Groups of automorphisms of rooted trees I

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Table of contents

Introduction

Motivation: famous problems in group theory

- Milnor's Problem $\Longrightarrow$ growth of a group.
- General Burnside Problem $\Longrightarrow$ finiteness properties of a group.

Some facts (no spoiler: see Alex's talk) about growth of groups

- Free groups of finite rank $k > 1$: exponential growth.
- Fundamental group $\pi_1(M)$ of a closed negatively curved Riemannian manifold: exponential growth.
- Finite groups: polynomial growth (of degree 0).
- (Gromov, 1981) A group is virtually nilpotent if and only if it has polynomial growth.

Milnor's question (1960):

Are there groups of intermediate growth between polynomial and exponential?

Grigorchuk's answer (1980):

Yes, the first ... Grigorchuk group.

About the General Burnside Problem

A still undecided point in the theory of discontinuous groups is whether the order of a group may be not finite, while the order of every operation it contains is finite.

W. Burnside (1902)

In modern terminology the general Burnside problem asks:

can a finitely generated periodic group be finite?

Are finitely generated periodic groups finite?

- Yes, for nilpotent groups.
- Yes, finitely generated periodic subgroups of the general linear group of degree $n > 1$ over the complex field.
- Yes, ... for many other classes of groups.
- Counterexample: the first Grigorchuk group.

To summarize

It seems that:

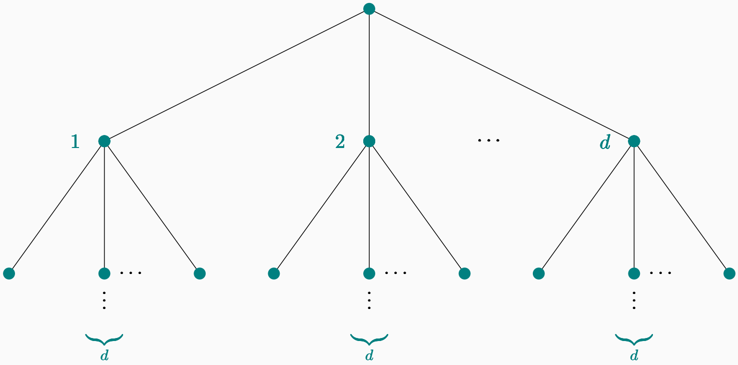
$$\text{Milnor's question} \bigcap \text{General Burnside Problem}$$

$$= \text{the first Grigorchuk group}, \ldots$$

Automorphisms of regular rooted trees

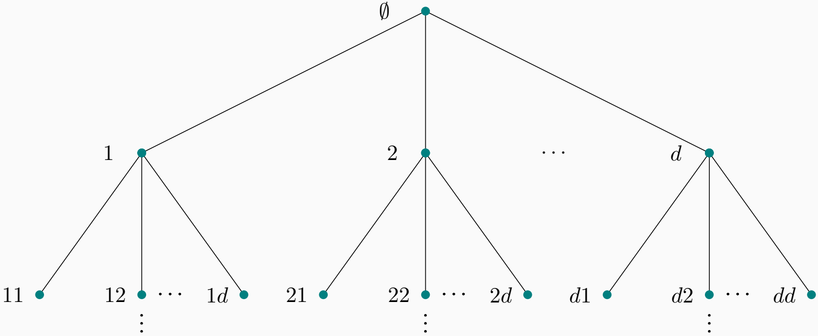
Regular rooted trees

Seriously: the regular rooted tree $\mathcal{T}_d$



Regular rooted trees

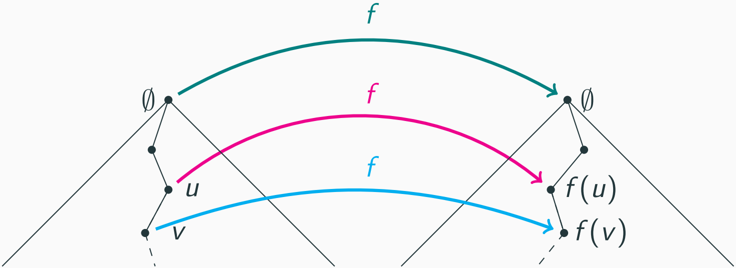
- The tree is infinite.
- The root is a distinguished (fixed) vertex.
- Regular: the number of descendants is the same at every level.
- A vertex is a word in the alphