ &= h\left[(a-f) - (b+g)P\right] \\ &= -h(b+g)P + h(a-f) \\ \frac{dP}{dt} + h(b+g)P &= h(a-f). \end{aligned}$$

This is a linear differential equation in P, so we can find $R(t) = \exp\left[\int h(b+g)\,dt\right] = \exp\left[h(b+g)t\right]$. Multiplying through by this R, we thus have

$$\begin{aligned} \frac{d}{dt}\left[e^{h(b+g)t}P\right] &= h(a-f)e^{h(b+g)t} \\ e^{h(b+g)t}P &= \int h(a-f)e^{h(b+g)t}\,dt + k \\ &= h(a-f)\frac{1}{h(b+g)}e^{h(b+g)t} + k, \end{aligned}$$

where k is a constant. Then we see that

$$\begin{aligned} P(t) &= \frac{a-f}{b+g} + ke^{-h(b+g)t} \\ &= P_\infty + ke^{-\gamma t}, \end{aligned}$$

where $P_\infty = \frac{a-f}{b+g}$ and $\gamma = h(b+g) > 0$. To fix k, observe that $P(0) = P_0$, so that

$$\begin{aligned} P_0 &= P_\infty + k \\ k &= P_0 - P_\infty \\ P(t) &= P_\infty + (P_0 - P_\infty)\, e^{-\gamma t}. \end{aligned}$$

This solution tells us that (i) the price has a constant solution $P(t) = P_\infty$, that is, if we start with $P_0 = P_\infty$, the price stays at P_∞ and (ii) if the initial price is different from P_∞, $P(t)$ tends exponentially towards P_∞ in all other cases.

3.5.2 Continuously compounded interest

Bank accounts sometimes pay interest on a “continuously-compounded” basis, where the interest due to you is calculated daily. Over the years, this *looks more or less like* interest being continually added to the account. If you withdraw money on a “continual” basis as well, then we can model this situation with a first-order differential equation:

$$\begin{pmatrix}\text{Rate of change of}\\ \text{money in the account}\end{pmatrix} = \begin{pmatrix}\text{Rate of accrual of}\\ \text{interest on the account}\end{pmatrix} - \begin{pmatrix}\text{Rate of withdrawal of}\\ \text{money from the account}\end{pmatrix}.$$

Example 3.6. *After a working life of careful saving and investing, your pension fund now totals £400,000. Your plan is to take out an investment that pays* 2.0% *interest annually, continuously compounded, and to withdraw £25,000 per year at a constant rate through the year. Assuming the interest rate does not change, how long do you expect the money to last?*[4]

Solution: We can set up a differential equation to model this situation. Let $M(t)$ be the amount of money (in units of £10,000) in the account, with t measured in years. Then the initial condition is $M(0) = M_0 = 40$, and the equation is

$$\frac{dM}{dt} = rM - I,$$

where r is the interest rate, and I is the income derived. Re-arranging this equation, we get

$$\frac{dM}{dt} - rM = -I.$$

[4]This question does not constitute financial advice. Seek professional assistance before making financial plans for your future. Past performance is no guide to future performance. May contain nuts.

This is a linear equation, with integrating factor $R(t) = \exp\left(-\int r\,dt\right) = \exp\left(-rt\right)$. Multiplying through, we find that

$$\begin{aligned}
\frac{d}{dt}(e^{-rt}M) &= -Ie^{-rt} \\
e^{-rt}M &= -I\int e^{-rt}\,dt \\
&= \frac{I}{r}e^{-rt} + C \\
M &= \frac{I}{r} + Ce^{rt}.
\end{aligned}$$

We can fix C by considering the initial condition $M(0) = M_0$:

$$\begin{aligned}
M_0 &= \frac{I}{r} + C \implies C = M_0 - \frac{I}{r} \\
\therefore \quad M(t) &= \frac{I}{r} + \left(M_0 - \frac{I}{r}\right)e^{rt}.
\end{aligned}$$

The money will run out at time $t = T$ when $M(T) = 0$, which is to say,

$$\begin{aligned}
0 &= \frac{I}{r} + \left(M_0 - \frac{I}{r}\right)e^{rT} \\
\frac{I}{r} &= \frac{I - rM_0}{r}e^{rT} \\
\frac{I}{I - rM_0} &= e^{rT} \\
\log\frac{I}{I - rM_0} &= rT \\
T &= \frac{1}{r}\log\frac{I}{I - rM_0}.
\end{aligned}$$

Now, in this case, $M_0 = 40$, $r = 0.02$ yr^{-1} and $I = 2.5$, so we find

$$\begin{aligned}
T &= \frac{1}{0.02}\log\frac{2.5}{2.5 - 0.02 \times 40} \\
&\approx 19.3 \text{ years}.
\end{aligned}$$

Note that the above calculation only makes sense when $I - rM_0 > 0$ so that we are taking the logarithm of a positive number. So what happens for the case $I - rM_0 < 0$? *(Hint: consider the sign of the second term in the particular solution.)*

Chapter 4

Second-order linear ODEs

Contents

In this chapter, we develop some basic theory and solution methods for second-order linear ODEs. We start by introducing the general form of these ODEs and define some basic concepts. We will see that the general solution of second-order linear ODEs comprises a linear combination of two linearly independent functions; there are some subtle connections to linear algebra here which we briefly touch on. The first method for finding a general solution, called **reduction of order**, relies on us already knowing one of these functions and is a robust method for dealing with non-constant coefficients. We then move on to **second-order linear ODEs with constant coefficients**, which may be familiar to some of you from school or college. Finally, we show that second-order ODEs can be written as *a system of two-dimensional ODEs*, and solve the problem using **phase plane analysis**. This is a powerful technique that gives a graphical interpretation of the solution, and can be further developed for solving nonlinear ODEs. This final part requires some basic knowledge from linear algebra, namely finding eigenvalues and eigenvectors of 2×2 matrices. Completion of this chapter allows us to move on to applications involving second-order ODEs.

4.1 Introduction

Second-order linear ODEs are of the form:

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = h(x), \tag{4.1}$$

where $f(x)$, $g(x)$ and $h(x)$ are given functions of x. The equation is second-order because the highest derivative is $\frac{d^2y}{dx^2}$, and it is linear because there are no products of y with itself or its derivatives. There are two important classifications:

- The equation is **homogeneous** if $h(x) \equiv 0$, otherwise it is **inhomogeneous**. Note that the word "homogeneous" is used in a sense different from that of section 2.2.1. Here it refers to the fact that all terms contain either y or its derivatives.
- The equation is **constant coefficient** if f and g are constants (rather than functions of x), otherwise it is **non-constant coefficient**.

The homogeneous equation corresponding to (4.1) is:

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = 0 \tag{4.2}$$

Note that $y = 0$ is always a solution of this homogeneous equation.

The general solution of (4.1) will contain two arbitrary constants (since the equation is second-order). The particular solution can be specified in two ways:

- Initial value problem (IVP): We are given the values of y and y' at a single value of x.
- Boundary value problem (BVP): We are given the values of $\alpha_i y + \beta_i y'$ for some constants α_i, β_i at two different values of $x = x_i$ $(i = 1, 2)$. Note that some of the α_i and β_i may be zero.

In the case of initial value problems, there is a useful theorem that tells us there is a unique solution of the ODE (4.1).

Theorem (for IVPs): Let $f(x)$, $g(x)$ and $h(x)$ be continuous on an interval I of the x-axis. If the initial condition is specified at a point in I, then the solution of equation (4.1) **exists** and is **unique**.

Example 4.1. *Show that*

$$y = C_1 \sin x + C_2 \cos x,$$

where C_1 and C_2 are arbitrary constants, satisfies the ODE

$$y'' + y = 0.$$

Hence solve the IVP:

$$y'' + y = 0, \qquad \textit{with } y(0) = 0 \textit{ and } y'(0) = 1,$$

and the BVP:

$$y'' + y = 0, \qquad \textit{with } y(0) = 0 \textit{ and } y(\pi) = 1.$$

Solution: Note that $f = 0$ and $g = 1$ are continuous functions of x, so the existence theorem for IVPs applies. To show that the given function satisfies the ODE, we need some derivatives:

$$y' = C_1 \cos x - C_2 \sin x, \qquad y'' = -C_1 \sin x - C_2 \cos x = -y,$$

so $y'' + y = 0$. For both the IVP and the BVP, we have $y(0) = 0$. Substitute $x = 0$ into the expression for $y(x)$, we find that:

$$y(0) = C_1 \sin 0 + C_2 \cos 0 = C_2 = 0$$

For the IVP, we have $y'(0) = C_1$, so $C_1 = 1$ and $y(x) = \sin x$: the solution exists and is unique, as expected.
For the BVP, we have $y(\pi) = C_1 \sin \pi = 0 \neq 1$, so we cannot have $y(\pi) = 1$: there is no solution. This is an example of an **ill-posed problem**, and is one reason why the existence theorem is only for IVPs. (See also Example (1.4))

4.1.1 Some special cases

Second-order ODEs, linear or not, are generally difficult to solve. There are some special cases when a second-order ODE can be transformed into a first-order ODE.

- When the dependent variable y does not appear in the equation, the substitution $u = y'$ leads to a first-oder ODE with $u(x)$ as the dependent variable. For example, the equation $xy'' + y' = (y')^2$ becomes the separable equation $xu' + u = u^2$ which we can solve for $u(x)$. We then obtain $y(x)$ by solving $y' = u$. The solution is $y = \frac{1}{C_1} \log |C_1 x + 1| + C_2$.

- For an autonomous equation, the independent variable x does not appear explicitly. If we let $u = y'(x)$ and consider u as a function of y, we can use the chain rule to write $y''(x) = u(y)u'(y)$ and obtain a first-order equation in which u is the dependent variable and y is the *independent* variable. For example, we can transform $y\frac{d^2y}{dx^2} = 2(\frac{dy}{dx})^2$ into $y\frac{du}{dy} = 2u$. Solving this equation for u and then the equation $y'(x) = u$ for y gives $y = (C_1 + C_2 x)^{-1}$.

4.2 Superposition principle and general solution

4.2.1 Superposition principle for linear ODEs

Theorem: Suppose that $y_1(x)$ and $y_2(x)$ are two solutions of the **homogeneous** linear ODE (4.2). Then $C_1 y_1 + C_2 y_2$ is also a solution of (4.2), where C_1 and C_2 are any constants. This combination $C_1 y_1 + C_2 y_2$ is called a linear superposition (or linear combination) of the two functions $y_1(x)$ and $y_2(x)$.

Proof. First we calculate the derivatives of the linear combination:

$$\begin{aligned}
y &= C_1 y_1 + C_2 y_2 \\
y' &= C_1 y_1' + C_2 y_2' \\
y'' &= C_1 y_1'' + C_2 y_2'' \\
\text{Then, LHS of } (4.2) &= y'' + f y' + g y \\
&= (C_1 y_1'' + C_2 y_2'') + f\,(C_1 y_1' + C_2 y_2') + g\,(C_1 y_1 + C_2 y_2) \\
&= C_1\,(y_1'' + f y_1' + g y_1) + C_2\,(y_2'' + f y_2' + g y_2) \\
&= 0 = \text{RHS of } (4.2)
\end{aligned}$$

So y satisfies (4.2). □

Note that this theorem does **not** hold for the inhomogeneous linear ODE (4.1) or a nonlinear ODE. For example, if y_0 is a solution of (4.1), $C_0 y_0$ is not a solution of (4.1). Similarly, the sum of two solutions of (4.1) is not a solution of (4.1). We do have the following theorem for an inhomogeneous linear ODE:

Theorem: Suppose that $y_0(x)$ is a solution of the inhomogeneous linear ODE (4.1), and that $y_1(x)$ is a solution of the homogeneous linear ODE (4.2). Then $y_0(x) + C_1 y_1(x)$ is a solution of (4.1).

4.2.2 General solution of a homogeneous linear ODE

Two functions $y_1(x)$ and $y_2(x)$ are linearly independent (on some interval) if

$$a_1 y_1(x) + a_2 y_2(x) = 0 \text{ implies } a_1 = 0 \text{ and } a_2 = 0.$$

When the above equation holds for some constants a_1, a_2 not both zero, we call $y_1(x)$ and $y_2(x)$ linearly dependent. An intuitive way to understand this is that if $y_1(x)$ and $y_2(x)$ are linearly dependent, then $y_1(x)$ and $y_2(x)$ are proportional to each other in the sense that

$$y_1(x) = k_1 y_2(x) \quad \text{or} \quad y_2(x) = k_2 y_1(x)$$

for some constants k_1, k_2.

We now focus on the homogeneous linear ODE (4.2). Suppose f and g are continuous. Then (4.2) has a general solution of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{4.3}$$

where y_1 and y_2 are linearly independent and both are solutions of (4.2). c_1 and c_2 are arbitrary constants.

Theorem: Every solution of the homogeneous linear ODE (4.2) can be obtained by assigning suitable values to c_1 and c_2 in the general solution (4.3).

In other words, the general solution (4.3) contains all solutions of (4.2). So the problem of finding all solutions of a second-order homogeneous linear ODE boils down to finding two solutions y_1 and y_2 that are linearly independent.

The theorems we learn in this section suggest the following strategy for finding a general solution of the inhomogeneous linear second-order ODE (4.1):

1. Find two linearly independent solutions y_1, y_2 of the corresponding homogeneous equation (4.2). The function $y_c = C_1 y_1 + C_2 y_2$ is thus a general solution of (4.2).

2. Find a solution y_p of the inhomogeneous equation (4.1).

3. Then the function $y = y_c + y_p = C_1 y_1 + C_2 y_2 + y_p$ is a solution of (4.1) and it contains two arbitrary constants. Therefore, it is a general solution of (4.1). It can be proved that y contains all solutions of (4.1).

While in principle this strategy can be applied to (4.1) even when $f(x)$ and $g(x)$ are functions of x, it is particularly useful for constant-coefficient equations because, as we shall in later sections, there are methods to determine y_1, y_2 and y_p when the coefficients are constant. But before we discuss that, we first introduce in the next section a technique called reduction of order which, when a certain condition is satisfied, can be applied to linear equations whose coefficients are not constant.

4.3 Reduction of order for linear ODEs

In general, the inhomogeneous equation (4.1) can be hard to solve. However, suppose we are given, or have managed to find, one solution of the homogeneous ODE (4.2)—perhaps by a lucky (educated!) guess. We can use this solution to turn the second-order linear ODE (4.1) into a first-order linear ODE, which we can then solve using the method of section 2.5. This technique is called **reduction of order**. We use the following procedure:

1. Let $y_1(x)$ be the solution of (4.2) that we know, so

$$y_1'' + f y_1' + g y_1 = 0 \tag{4.4}$$

Now let

$$y(x) = u(x) y_1(x),$$

where $u(x)$ is an unknown function of x. We will derive a first-order ODE for $u'(x)$.

2. Calculate derivatives up to order 2 (using product rule since u and y_1 are both functions of x):

$$\begin{aligned} y &= uy_1, \\ y' &= u'y_1 + uy_1', \\ y'' &= u''y_1 + 2u'y_1' + uy_1''. \end{aligned}$$

3. Substitute y, y', y'' into (4.1), noting that the u term should cancel exactly:

$$\begin{aligned} y'' + fy' + gy &= (u''y_1 + 2u'y_1' + uy_1'') + f\,(u'y_1 + uy_1') + g\,(uy_1) \\ &= (y_1)\,u'' + (2y_1' + fy_1)\,u' + (y_1'' + fy_1' + gy_1)\,u \\ &= (y_1)\,u'' + (2y_1' + fy_1)\,u' \\ &= h(x). \end{aligned}$$

4. Note that u itself disappears in the equation above because of (4.4). Therefore from our discussion in section (4.1.1), we define a new dependent variable $v = u'$ and write an ODE for v:

$$y_1 v' + (2y_1' + fy_1)\,v = h.$$

5. This is a first-order linear ODE, which can be solved by the method of section 2.5 (using the Integrating Factor method). Verify that the solution $v(x)$ is correct.

6. Once we have the general solution $v(x)$ (with one arbitrary constant C_1), we integrate once more to find $u(x) = \int v\,dx + C_2$, remembering to add the second arbitrary constant C_2. Now we have $u(x)$.

7. Hence we can find the general solution $y(x)$ of (4.1) by calculating $y = uy_1$.

8. We can use the initial values or boundary values to calculate the values of the two arbitrary constants C_1 and C_2, and hence find the particular solution.

9. Lastly, we can verify that the particular solution satisfies (4.1) and the initial or boundary conditions.

Example 4.2. *Solve the boundary value problem*

$$xy'' + 2(x+1)y' + (x+2)y = x^2e^{-x} \qquad \textit{with } y(1) = y(2) = 0,$$

given that $y_1 = e^{-x}$ is a solution of the corresponding homogeneous problem.

Solution:

1. First, verify that $y_1(x) = e^{-x}$ is a solution of the corresponding homogeneous problem. Computing the derivatives:
$$y_1 = e^{-x}, \qquad y_1' = -e^{-x}, \qquad y_1'' = e^{-x},$$
we have that:
$$\begin{aligned} xy_1'' + 2(x+1)y_1' + (x+2)y_1 &= xe^{-x} - 2(x+1)e^{-x} + (x+2)e^{-x} \\ &= 0 \qquad \text{OK.} \end{aligned}$$
Now let $y(x) = u(x)y_1(x) = ue^{-x}$ where $u(x)$ is an unknown function of x.

2. Calculate higher-order derivatives:
$$\begin{aligned} y &= ue^{-x}, \\ y' &= u'e^{-x} - ue^{-x}, \\ y'' &= u''e^{-x} - 2u'e^{-x} + ue^{-x}. \end{aligned}$$

3. Substitute y into the given ODE, noting that the u term should cancel exactly:
$$\begin{aligned} x\left(u''e^{-x} - 2u'e^{-x} + ue^{-x}\right) + 2(x+1)\left(u'e^{-x} - ue^{-x}\right) + (x+2)ue^{-x} &= x^2e^{-x} \\ xu''e^{-x} + \left(-2x + 2(x+1)\right)u'e^{-x} + \left(x - 2(x+1) + (x+2)\right)ue^{-x} &= x^2e^{-x} \\ xu''e^{-x} + 2u'e^{-x} &= x^2e^{-x}. \\ xu'' + 2u' &= x^2. \end{aligned}$$

4. Note that u itself does not appear in this equation. We can therefore define a new dependent variable $v = u'$ and write an ODE for v:
$$xv' + 2v = x^2$$
or, dividing by x:
$$v' + \frac{2}{x}v = x.$$

5. This is a first-order linear ODE which can be solved using the Integrating Factor method. The Integrating Factor is:
$$R(x) = \exp\left(\int \frac{2}{x}\,dx\right) = \exp(2\log|x|) = x^2$$
Multiply the ODE by $R(x)$ and integrate:
$$\begin{aligned} x^2v' + 2xv = (x^2v)' &= x^3 \\ x^2v &= \frac{x^4}{4} + C_2 \\ v &= \frac{x^2}{4} + \frac{C_2}{x^2}. \end{aligned}$$
Verify that the solution $v(x)$ is correct (exercise).

6. Since $u' = v$, we integrate once more to find $u(x)$:

$$u = \int v dx + C_1 = \int \left(\frac{x^2}{4} + \frac{C_2}{x^2} \right) dx + C_1 = \frac{x^3}{12} - \frac{C_2}{x} + C_1,$$

remembering to add the second arbitrary constant C_1.

7. The general solution $y(x)$ of the given ODE is:

$$y = uy_1 = \left(\frac{x^3}{12} - \frac{C_2}{x} + C_1 \right) e^{-x}.$$

8. We can use the given boundary values to determine C_1 and C_2, and hence find the particular solution:

$$\begin{aligned} y(1) &= \left(\frac{1}{12} - C_2 + C_1 \right) e^{-1} = 0; \\ y(2) &= \left(\frac{8}{12} - \frac{C_2}{2} + C_1 \right) e^{-2} = 0. \end{aligned}$$

These lead to a set of simultaneous equations for C_1 and C_2:

$$\begin{aligned} 1 + 12C_1 - 12C_2 &= 0 \\ 8 + 12C_1 - 6C_2 &= 0 \end{aligned}$$

$$\therefore \qquad C_1 = -\frac{5}{4} \qquad \text{and} \qquad C_2 = -\frac{7}{6}.$$

The particular solution is therefore:

$$y(x) = \left(\frac{x^3}{12} + \frac{7}{6x} - \frac{5}{4} \right) e^{-x}.$$

9. We can verify that the answer satisfies the ODE and the boundary conditions:

$$\begin{aligned} y(1) &= \left(\frac{1}{12} + \frac{7}{6} - \frac{5}{4} \right) e^{-1} = 0 \\ y(2) &= \left(\frac{8}{12} + \frac{7}{12} - \frac{5}{4} \right) e^{-2} = 0 \\ y' &= \left(-\frac{x^3}{12} + \frac{x^2}{4} - \frac{7}{6x} - \frac{7}{6x^2} + \frac{5}{4} \right) e^{-x} \\ y'' &= \left(\frac{x^3}{12} - \frac{x^2}{2} + \frac{x}{2} - \frac{7}{6x} + \frac{7}{3x^2} + \frac{7}{3x^3} - \frac{5}{4} \right) e^{-x} \\ xy'' + 2(x+1)y' + (x+2)y &= \cdots = x^2 e^{-x} \qquad \text{(check!)} \end{aligned}$$

Note that the general solution can be written as

$$y = C_1 e^{-x} - C_2 \frac{e^{-x}}{x} + \frac{x^3 e^{-x}}{12}$$

which is of the form $y = C_1 y_1 + C_2 y_2 + y_p$. Convince yourself that $y_p = \frac{x^3 e^{-x}}{12}$ satisfies the given inhomogeneous equation while $y_1 = e^{-x}$ and $y_2 = \frac{e^{-x}}{x}$ are linearly independent and satisfy the corresponding homogeneous equation. This is consistent with the discussion at the end of the previous section.

4.4 Second-order constant coefficient linear ODEs

Second-order linear ODEs with constant coefficients are of the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = h(x), \tag{4.5}$$

where a and b are constants (assumed to be real), and $h(x)$ is a given function of x. Note that the coefficient of y'' may be a real number not equal to 1, but we can always divide through so that the coefficient of y'' is 1. If $h = 0$, the ODE is homogeneous:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0. \tag{4.6}$$

4.4.1 The second-order homogeneous problem

We start with the homogeneous equation (4.6) in this section. Motivated by the fact that when we differentiate $e^{\lambda x}$, a multiplicative constant λ is introduced in front of the exponential:

$$(e^{\lambda x})' = \lambda e^{\lambda x} \qquad (e^{\lambda x})'' = \lambda^2 e^{\lambda x},$$

we **postulate** that the solution to (4.6) is of the form

$$y = e^{\lambda x}.$$

Substitute the above function and its derivatives into (4.6), we get

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = \left(\lambda^2 + a\lambda + b\right) e^{\lambda x} = 0.$$

Since $e^{\lambda x} > 0$, we must have

$$\lambda^2 + a\lambda + b = 0. \tag{4.7}$$

Hence, $y = e^{\lambda x}$ is a solution of (4.6) if λ is a solution of (4.7). This quadratic equation is called the **characteristic equation** of the ODE (4.6). In general (4.7) has two roots, λ_1 and λ_2. Using the quadratic formula, these roots are:

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a \pm \sqrt{\Delta}}{2}$$

where Δ is the **discriminant**

$$\Delta = a^2 - 4b.$$

Clearly the form of λ is determined by the sign Δ, and so we consider three possible cases: (i) $\Delta > 0$, (ii) $\Delta = 0$, and (iii) $\Delta < 0$.

(i) Real distinct roots. If $\Delta = a^2 - 4b > 0$, there are two real distinct roots:

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

Therefore the two functions,

$$y_1 = e^{\lambda_1 x}, \qquad y_2 = e^{\lambda_2 x}$$

are solutions of (4.6). Furthermore, they are linearly independent. So we have the following general solution for (4.6):

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

(ii) Real repeated root. If $\Delta = a^2 - 4b = 0$, there is one real (repeated) root:

$$\lambda = \frac{-a}{2}.$$

This gives us one solution y_1 to (4.6),

$$y_1 = e^{\lambda x} = e^{-\frac{a}{2}x}.$$

We can now use the reduction of order method of section 4.3 to find a general solution to (4.6). Write $y = u(x)y_1$, where u is an unknown function. First, compute derivatives of y up to order 2:

$$\begin{aligned} y &= ue^{\lambda x}, \\ y' &= u'e^{\lambda x} + \lambda u e^{\lambda x}, \\ y'' &= u''e^{\lambda x} + 2\lambda u' e^{\lambda x} + \lambda^2 u e^{\lambda x}, \end{aligned}$$

and then substitute these expressions into (4.6):

$$\begin{aligned} \left(u'' + 2\lambda u' + \lambda^2 u\right) e^{\lambda x} + \left(au' + a\lambda u\right) e^{\lambda x} + bue^{\lambda x} &= 0 \\ \left(u'' + (2\lambda + a)u' + (\underbrace{\lambda^2 + a\lambda + b}_{=0 \text{ from (4.7)}})\right) e^{\lambda x} &= 0. \end{aligned}$$

So we have a linear ODE for u:

$$u'' + (2\lambda + a)u' = 0.$$

Since $\lambda = -a/2$, it follows that $2\lambda + a = 0$ and so the equation we have to solve is simply

$$u'' = 0$$

Integrating (twice) gives the solution $u = C_1 + C_2x$, and therefore the general solution is $y = uy_1 = (C_1 + C_2x)e^{\lambda x}$ or,

$$y = C_1e^{\lambda x} + C_2xe^{\lambda x}$$

with $\lambda = -a/2$. Note that $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$ are linearly independent.

(iii) Complex roots. If $\Delta = a^2 - 4b < 0$, there are two distinct complex roots. In this case, let

$$p = \frac{-a}{2} \qquad \text{and} \qquad q = \frac{\sqrt{4b - a^2}}{2}$$

so

$$\lambda_1 = p + iq, \qquad \text{and} \qquad \lambda_2 = p - iq.$$

As in the case of real distinct roots, we can write:

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x} = Ae^{(p+iq)x} + Be^{(p-iq)x}$$

In general, A and B will be complex numbers in this expression. If we want to be sure of a real solution, we can tidy this up:

$$\begin{aligned} y &= e^{px}\left(Ae^{iqx} + Be^{-iqx}\right) \\ &= e^{px}\left(A(\cos(qx) + i\sin(qx)) + B(\cos(qx) - i\sin(qx))\right) \\ &= e^{px}\left((A+B)\cos(qx) + i(A-B)\sin(qx)\right) \\ &= C_1e^{px}\cos(qx) + C_2e^{px}\sin(qx) \end{aligned}$$

where $C_1 = A + B$ and $C_2 = i(A - B)$. For real initial or boundary conditions, C_1 and C_2 will be real, in spite of appearances (because A and B are complex).

Summary: For the homogeneous second-order linear ODE with constant coefficients (4.6), the characteristic equation is

$$\lambda^2 + a\lambda + b = 0.$$

There are three possible cases:

(i) If there are two real roots, λ_1 and λ_2, then the general solution is

$$y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}.$$

(ii) If there is one repeated real root, λ, then the general solution is

$$y = (C_1 + C_2x)e^{\lambda x}.$$

(iii) If there are two complex roots, $p \pm iq$, then the general solution is

$$y = e^{px}[C_1 \cos(qx) + C_2 \sin(qx)].$$

Example 4.3. *Find a general solution of the ODE:*

$$y'' + 3y' + 2y = 0.$$

Solution: We find the characteristic equation by substituting $y = e^{\lambda x}$ into the ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned} \lambda^2 e^{\lambda x} + 3\lambda e^{\lambda x} + 2e^{\lambda x} &= 0 \\ \lambda^2 + 3\lambda + 2 &= 0 \end{aligned}$$

Solve this equation for λ:

$$(\lambda + 1)(\lambda + 2) = 0, \qquad \text{so } \lambda = -1 \text{ or } \lambda = -2.$$

Note that these are real and distinct, so the general solution is

$$y = C_1 e^{-x} + C_2 e^{-2x}$$

Example 4.4. *Find a general solution of the ODE:*

$$y'' + 4y' + 4y = 0.$$

Solution: We find the characteristic equation by substituting $y = e^{\lambda x}$ into the ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned} \lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} + 4e^{\lambda x} &= 0 \\ \lambda^2 + 4\lambda + 4 &= 0 \end{aligned}$$

Solve this equation for λ:

$$(\lambda + 2)^2 = 0, \qquad \text{so } \lambda = -2.$$

Note that this is a repeated real root, so the general solution is

$$y = (C_1 + C_2 x)e^{-2x}$$

Example 4.5. *Find a solution of the ODE:*

$$4y'' + 4y' + 17y = 0.$$

Solution: We find the characteristic equation by substituting $y = e^{\lambda x}$ into the ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned} 4\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} + 17e^{\lambda x} &= 0 \\ 4\lambda^2 + 4\lambda + 17 &= 0 \end{aligned}$$

Solve this equation for λ:

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4 \times 4 \times 17}}{2 \times 4} = \frac{-4 \pm \sqrt{-256}}{8} = \frac{-4 \pm 16i}{8} = -\frac{1}{2} \pm 2i$$

Note that these are complex: the real part is $-\frac{1}{2}$ and the imaginary part is ± 2. So the general solution is

$$y = (C_1 \cos 2x + C_2 \sin 2x)\, e^{-x/2}$$

4.4.2 The second-order inhomogeneous problem

We now turn to the inhomogeneous problem (4.5):

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = h(x).$$

From the theorems in section 4.2, we know that the general solution of an inhomogeneous equation has the form

$$y = y_c + y_p$$

where y_c is a general solution of the corresponding homogeneous equation and y_p is any particular solution of the inhomogeneous equation. y_c is sometimes called the complementary function of (4.5). We have already learned how to obtain y_c for equations with constant coefficients. We now discuss how to determine y_p.

If $h(x)$ in (4.5) involves exponential functions, polynomials, sine or cosine (a common characteristics of these functions is their derivatives are of the same kind as the functions themselves), the **method of undetermined coefficients** can be applied to compute y_p.[1] The basic idea is to guess that y_p has a similar form to $h(x)$ but with unknown coefficients. We then substitute this guess into (4.5) to determine the coefficients. The detailed rules for choosing y_p are:

1. If $h(x)$ is an exponential function, a polynomial (including the constant x^0), the sine or cosine function, choose y_p according to Table 4.1. If $h(x)$ is a product of two of these functions, form a product from the corresponding choices. Then check Rule 2.

   Note that whenever $h(x)$ involves a sine or a cosine, **both sine and cosine** should be included in the choice as indicated in Table 4.1, e.g. if $h(x) = e^{kx} \sin \omega x$, then $y_p = e^{kx}(A \sin \omega x + B \cos \omega x)$.

[1] There is a general method called the variation of parameters that works for any $h(x)$.

functions in $h(x)$	y_p
e^{kx}	Ae^{kx}
$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0$
$\sin \omega x$ or $\cos \omega x$	$A \sin \omega x + B \cos \omega x$

Table 4.1: Method of undetermined coefficients

2. If $h(x)$ is a solution of the corresponding homogeneous equation (4.6), multiply the choice from Rule 1 by x. If the characteristic equation has repeated root, multiply by x^2 instead.

3. If $h(x)$ is a sum of more than one function, add together the corresponding choices obtained from Rule 1 and Rule 2.

So now we have a complete method for solving second-order linear constant-coefficient ODEs:

1. Find the characteristic equation and then a general solution y_c of the corresponding homogeneous problem. y_c contains two arbitrary constants.

2. Find a particular solution y_p of the inhomogeneous equation using the method of undetermined coefficients. This should contain no arbitrary constants.

3. The general solution is $y = y_c + y_p$.

4. Use the initial values or the boundary values to determine the two arbitrary constants in the complete solution $y_c + y_p$.

5. Verify your solution is correct.

Example 4.6. *Solve the initial value problems:*

$$\begin{array}{lll} (a) & y'' + 5y' + 4y = 4x^2 + 10x + 2, & \textit{with } y(0) = 1 \textit{ and } y'(0) = 2 \\ (b) & y'' + 5y' + 4y = \sin x, & \textit{with } y(0) = 1 \textit{ and } y'(0) = 2 \\ (c) & y'' + 5y' + 4y = e^{-x}, & \textit{with } y(0) = 1 \textit{ and } y'(0) = 2 \end{array}$$

Solution:
(a) We find the characteristic equation by substituting $y = e^{\lambda x}$ into the **homogeneous version** of the given ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned} \lambda^2 e^{\lambda x} + 5\lambda e^{\lambda x} + 4e^{\lambda x} &= 0 \\ \lambda^2 + 5\lambda + 4 &= 0 \end{aligned}$$

Solve this equation for λ:

$$(\lambda + 1)(\lambda + 4) = 0, \qquad \text{so } \lambda = -1 \text{ or } \lambda = -4$$

Note that these are real and distinct, so the complementary function is

$$y_c = C_1 e^{-x} + C_2 e^{-4x}$$

$h(x)$ is a quadratic function of x, so we choose a general quadratic function for y_p:

$$y_p = Ax^2 + Bx + D$$

We need some derivatives:

$$y_p' = 2Ax + B \qquad \text{and} \qquad y_p'' = 2A$$

Substitute these into the given ODE:

$$\begin{aligned} 2A + 5(2Ax + B) + 4(Ax^2 + Bx + D) &= 4x^2 + 10x + 2 \\ 4Ax^2 + (4B + 10A)x + (4D + 5B + 2A) &= 4x^2 + 10x + 2 \end{aligned}$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is when the different terms on the LHS and the RHS have the same coefficients. There are three terms: the constant term, the x term and the x^2 term. Matching the coefficients of these terms on the LHS and the RHS, we get:

$$\begin{aligned} \text{constant term:} &\qquad 2A + 5B + 4D = 2 \\ x \text{ term:} &\qquad 10A + 4B = 10 \\ x^2 \text{ term:} &\qquad 4A = 4 \end{aligned}$$

which we can solve:

$$A = 1, \qquad B = \frac{1}{4}(10 - 10A) = 0, \qquad D = \frac{1}{4}(2 - 2A - 5B) = 0$$

so a particular solution to the given ODE is

$$y_p = x^2$$

and the general solution is:

$$y = C_1 e^{-x} + C_2 e^{-4x} + x^2$$

For dealing with the initial values, we need a derivative:

$$y' = -C_1 e^{-x} - 4C_2 e^{-4x} + 2x$$

and the initial values give us:

$$\begin{aligned} y(0) = C_1 + C_2 = 1 \\ y'(0) = -C_1 - 4C_2 = 2 \\ \implies C_1 = 2 \quad \text{and} \quad C_2 = -1 \end{aligned}$$

So,

$$y(x) = 2e^{-x} - e^{-4x} + x^2$$

(b) The complementary function y_c is as in (a). Since now $h(x) = \sin x$, we choose a general combination of sines and cosines for y_p:

$$y_p = A\cos x + B\sin x$$

We need some derivatives:

$$y_p' = -A\sin x + B\cos x, \qquad \text{and} \qquad y_p'' = -A\cos x - B\sin x$$

Substitute these into the given ODE:

$$\begin{aligned} -A\cos x - B\sin x + 5(-A\sin x + B\cos x) + 4(A\cos x + B\sin x) &= \sin x \\ (3A + 5B)\cos x + (3B - 5A)\sin x &= \sin x \end{aligned}$$

This is an identity. Matching the coefficients of the sine term and the cosine term on the two sides of the equation gives:

$$\begin{aligned} \text{sine term:} &\qquad -5A + 3B = 1 \\ \text{cosine term:} &\qquad 3A + 5B = 0 \end{aligned}$$

which we can solve:

$$A = -\frac{5}{34}, \quad B = \frac{3}{34}$$

so

$$y_p = \frac{3\sin x - 5\cos x}{34}$$

and the general solution is:

$$y = C_1 e^{-x} + C_2 e^{-4x} + \frac{3\sin x - 5\cos x}{34}$$

For dealing with the initial values, we need a derivative:

$$y' = -C_1 e^{-x} - 4C_2 e^{-4x} + \frac{3\cos x + 5\sin x}{34}$$

and the initial values give us:

$$\begin{aligned} y(0) &= C_1 + C_2 - \frac{5}{34} = 1 \\ y'(0) &= -C_1 - 4C_2 + \frac{3}{34} = 2 \\ \implies C_1 &= \frac{13}{6} \quad \text{and} \quad C_2 = -\frac{52}{51} \end{aligned}$$

So,

$$y(x) = \frac{13}{6}e^{-x} - \frac{52}{51}e^{-4x} + \frac{3\cos x + 5\sin x}{34}$$

(c) The complementary function y_c is as in (a). Now $h(x) = e^{-x}$ **is a solution of the homogeneous version of the given ODE** (as can be seen by assigning $C_1 = 1$ and $C_2 = 0$ to y_c), so according to Rule 2 we assume:

$$y_p = Axe^{-x}$$

(As an exercise, try $y_p = Ae^{-x}$ and show that it does not work.) We need some derivatives:

$$\begin{aligned} y_p' &= Ae^{-x} - Axe^{-x} = A(1-x)e^{-x} \\ y_p'' &= -Ae^{-x} - A(1-x)e^{-x} = A(x-2)e^{-x} \end{aligned}$$

Substitute these into the given ODE:

$$\begin{aligned} A(x-2)e^{-x} + 5A(1-x)e^{-x} + 4Axe^{-x} &= e^{-x} \\ 3Ae^{-x} &= e^{-x} \end{aligned}$$

Matching coefficients of the e^{-x} terms on both sides of the equation, we get:

$$3A = 1 \quad \Longrightarrow \quad A = \frac{1}{3}$$

so

$$y_p = \frac{1}{3}xe^{-x}$$

and the general solution is:

$$y = C_1e^{-x} + C_2e^{-4x} + \frac{1}{3}xe^{-x}$$

For dealing with the initial values, we need a derivative:

$$y' = -C_1e^{-x} - 4C_2e^{-4x} + \frac{1}{3}(1-x)e^{-x}$$

and the initial values give us:

$$y(0) = C_1 + C_2 = 1$$
$$y'(0) = -C_1 - 4C_2 + \frac{1}{3} = 2$$
$$\Longrightarrow \; C_1 = \frac{17}{9} \quad \text{and} \quad C_2 = -\frac{8}{9}$$

Therefore,

$$y(x) = \frac{17}{9}e^{-x} - \frac{8}{9}e^{-4x} + \frac{1}{3}xe^{-x}$$

Example 4.7. *Find the general solution of*

$$y'' + 6y' + 9y = 2e^{-3x} + \cos 3x$$

Solution: The characteristic equation is:

$$\lambda^2 + 6\lambda + 9 = 0$$

Solve this equation for λ:

$$(\lambda + 3)^2 = 0, \qquad \text{so } \lambda = -3$$

Note that this is a repeated real root, so the complementary function is

$$y_c = (C_1 + C_2 x)e^{-3x}$$

Here $h(x)$ is the sum of two functions $2e^{-3x}$ and $\cos 3x$. According to Rule 3, y_p will have two parts:

1. For $2e^{-3x}$, it is **a solution of the homogeneous version** of the given ODE (take $C_1 = 2, C_2 = 0$ in y_c) and the characteristic equation has a **repeated root**, so following Rule 2, we choose Ax^2e^{-3x}.
2. For $\cos 3x$, it is not a solution of the homogeneous version of the given ODE, so from Table 4.1, we choose $B\cos 3x + C\sin 3x$.

Putting everything together, we have

$$y_p = Ax^2e^{-3x} + B\cos 3x + C\sin 3x$$

We need some derivatives:

$$\begin{aligned} y_p' &= A(2x - 3x^2)e^{-3x} - 3B\sin 3x + 3C\cos 3x \\ y_p'' &= A(2 - 12x + 9x^2)e^{-3x} - 9B\cos 3x - 9C\sin 3x \end{aligned}$$

Substitute these into the ODE and simplify:

$$2Ae^{-3x} + 18C\cos 3x - 18B\sin 3x = 2e^{-3x} + \cos 3x$$

This is an identity that is true for all values of x. Therefore matching coefficients on both sides gives

$$\begin{aligned} 2A &= 2 \\ 18C &= 1 \\ -18B &= 0 \\ \implies \quad A = 1, \quad B &= 0, \quad C = \frac{1}{18} \end{aligned}$$

so

$$y_p = x^2 e^{-3x} + \frac{1}{18}\sin 3x$$

and the general solution is:

$$y = (C_1 + C_2 x + x^2)e^{-3x} + \frac{1}{18}\sin 3x$$

Note that $h(x)$ involves $\cos 3x$ but $\sin 3x$ appears in the general solution. This shows the importance of including both sine and cosine in the choice of y_p during the calculation.

Example 4.8. *Find the general solution of*

$$y'' + 4y = 4e^x \sin 2x$$

Solution: The characteristic equation is

$$\lambda^2 + 4 = 0$$

Solve this equation for λ:

$$\lambda^2 = -4, \qquad \text{so } \lambda = \pm 2i.$$

Note that these are complex roots: the real part is 0 and the imaginary part is 2. So the complementary function is

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

Here $h(x)$ involves the product of two functions: e^x and $\sin 2x$. Also note that $h(x)$ is **not** a solution of the homogeneous version of the given ODE. So we form a combination of (i) the product of e^x and $\cos 2x$ and (ii) the product of e^x and $\sin 2x$:

$$y_p = e^x(A\cos 2x + B\sin 2x)$$

We need some derivatives:

$$\begin{aligned} y_p' &= e^x[(A+2B)\cos 2x + (B-2A)\sin 2x] \\ y_p'' &= e^x[(4B-3A)\cos 2x - (4A+3B)\sin 2x] \end{aligned}$$

Substitute these into the ODE and simplify:

$$(A+4B)e^x\cos 2x + (B-4A)e^x \sin 2x = 4e^x \sin 2x$$

This is an identity that is true for all values of x. Therefore matching coefficients on both sides give

$$\begin{aligned} A + 4B &= 0 \\ B - 4A &= 4 \end{aligned}$$

Solving these equations, we get

$$A = -\frac{16}{17} \quad \text{and} \quad B = \frac{4}{17}$$

so

$$y_p = \frac{4}{17} e^x (\sin 2x - 4\cos 2x)$$

and the general solution is:

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{4}{17} e^x (\sin 2x - 4\cos 2x)$$

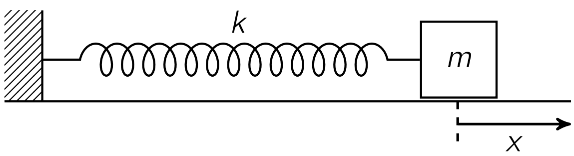
Chapter 5

Applications of second-order ODEs and phase plane analysis

Contents

In this chapter, we apply the theory and solution methods of second-order ODEs from chapter 4 to two application areas. As in chapter 3, the goal here is not just to solve an ODE but also to formulate the ODE that must be solved using appropriate modelling assumptions. We cover harmonic oscillation in a spring-mass system and in an LRC electric circuit. In the final section of the chapter, we discuss two-dimensional systems of ODEs.

5.1 Spring–mass system



We consider the system in the above figure where a mass m resting on a horizontal surface is connected to a spring. If, say, the mass is pulled to the right and then released, it will move horizontally under the action of the spring. We want to predict this motion.

The first thing we need to do is to set up a coordinate system for the problem. Let x be the position of the mass. We put the origin $x = 0$ at the position where the spring is at its natural length, i.e. the spring is not extended nor compressed. This is called the **equilibrium position**. We take the rightward direction to be positive as shown in the figure. The velocity of the mass, which is the rate of change of position (with time), is then

$$v(t) = \frac{dx}{dt} = \dot{x}(t).$$

It is common to use the dot notation to denote time derivatives. The acceleration, which is the rate of change of velocity, is

$$a(t) = \dot{v}(t) = \ddot{x}(t) = \frac{d^2x}{dt^2}.$$

The motion of the mass is governed by **Newton's second law of motion**, which states that the time-dependent acceleration of an object is (i) directly proportional the net force F acting upon the object and (ii) inversely proportional to the mass of the object. Without loss of generality, we can take the proportionality constant to be 1, then $a = F/m$. This means that: as the net force acting upon an object is increased, the acceleration of the object is increased; and as the mass of an object is increased, the acceleration of the object is decreased.

Newton's second law of motion is most commonly written as

$$F = ma.$$

We assume the mass m is constant. Since $a = \ddot{x}$, if we know how the force $F(x, t)$ changes as a function of position and time, we can formulate an ODE to solve for the position x of an object.

5.1.1 Simple harmonic motion

So what forces are acting on the mass? We first consider the simple situation where the horizontal surface is smooth, which means there is no friction between the mass and the surface. Then there are three forces acting on the mass: (i) gravity, (ii) the normal reaction from the horizontal surface and (iii) the force due to the spring, which is called the **tension**. Because the surface is horizontal, forces (i) and (ii) cancel each other. So there is no vertical motion and the mass moves horizontally due to the tension from the spring.

Now we need a mathematical model for the tension T. Provided that the spring is not deformed (extended or compressed) too much, **Hooke's law** states that the spring resists the deformation with a tension proportional to the amount of deformation. Since we put the equilibrium position at $x = 0$, the Hooke's law for the spring is

$$T = -kx$$

where the proportionality constant $k > 0$ is called the spring constant. The "$-$" sign means the tension is always in the opposite direction of the deformation (check it!).

Putting the model for the tension into Newton's second law of motion, we obtain the ODE that governs the position of the mass:

$$m\ddot{x} + kx = 0.$$

Because both k and m are positive, we can define a new constant, called the **natural frequency**, $\omega_0 = \sqrt{k/m}$ and rewrite the ODE as

$$\ddot{x} + \omega_0^2 x = 0. \tag{5.1}$$

This is called the equation of motion and it is a homogeneous linear second-order ODE with constant coefficients. We now solve (5.1) to find $x(t)$. The characteristic equation for (5.1) is:

$$\lambda^2 + \omega_0^2 = 0$$

and its roots are

$$\lambda = \pm i\omega_0.$$

Therefore,

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

Differentiating with respect to t, we also obtain

$$\dot{x}(t) = -\omega_0 C_1 \sin \omega_0 t + \omega_0 C_2 \cos \omega_0 t.$$

Next we need two conditions to determine the arbitrary constants C_1 and C_2. For example, if we assume at $t = 0$, the mass is displaced to $x(0) = A$, and then it is *released from rest*, mathematically it means $\dot{x}(0) = 0$. These two conditions lead to

$$C_1 = A \quad \text{and} \quad C_2 = 0.$$

So we have

$$x(t) = A \cos \omega_0 t.$$

We see that the mass oscillates sinusoidally about the equilibrium position. This is known as simple harmonic oscillation. A is the **amplitude** and ω_0 is the **angular frequency** of the oscillation.

5.1.2 Damped harmonic oscillation

We now consider the situation when the surface is not smooth. We need a model for the frictional force f between the mass and the surface. We note that friction always opposes the instantaneous motion. And if the velocity is not too large, the frictional force can be assumed to be proportional to the velocity. Therefore,

$$f = -bv = -b\dot{x}$$

where $b > 0$ is the proportionality constant. The "$-$" sign means that if the mass is moving to the right ($\dot{x} > 0$), the frictional force is pointing to the left ($f < 0$) and vice versa. From Newton's second law of motion,

$$T + f = ma,$$

we obtain the equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0.$$

We define the damping coefficient $\mu = \frac{b}{2m}$ and rewrite the equation of motion as:

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = 0. \tag{5.2}$$

Remember that ω_0 is the angular frequency of the undamped oscillation in (5.1). The characteristic equation for (5.2) is:

$$\lambda^2 + 2\mu\lambda + \omega_0^2 = 0,$$

which has roots:

$$\lambda = -\mu \pm \sqrt{\mu^2 - \omega_0^2}.$$

We know from chapter 4 that the form of the solution $x(t)$ depends on the sign of the discriminant $\Delta = \mu^2 - \omega_0^2$. In other words, $x(t)$ depends on the strength of the damping μ compared to ω_0. We consider three possible cases: (i) $\mu > \omega_0$ (real distinct roots), (ii) $\mu = \omega_0$ (real repeated root) and (iii) $\mu < \omega_0$ (complex roots).

(i) Real distinct roots: If the damping is so large that $\mu > \omega_0$, then we have two distinct real roots,

$$\lambda_1 = -\mu + \sqrt{\mu^2 - \omega_0^2} \quad \text{and} \quad \lambda_2 = -\mu - \sqrt{\mu^2 - \omega_0^2}$$

with $\lambda_2 < \lambda_1 < 0$. Therefore, the general solution is:

$$x(t) = e^{-\mu t}\left(C_1 e^{\sqrt{\mu^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\mu^2-\omega_0^2}\,t}\right).$$

This is called *overdamped*. The details of the motion depend on the initial conditions, hence the values of C_1 and C_2. Generally, since both roots are negative, the solution decays to zero with no oscillations. Although for some initial conditions, it is possible for $x(t)$ to cross the equilibrium position $x = 0$ once before approaching $x = 0$ at long time.

(ii) Real repeated root: If $\mu = \omega_0$, there is one repeated real root:

$$\lambda = -\mu,$$

and the general solution is

$$x(t) = (C_1 + C_2 t)e^{-\mu t}.$$

Similar to the overdamped case, $x(t)$ decays to $x = 0$ (with the possibility of crossing $x = 0$ once). For the same initial conditions, it does so in a more rapid manner than the overdamped case. This is called *critically damped*. The damping is just large enough to prevent oscillation without slowing down the decay excessively.

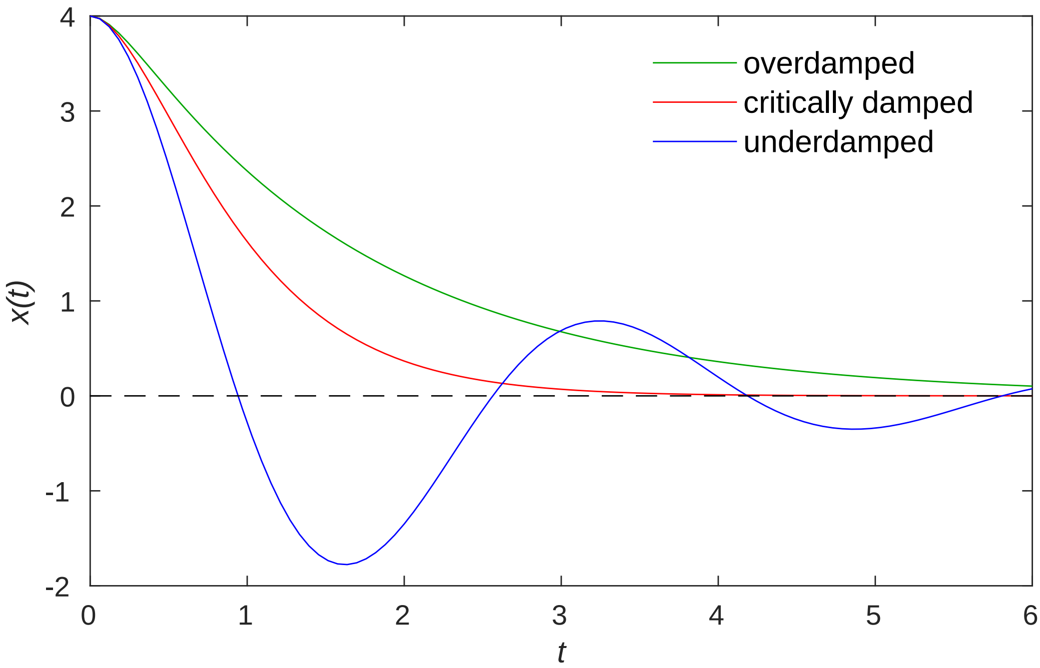


Figure 5.1: Particular solutions of damped harmonic motion. The initial conditions are $x(0) = 4$ and $\dot{x}(0) = 0$. We set $\omega_0 = 2$. The three different cases are (i) $\mu = 3.5$ (overdamped), (ii) $\mu = \omega_0$ (critically damped) and (iii) $\mu = 0.5$ (underdamped).

(iii) Complex roots: If the damping is so small that $0 < \mu < \omega_0$, we have two complex roots,

$$\lambda_\pm = -\mu \pm i\sqrt{\omega_0^2 - \mu^2}$$

and the general solution can be expressed in the following form:

$$x(t) = e^{-\mu t}\left(C_1 \cos \omega_* t + C_2 \sin \omega_* t\right),$$

where $\omega_* = \sqrt{\omega_0^2 - \mu^2} < \omega_0$. This is called *underdamped*. The solution oscillates with an angular frequency less than that of the undamped simple harmonic motion while its amplitude decreases exponentially.

Figure 5.1 illustrates the three different cases by plotting some particular solutions of $x(t)$. In all cases, the mass comes to a stop at long enough time.

5.1.3 Forced oscillation

Let us look at what happens if the mass is being driven by a time-varying external force F. We assume F varies sinusoidally with time t and has amplitude F_0 and angular frequency ω:

$$F(t) = F_0 \cos \omega t.$$

There are three forces acting on the mass. From Newton's second law of motion,

$$F + T + f = ma,$$

we obtain the equation of motion for the mass,

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t. \tag{5.3}$$

Defining $F_* = F_0/m$ and dividing by m gives,

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = F_* \cos \omega t. \tag{5.4}$$

This is an *inhomogeneous* linear second-order ODE with constant coefficients. We have learned that its solution consists of two parts:

$$x = x_c + x_p.$$

The first part x_c is a general solution of (5.2) which we have already obtained in section 5.1.2. Here we focus on how to find a particular solution x_p of (5.4). As $\cos \omega t$ is not a solution of (5.2), according to Table 4.1, we choose

$$x_p = B \cos \omega t + C \sin \omega t. \tag{5.5}$$

Differentiate with respect to t:

$$\dot{x}_p = -\omega B \sin \omega t + \omega C \cos \omega t \quad \text{and} \quad \ddot{x}_p = -\omega^2 B \cos \omega t - \omega^2 C \sin \omega t.$$

Substitute $x_p, \dot{x}_p, \ddot{x}_p$ into (5.4) and simplify:

$$(-\omega^2 B + 2\mu\omega C + \omega_0^2 B) \cos \omega t + (-\omega^2 C - 2\mu\omega B + \omega_0^2 C) \sin \omega t = F_* \cos \omega t.$$

Equate coefficients on both sides:

$$\begin{aligned} (\omega_0^2 - \omega^2)B + 2\mu\omega C &= F_*, \\ -2\mu\omega B + (\omega_0^2 - \omega^2)C &= 0. \end{aligned}$$

Solving this set of simultaneous equations then gives,

$$\begin{aligned} B &= \frac{(\omega_0^2 - \omega^2)F_*}{(\omega_0^2 - \omega^2)^2 + 4\mu^2\omega^2}, \\ C &= \frac{2\mu\omega F_*}{(\omega_0^2 - \omega^2)^2 + 4\mu^2\omega^2}. \end{aligned}$$

Therefore,

$$x_p = \frac{(\omega_0^2 - \omega^2)F_*}{(\omega_0^2 - \omega^2)^2 + 4\mu^2\omega^2} \cos \omega t + \frac{2\mu\omega F_*}{(\omega_0^2 - \omega^2)^2 + 4\mu^2\omega^2} \sin \omega t.$$

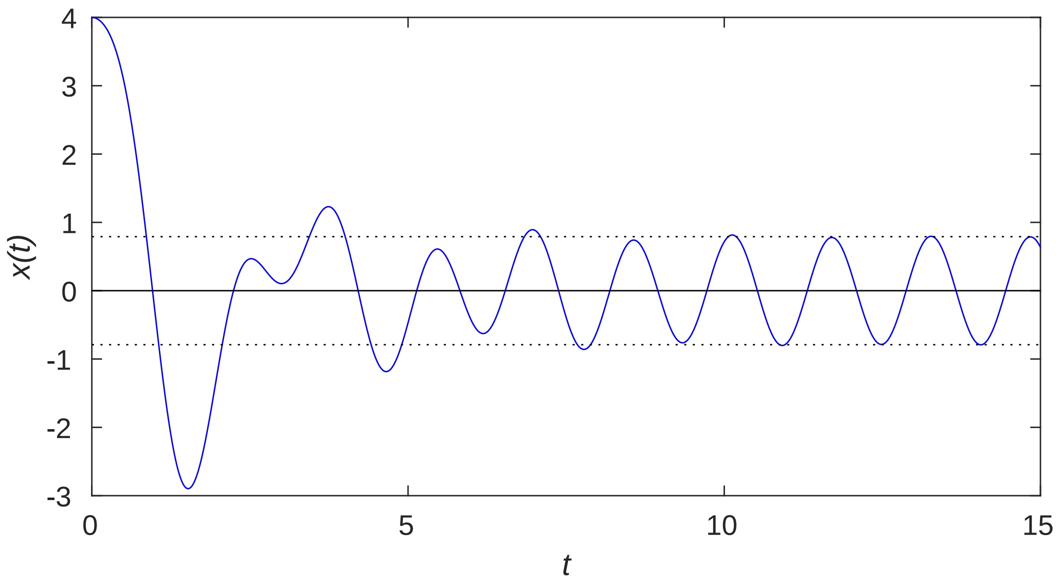


Figure 5.2: An example of forced harmonic oscillation. The initial conditions are $x(0) = 4$ and $\dot{x}(0) = 0$. We set $\omega_0 = 2, \mu = 0.5, F_* = 10, \omega = 2\omega_0$ in (5.4). The dotted lines indicate the amplitude A in (5.7).

What does this solution tell us about the motion of the mass? We know from section 5.1.2 that $x_c \to 0$ as $t \to \infty$. On the other hand, x_p is oscillatory for all t. So after sufficiently long time, the position of the mass $x = x_c + x_p \approx x_p$. Hence x_c is called the **transient solution** and x_p is called the **steady-state solution**.

Next we ask: what are the amplitude and angular frequency of the steady-state oscillation? To answer this question, we rewrite x_p using the subsidiary angle formula as follows. Let

$$B = A\cos\phi,$$
$$C = A\sin\phi,$$

so that x_p in (5.5) can be written as

$$x_p = A\cos(\omega t - \phi). \tag{5.6}$$

Solving for A and ϕ in terms of B and C gives,

$$A = \sqrt{B^2 + C^2} = \frac{F_*}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\mu^2\omega^2}}, \tag{5.7a}$$

$$\phi = \tan^{-1}\frac{C}{B} = \tan^{-1}\frac{2\mu\omega}{\omega_0^2 - \omega^2}. \tag{5.7b}$$

So we see that x_p oscillates at the same frequency ω as the external force $F(t)$ but with phase lag ϕ and amplitude A given by (5.7). Figure 5.2 shows an example of the full solution $x = x_c + x_p$.

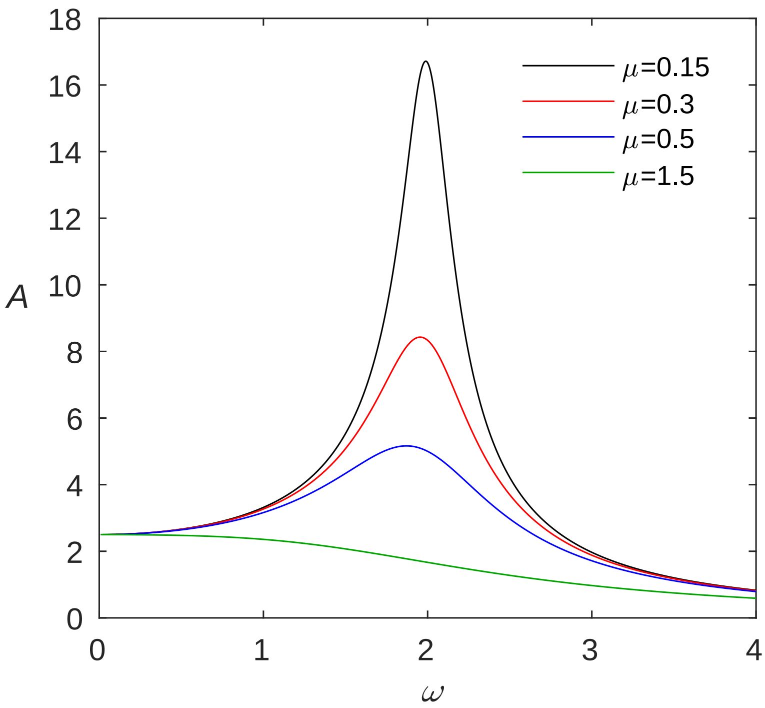


Figure 5.3: Amplitude A of the steady-state solution in forced harmonic oscillation given in (5.7) as a function of the driving frequency ω. In this plot, $\omega_0 = 2$ and $F_* = 10$. The different curves are for different damping coefficient μ.

If we consider $F_*(= F_0/m)$ and $\omega_0(= \sqrt{k/m})$ fixed, then the amplitude A depends on the two parameters ω and μ. If μ is not too large, an interesting phenomenon called **resonance** occurs as we varies ω. Figure 5.3 plots A as a function ω for several values of μ. We see that for small values of μ, if we drive the system at the "right" frequency, the amplitude of the steady-state oscillation becomes very large. And A attains a maximum value at a particular frequency ω_R called the **resonance frequency**. Let us calculate ω_R at a fixed μ by setting

$$\left.\frac{dA}{d\omega}\right|_{\omega=\omega_R} = 0$$

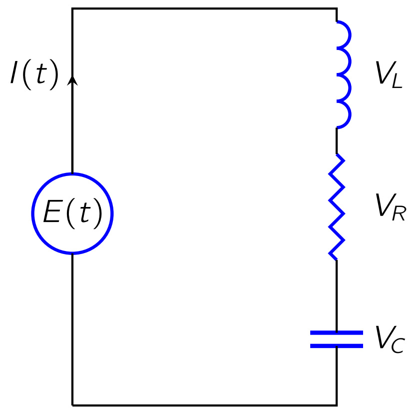
$$\Longrightarrow \quad \frac{2\omega_R(\omega_0^2 - \omega_R^2) - 4\mu^2\omega_R}{[(\omega_0^2 - \omega_R^2)^2 + 4\mu^2\omega_R^2]^{3/2}} = 0.$$

Therefore the resonance frequency is

$$\omega_R = \sqrt{\omega_0^2 - 2\mu^2}.$$

The above relation shows that no resonance occurs if $\mu > \omega_0/\sqrt{2}$ for then ω_R becomes imaginary. In such case, the amplitude A does not have a peak (local maximum) as a function ω. It simply decreases monotonically with ω.

5.2 Resonant electric circuits



A resonant electric circuit (or RLC circuit) consists of three circuit elements—a **resistor**, an **inductor** and a **capacitor**—driven by an imposed voltage $E(t)$. We consider an AC voltage that varies sinusoidally with time t:

$$E(t) = E_0 \sin \omega t.$$

The voltage drop across the individual element is governed by the following laws.

$$\begin{aligned} &\text{Voltage drop across a resistor:} && V_R = IR, \quad \text{(Ohm's law)} \\ &\text{Voltage drop across an inductor:} && V_L = L\frac{dI}{dt}, \\ &\text{Voltage drop across a capacitor:} && V_C = \frac{Q}{C}, \end{aligned}$$

where I is the current passing through each element. In the above relations, the resistance R, the inductance L and the capacitance C are constant. The charge Q in the capacitor is related to the current by

$$I = \frac{dQ}{dt},$$

so

$$\frac{dV_C}{dt} = \frac{1}{C} I.$$

By **Kirchhoff's voltage law**, the sum of the voltages across each element is equal to the imposed voltage $E(t)$. Therefore,

$$V_L + V_R + V_C = E(t).$$

And by **Kirchhoff's current law**, the current $I(t)$ is the same through each element. Hence substituting the three laws on voltage drop into the above equation and differentiating with respect to time t, we obtain:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C} I = \omega E_0 \cos \omega t. \tag{5.8}$$

This is an inhomogeneous linear second-order constant-coefficient ODE for $I(t)$. Notice that it has the exact same form as the equation of motion (5.3) for forced harmonic oscillation. Thus there is an analogy between the two systems: L corresponds to m, R corresponds to b, $1/C$ corresponds to k, ωE_0 corresponds to F_0 and finally the current I corresponds to the position x of the mass. All the results we have developed for the spring–mass system, including the resonance phenomenon, can be adapted to the RLC circuit. It is remarkable that an electrical system and a mechanical system, which are so different in nature, share the same mathematical model.

5.3 Two-dimensional ODEs: phase plane analysis

5.3.1 Systems of first-order ODEs

In this section, we discuss **two-dimensional systems** of first-order ODEs. Such systems involve **two** dependent variables, x and y, both depend on a single independent variable t. It has the following general form:

$$\frac{dx}{dt} = F(x, y, t), \tag{5.9a}$$

$$\frac{dy}{dt} = G(x, y, t). \tag{5.9b}$$

We first restrict ourselves to homogeneous linear systems with constant coefficients:

$$\frac{dx}{dt} = a_{11}x + a_{12}y, \tag{5.10a}$$

$$\frac{dy}{dt} = a_{21}x + a_{22}y, \tag{5.10b}$$

where a_{ij} are constants. The goal is to determine the solutions $x(t)$ and $y(t)$, typically given some initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$. The difficulty with solving these systems is that the equations for x and y are **coupled**, i.e., the solution of the first equation for $x(t)$ relies on knowing $y(t)$, which is determined by the second equation, which in turn relies on knowing $x(t)$!

The system (5.10) can be conveniently written in matrix notation:

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\text{or} \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{5.11}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

We illustrate how to solve this system with the following example.

Example 5.1. *Find the general solution of*

$$\begin{aligned}\dot{x} &= 4x - 2y,\\ \dot{y} &= x + y.\end{aligned}$$

Solution: We write the system using the matrix notation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}.$$

From our experiences with one-dimensional ODEs, we guess the solution is of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t} \quad \text{where} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

for some constants v_1, v_2, λ. We determine $\mathbf{v}$ and λ by substituting the above assumed form into the system,

$$\begin{aligned} \lambda\mathbf{v}e^{\lambda t} &= \mathbf{A}\mathbf{v}e^{\lambda t}\\ \implies \quad \mathbf{A}\mathbf{v} &= \lambda\mathbf{v}.\end{aligned}$$

So we see that $\mathbf{v}$ is the eigenvector and λ is the eigenvalue of $\mathbf{A}$. We find λ by solving

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= 0 \qquad (\mathbf{I} \text{ is the identity matrix})\\ \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} &= 0\\ \lambda^2 - 5\lambda + 6 &= 0\end{aligned}$$

Hence the two eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

Let $\mathbf{v}^{(1)}$ be the eigenvector corresponding to the first eigenvalue $\lambda_1 = 3$, then $\mathbf{A}\mathbf{v}^{(1)} = \lambda_1\mathbf{v}^{(1)}$ implies

$$v_1^{(1)} - 2v_2^{(1)} = 0$$

and a solution is $v_1^{(1)} = 2$, $v_2^{(1)} = 1$. So a eigenvector is $\mathbf{v}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Similarly, for the second eigenvalue $\lambda_2 = 2$, we have

$$v_1^{(2)} - v_2^{(2)} = 0$$

and a solution is $v_1^{(2)} = 1$, $v_2^{(2)} = 1$. So a eigenvector is $\mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Therefore we have found two solutions to the system:

$$\mathbf{v}^{(1)} e^{\lambda_1 t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} \quad \text{and} \quad \mathbf{v}^{(2)} e^{\lambda_2 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The general solution is a linear combination of these two solutions,

$$\begin{aligned} \mathbf{x} &= C_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + C_2 \mathbf{v}^{(2)} e^{\lambda_2 t} \\ &= C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \end{aligned}$$

or

$$\begin{aligned} x(t) &= 2C_1 e^{3t} + C_2 e^{2t}, \\ y(t) &= C_1 e^{3t} + C_2 e^{2t}. \end{aligned}$$

We do not consider the special cases when $\lambda_1 = \lambda_2$ or when one of the eigenvalues is zero in this course.

5.3.2 Relation to second-order ODEs

A single (one-dimensional) second-order linear homogeneous ODE with constant coefficients can be converted into a system of two-dimensional first-order ODEs. Consider the second-order ODE

$$\ddot{x} + f\dot{x} + gx = 0, \tag{5.12}$$

where f and g are constants. Introduce a new *dependent* variable $y(t)$ defined by:

$$y = \dot{x}.$$

Since $\ddot{x} = \dot{y}$, (5.12) can be written as:

$$\dot{y} = -fy - gx.$$

Then we see that (5.12) is equivalent to the following two-dimensional system of first-order ODEs with x and y as the two dependent variables:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -gx - fy. \end{aligned}$$

In matrix notation, we have:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g & -f \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.13}$$

It is clear that this two-dimensional first-order system is a particular case of the general form (5.11) with $a_{11} = 0$, $a_{12} = 1$, $a_{21} = -g$ and $a_{22} = -f$.

The converse also holds in most cases[1]: a two-dimensional system (5.10) can be converted into a single second-order linear ODE. Starting from (5.10), we differentiate the first equation:

$$\ddot{x} = a_{11}\dot{x} + a_{12}\dot{y},$$

substitute $\dot{y}$ from the second equation:

$$\ddot{x} = a_{11}\dot{x} + a_{12}(a_{21}x + a_{22}y),$$

and finally eliminate y by writing it in terms of x and $\dot{x}$ using the first equation:

$$\ddot{x} = a_{11}\dot{x} + a_{12}\left(a_{21}x + a_{22}\frac{\dot{x} - a_{11}x}{a_{12}}\right).$$

Rearranging gives:

$$\ddot{x} + (-a_{11} - a_{22})\,\dot{x} + (a_{11}a_{22} - a_{12}a_{21})\,x = 0,$$

or equivalently:

$$\ddot{x} + f\dot{x} + gx = 0$$

where $f = -a_{11} - a_{22}$ and $g = a_{11}a_{22} - a_{12}a_{21}$. Hence we have converted the first-order system (5.10) into a single second-order ODE.

5.3.3 Phase plane

In Example 5.1, we find the solution $x(t)$ and $y(t)$ of a two-dimensional system. This solution describes how x and y change with t. Here we show that we can also examine the solutions of the *autonomous* two-dimensional system (5.10) on the xy-plane. We call this plane of which the two axes represent the two dependent variables the **phase plane**.

At some particular time t, $(x(t), y(t))$ is a point on the phase plane. As we vary t, $(x(t), y(t))$ traces out a *solution curve*, or a **trajectory**. The sense of increasing t is called the positive sense of the trajectory and is usually indicated by an arrow. This defines an **orientation** for a trajectory. Different particular solutions of (5.10), which correspond to different initial conditions, trace out different trajectories. A diagram showing one or more trajectories is called a **phase portrait** of the system. The phase portrait allows us to study not just one particular solution but the whole family of solutions of (5.10). This is demonstrated in the next example.

Example 5.2. *Sketch and interpret the phase portrait of the simple harmonic oscillation governed by*

$$\ddot{x} + x = 0.$$

[1] In certain cases, such as a 'star' node, a two-dimensional first-order system can not be written as a second-order ODE. This is beyond the scope of this course.

Solution: To write the given second-order ODE as a two-dimensional first-order system, we define $v(t) = \dot{x}(t)$ (the velocity). Hence,

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -x. \end{aligned}$$

Or in matrix notation,

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}.$$

We can find x and v by solving the given second-order ODE directly as in Section 5.1.1. Alternatively, we can solve the above two-dimensional system as in Example 5.1. Either way, for the initial conditions $(x(0), v(0)) = (x_0, v_0)$, the solution is

$$\begin{aligned} x(t) &= x_0 \cos t + v_0 \sin t, \\ v(t) &= -x_0 \sin t + v_0 \cos t. \end{aligned}$$

Following the convention in classical mechanics, we put x on the horizontal axis and v on the vertical axis of our phase plane. To construct trajectories for different initial conditions on this xv-plane, we need to find $v(x)$ as a function of x. One way to do so is to eliminate t from $x(t)$ and $v(t)$,

$$\begin{aligned} x^2 + v^2 &= (x_0 \cos t + v_0 \sin t)^2 + (-x_0 \sin t + v_0 \cos t)^2 \\ &= x_0^2 + v_0^2. \end{aligned}$$

So for a given initial condition (x_0, v_0), the trajectory is a circle with centre at the origin and radius $\sqrt{x_0^2 + v_0^2}$. We can also obtain $v(x)$ directly from the (autonomous) second-order governing ODE by writing $\ddot{x} = v\frac{dv}{dx}$ (see Section 4.1.1). Then,

$$\begin{aligned} & v\frac{dv}{dx} + x = 0 \\ & \int_{v_0}^{v} \tilde{v}\, d\tilde{v} = -\int_{x_0}^{x} \tilde{x}\, d\tilde{x} \qquad \text{(see Example 2.3)} \\ & \left[\frac{\tilde{v}^2}{2}\right]_{v_0}^{v} = -\left[\frac{\tilde{x}^2}{2}\right]_{x_0}^{x} \\ \implies \qquad & x^2 + v^2 = x_0^2 + v_0^2 \end{aligned}$$

and we again see the trajectories are circles. But how about the orientation?

The right panel of Fig. 5.4 plots several trajectories on the phase plane. Let us take a closer look at the red one which has the initial condition $(x_0, v_0) = (0, 1)$. The left panel of Fig. 5.4 shows $x(t)$ and $v(t)$ for this trajectory. For concreteness, we discuss this solution in terms of the spring–mass system.

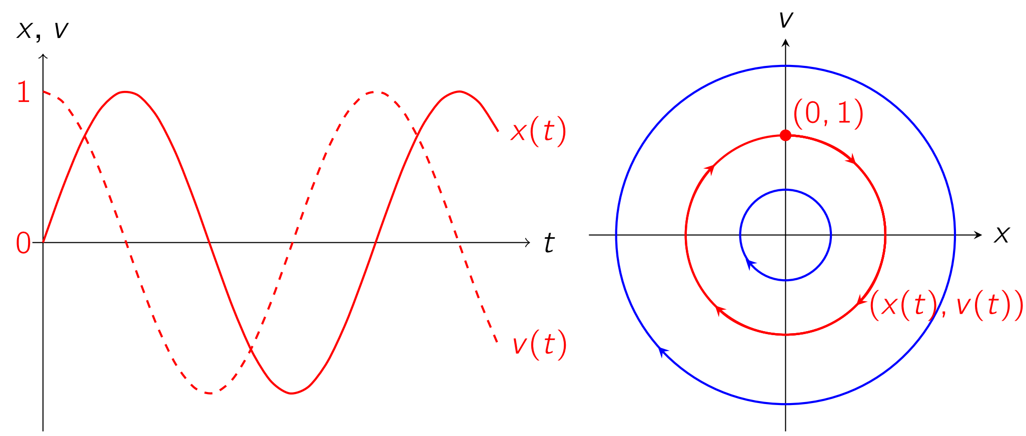


Figure 5.4: Phase portrait of the simple harmonic oscillator in Example 5.2.

1. In the upper half-plane, the velocity is positive $v > 0$. So x is increasing with t as indicated by the arrow on the trajectory. This corresponds to the portion of the oscillation when the mass is moving from the leftmost position $x = -1$ to the rightmost position $x = 1$.

2. In the lower half-plane, the velocity is negative $v < 0$. So x is decreasing with t and the mass is moving from $x = 1$ to $x = -1$.

3. The points $(-1, 0)$ and $(1, 0)$ are the turning points at which $v = 0$. At these points the mass is momentarily at rest as the motion switches direction. Between the two turning points, the *magnitude* of the velocity $|v|$ increases from $|v| = 0$ to the maximum $|v| = 1$ when the mass is at the equilibrium position $x = 0$ and then decreases to $|v| = 0$ again.

So we see that the trajectories of the simple harmonic oscillation in this example are circles on the xv-plane going in the clockwise direction. Generally, the trajectories of any periodic motion are closed curves on the phase plane.

5.3.4 Fixed points

The two-dimensional system (5.9) is *autonomous* if F and G do not depend on t explicitly. For example, the homogeneous linear system (5.10) is autonomous. For an autonomous system, a point at which both $\dot{x} = 0$ and $\dot{y} = 0$ is called a **fixed point**[2]. A fixed point is therefore a pair of values (x_*, y_*) that solves $F(x, y) = 0$ and $G(x, y) = 0$ simultaneously. By definition, fixed points are **constant solutions**. If $(x(0), y(0))$ is a fixed point, $(x(t), y(t))$ stays at the fixed point for all $t > 0$. So a fixed point is represented by a **single point** on the phase-plane.

[2]Fixed points are also sometimes known as critical points or equilibrium points.

For the homogeneous system (5.10), there is only one fixed point at the origin $(0,0)$. The following example considers an inhomogeneous system where the inhomogeneous term is constant.

Example 5.3. *Determine the fixed points of the following inhomogeneous system:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

Solution: To find the fixed point, we set $\dot{x} = 0$ and $\dot{y} = 0$ which leads to the following pair of equations:

$$\begin{aligned} 4x - 2y &= 0, \\ x + y - 3 &= 0. \end{aligned}$$

So there is a single fixed point at $(1,2)$.

Let us consider the following inhomogeneous linear system with a *constant* inhomogeneous term:

$$\dot{x} = a_{11}x + a_{12}y + h_1, \tag{5.14a}$$

$$\dot{y} = a_{21}x + a_{22}y + h_2, \tag{5.14b}$$

where a_{ij} and h_i are constants. Apart from some exceptional cases, this system has one fixed point. Once this fixed point is found, the system can be solved as follows. Let (x_*, y_*) be the fixed point. Make the change of variables: $\tilde{x} = x - x_*$ and $\tilde{y} = y - y_*$ and substitute into the system:

$$\begin{aligned} \dot{\tilde{x}} &= a_{11}\tilde{x} + a_{12}\tilde{y} + (a_{11}x_* + a_{12}y_* + h_1), \\ \dot{\tilde{y}} &= a_{21}\tilde{x} + a_{22}\tilde{y} + (a_{21}x_* + a_{22}y_* + h_2). \end{aligned}$$

By the definition of fixed points, the expressions inside the brackets vanish and so $(\tilde{x}, \tilde{y})$ satisfies $\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}}$ (in our standard matrix notation). This is the homogeneous version of the original system and its general solution is

$$\tilde{\mathbf{x}} = C_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + C_2 \mathbf{v}^{(2)} e^{\lambda_2 t}$$

where λ_i and $\mathbf{v}^{(i)}$ are respectively the eigenvalues and eigenvectors of $\mathbf{A}$. Reverting back to the original variables, we thus have

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} e^{\lambda_2 t} + \begin{pmatrix} x_* \\ y_* \end{pmatrix}$$

Example 5.4. *Solve the inhomogeneous system in Example 5.3.*

Solution: The corresponding homogeneous system has already been solved in Example 5.1. And in Example 5.3, we see that the fixed point is at $(1,2)$. Therefore, the general solution of the inhomogeneous system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

5.3.5 Classification of fixed points

We may classify fixed points from two different aspects, one based on their **stability** and another based on the **geometric shape of the trajectories** in their neighbourhood.

In the previous section, we learnt that if the system is initially at a fixed point, it stays at the fixed point for all future times. But what happens if we start at a point near the fixed point, does the trajectory move towards or away from it? This question concerns the stability of a fixed point. We first introduce the following two concepts:

- A fixed point $\mathbf{x}_*$ is **Lyapunov stable** if all trajectories that at some instant are sufficiently close to $\mathbf{x}_*$ remain close to it at all future times.
- A fixed point $\mathbf{x}_*$ is **attracting** if all trajectories that at some instant are sufficiently close to $\mathbf{x}_*$ approach it as $t \to \infty$.

Note that Lyapunov stable and attracting are two separate notions and neither one implies the other. We now classify the stability of a fixed point as follows. A fixed point is:

- **stable** (or asymptotically stable) if it is both Lyapunov stable and attracting;
- **neutrally stable** if it is Lyapunov stable but not attracting;
- **unstable** if it is not Lyapunov stable.

We can also classify fixed points based on the pattern of the trajectories near it. A fixed point is a:

- **node** if *every* trajectory approaches it in a *definite direction* as $t \to \infty$ or $t \to -\infty$;
- **saddle** if some trajectories approach it as $t \to \infty$ and some trajectories approach it as $t \to -\infty$;
- **centre** if the trajectories form closed curves that enclose it;
- **spiral** if the trajectories spiral about it (i.e. individual trajectory does not approach the fixed point in a definite direction).

Notice that "as $t \to -\infty$" means we are moving backwards in t. So approaching a fixed point as $t \to -\infty$ is equivalent to moving away from the fixed point as $t \to \infty$.

We now focus on the linear system (5.14). We have seen that its general solution depends on the eigenvalues and eigenvectors of the coefficient matrix **A**. So it is not surprising that the type of its fixed point can be deduced from the eigenvalues of **A**. There are six different cases as illustrated by the following examples. We do not consider the special cases when the eigenvalues are equal (star nodes or improper nodes) or when one of the eigenvalue is zero (non-isolated nodes) in this course. In all of the examples below, without loss of generality, we consider homogeneous systems, so the fixed point is at $(0, 0)$.

Example 5.5. Stable Node (distinct, real and negative eigenvalues)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \textit{with} \qquad \mathbf{A} = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix}.$$

We first find the general solution. The eigenvalues of **A** are given by:

$$\mathrm{Det}(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -3-\lambda & -1 \\ -1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = (\lambda+2)(\lambda+4) = 0.$$

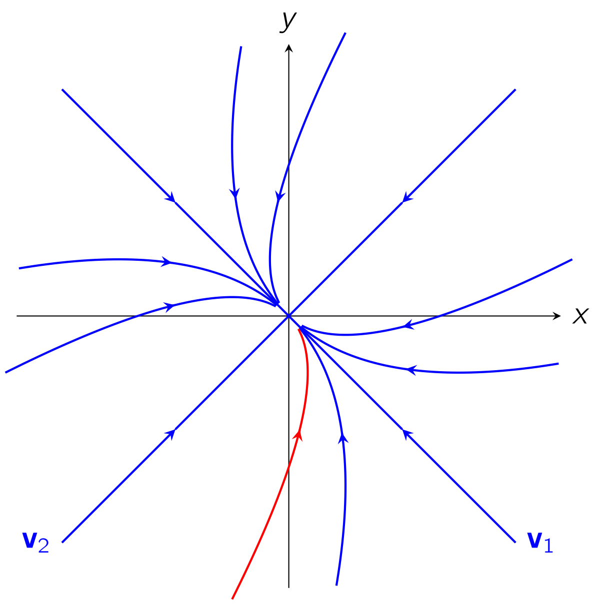
Therefore $\lambda_1 = -2$ and $\lambda_2 = -4$. The eigenvectors corresponding to these eigenvalues are:

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So the general solution is:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}.$$

Since both eigenvalues are real and negative, all solutions approach the fixed point at $(0,0)$ as $t \to +\infty$. So the fixed point is stable. The phase portrait below shows that the origin is a stable node.



Equipped with the eigenvalues and eigenvectors (and hence the general solution), we can make a qualitative sketch of the phase portrait as follows.

1. The two eigenvectors define two special directions. When one of the arbitrary constant is zero, the particular solution becomes $\mathbf{x} = C_i \mathbf{v}^{(i)} e^{\lambda_i t}$ $(i = 1, 2)$. This

solution is parallel to $\mathbf{v}^{(i)}$ and its trajectory is a *straight line* through the origin with slope $v_2^{(i)}/v_1^{(i)}$. In particular, when $C_2 = 0$:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}.$$

As t increases, this solution moves towards $(0, 0)$ along the straight line $y = -x$. Similarly, when $C_1 = 0$, the solution approaches $(0, 0)$ along the straight line $y = x$.

2. The two straight lines through the fixed point and parallel to $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ divide the phase plane into four regions. Within each region, a trajectory represents a particular solution with non-zero C_1 and C_2. We can deduce the shape of these trajectories by considering their behaviour as $t \to \pm\infty$. As $t \to +\infty$, e^{-4t} tends to 0 more rapidly than e^{-2t} (since $-4 < -2 < 0$), so

$$\mathbf{x}(t) \sim C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} \text{ for large positive } t.$$

This means that as $\mathbf{x}(t)$ get close to $(0, 0)$ at large positive t, the trajectories become nearly parallel to $\mathbf{v}_1$. Conversely, as $t \to -\infty$, i.e. as we go backwards in t and $\mathbf{x}(t)$ is moving away from $(0, 0)$, e^{-4t} *increases* more rapidly than e^{-2t}, so

$$\mathbf{x}(t) \sim C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t} \text{ for large negative } t.$$

Therefore the trajectories are nearly parallel to $\mathbf{v}_2$ when $\mathbf{x}$ is far away from the origin. For example, consider the single trajectory in red: far away from the origin, the trajectory is nearly parallel to $\mathbf{v}_2$. As t increases, the trajectory bends towards the origin and as it does so, it becomes nearly parallel to $\mathbf{v}_1$.

Example 5.6. Unstable Node (distinct, real and positive eigenvalues)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \textit{with} \qquad \mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}$ are given by

$$\text{Det}(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0.$$

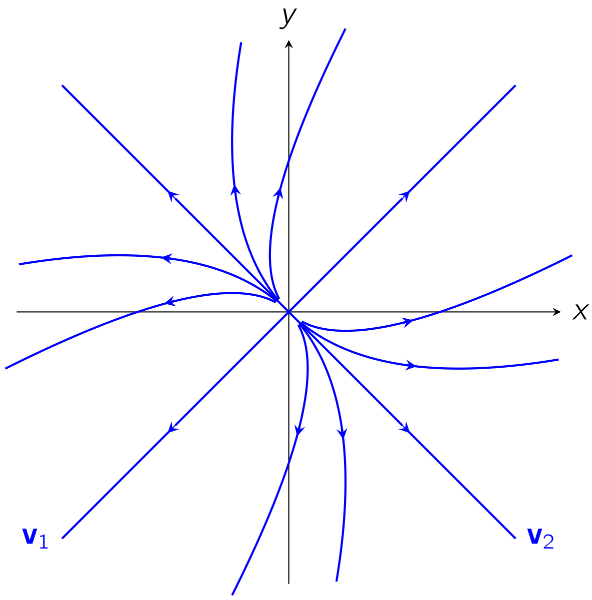
The roots are $\lambda_1 = 4$ and $\lambda_2 = 2$. The eigenvectors are:

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So the general solution is:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

Since both eigenvalues are real and positive, all solutions go away from the fixed point at $(0,0)$ as $t \to +\infty$. So the fixed point is unstable. The phase portrait below shows that the origin is an unstable node.



Except for the straight-line trajectories (which are parallel to one of the eigenvectors at all t), all other trajectories are nearly parallel to $\mathbf{v}_2$ when they are close to the origin and then bend to become nearly parallel to $\mathbf{v}_1$ when $\mathbf{x}(t)$ moves away from the origin as t increases.

Example 5.7. Saddle (real eigenvalues with opposite sign)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \textit{with} \qquad \mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

In this case, the characteristic equation for the eigenvalues is:

$$\mathrm{Det}(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8 = (\lambda+2)(\lambda-4) = 0,$$

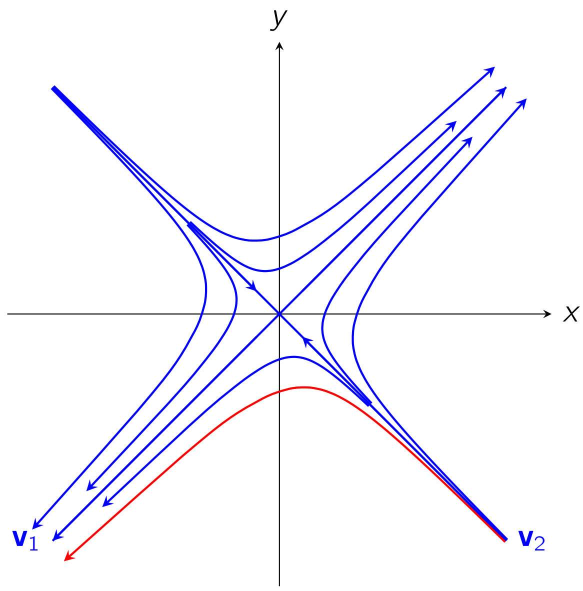
so the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -2$. The eigenvectors are:

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So the general solution is:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}.$$

Since $\lambda_2 = -2 < 0$, we see that those solutions with $C_1 = 0$ approach the fixed point at $(0,0)$ as $t \to \infty$. However, because $\lambda_1 = 4 > 0$, all other solutions move away from the origin at large positive t. So the origin is a saddle. By definition, saddles are always unstable. The phase portrait is shown below.



As before, we can sketch this phase portrait using knowledge of the eigenvalues and eigenvectors.

1. We again start with those solutions where one of the arbitrary constants is zero. When $C_2 = 0$, $\mathbf{x}$ is parallel to $\mathbf{v}^{(1)}$ for all t. Furthermore, $\lambda_1 = 4 > 0$, so the solutions move along the straight line $y = x$ away from $(0,0)$ as t increases. By similar argument, when $C_1 = 0$, the trajectory is the straight line $y = -x$, which is parallel to $\mathbf{v}^{(2)}$. However because $\lambda_2 = -2 < 0$, these solutions move towards $(0,0)$ as t increases.

2. For solutions with non-zero C_1 and C_2, we analyse the behaviour of $\mathbf{x}(t)$ as $t \to \pm\infty$. For large *negative* t, $e^{4t} \to 0$ and the solution is dominated by the second term:

$$\mathbf{x}(t) \sim C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}.$$

   So $\mathbf{x}$ is nearly parallel to $\mathbf{v}^{(2)}$, far away from the origin ($e^{-2t} \gg 1$) but moving towards it ($\lambda_2 < 0$). On the other hand, for large *positive* t, $e^{-2t} \to 0$ and the first term dominates:

$$\mathbf{x}(t) \sim C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

So $\mathbf{x}$ is nearly parallel to $\mathbf{v}^{(1)}$, far away from the origin ($e^{4t} \gg 1$) and moving further away from it ($\lambda_4 > 0$). The trajectory in red shows how the transition between these two behaviour happens: as t increases, $\mathbf{x}$ initially moves towards the origin along a direction nearly parallel to $\mathbf{v}^{(2)}$; as $\mathbf{x}$ approaches the origin, its trajectory bends away from $\mathbf{v}^{(2)}$ towards $\mathbf{v}^{(1)}$; as t increases further, $\mathbf{x}$ moves away from the origin along a direction nearly parallel to $\mathbf{v}^{(1)}$.

Example 5.8. Centre (complex eigenvalues with zero real part)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \textit{with} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To find the eigenvalues, the characteristic equation is:

$$\mathrm{Det}(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0,$$

The roots are $\lambda_{\pm} = \pm i$. The eigenvectors are also complex:

$$\mathbf{v}^{(+)} = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(-)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

One way to write the general solution is:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + B \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}.$$

We can also express the general solution in terms of real functions:

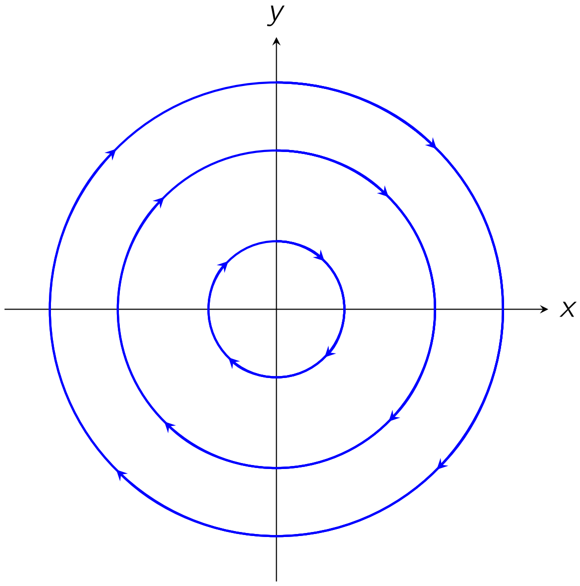
$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} A + B \\ i(A - B) \end{pmatrix} \cos t + \begin{pmatrix} i(A - B) \\ -(A + B) \end{pmatrix} \sin t \\ &= \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cos t + \begin{pmatrix} C_2 \\ -C_1 \end{pmatrix} \sin t \end{aligned}$$

where we have defined the new arbitrary constants $C_1 = A + B$ and $C_2 = i(A - B)$. Eliminatng t from $x(t)$ and $y(t)$ (see Example 5.2), we obtain the equation for the trajectories on the phase plane:

$$x^2 + y^2 = C_1^2 + C_2^2$$

which represents a family of circles centred at $(0,0)$ with radius $\sqrt{C_1^2 + C_2^2}$. To determine the orientation of the trajectories, substitute a point, say $(x, y) = (1, 1)$, into the system of equations. This gives $(\dot{x}, \dot{y}) = (1, -1)$. So at $(1, 1)$, x is increasing while y is decreasing. Therefore the trajectories are clockwise.

The phase portrait is shown below and we see that the fixed point at the origin is a centre. By definition, centres are neutrally stable.



Note that if the eigenvalues are $\pm ai$ for some real constant a, the trajectories are concentric ellipses.

Example 5.9. Stable Spiral (complex eigenvalues with negative real part)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \textit{with} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$$

To find the eigenvalues, the characteristic equation is:

$$\mathrm{Det}(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0.$$

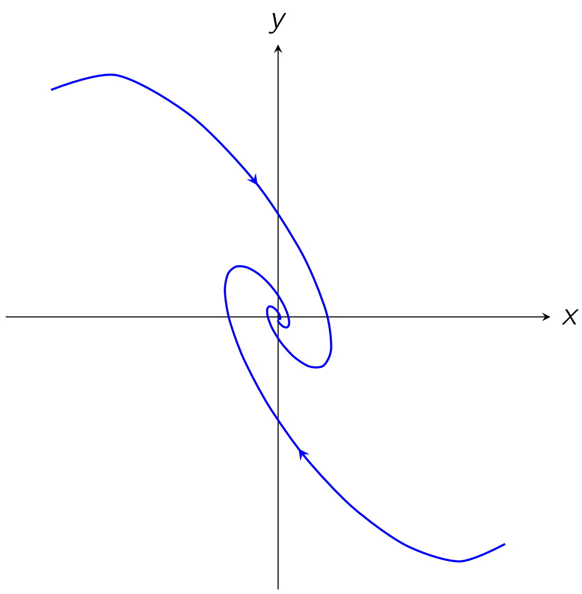
The roots are $\lambda_\pm = -1 \pm i$. The eigenvectors are also complex:

$$\mathbf{v}^{(+)} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(-)} = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}.$$

In terms of real functions, the general solution is

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 - C_1 \end{pmatrix} e^{-t} \cos t + \begin{pmatrix} C_2 \\ -C_1 - C_2 \end{pmatrix} e^{-t} \sin t.$$

The phase portrait is shown below and we see that the fixed point at $(0, 0)$ is a stable spiral. The spiralling behaviour stems from the periodic functions $\cos t$ and $\sin t$ in the solutions. The negative real parts of the eigenvalues leads to the factor e^{-t}. So $x \to 0$ and $y \to 0$ and the solutions spiral towards the fixed point as $t \to +\infty$.



Example 5.10. Unstable Spiral (complex eigenvalues with positive real part)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \textit{with} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$$

To find the eigenvalues, the characteristic equation is:

$$\begin{vmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = 0.$$

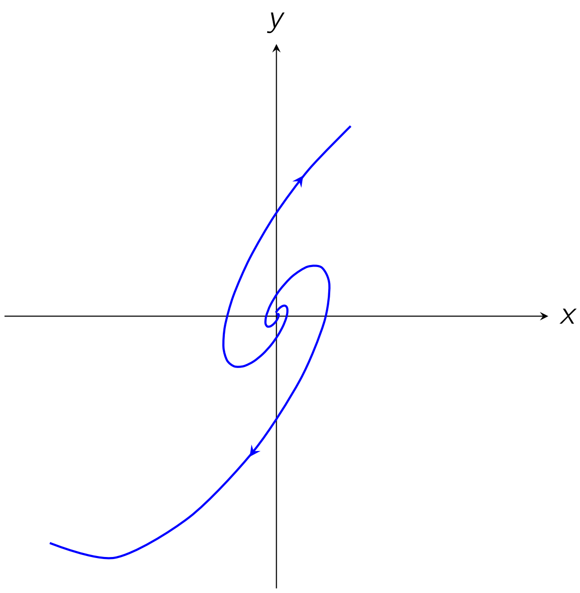
The roots are $\lambda_\pm = 1 \pm i$, The eigenvectors are also complex:

$$\mathbf{v}^{(+)} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(-)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}.$$

In terms of real functions, the general solution is

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_1 + C_2 \end{pmatrix} e^t \cos t + \begin{pmatrix} C_2 \\ C_2 - C_1 \end{pmatrix} e^t \sin t.$$

The phase portrait is shown below and we see that the fixed point at $(0, 0)$ is an unstable spiral. The positive real parts of the eigenvalues give rise to the factor e^t. So the solutions spiral away from the fixed point at the origin as $t \to +\infty$.



Summary. For an autonomous linear two-dimensional system of first-order ODEs:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \qquad \text{with} \qquad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

The fixed point can be classified by the eigenvalues of **A** as follows.

Real eigenvalues λ_1 and λ_2:

$\lambda_2 < \lambda_1 < 0$	distinct real negative eigenvalues: stable node
$0 < \lambda_2 < \lambda_1$	distinct real positive eigenvalues: unstable node
$\lambda_2 < 0 < \lambda_1$	real eigenvalues of opposite sign: saddle

Complex eigenvalues $\lambda_\pm = p \pm iq$:

$p < 0$	complex eigenvalues with negative real part: stable spiral
$p = 0$	complex eigenvalues with zero real part: centre
$p > 0$	complex eigenvalues with positive real part: unstable spiral

(excluding the special cases when $\lambda_1 = \lambda_2$ or when one of the eigenvalues is zero)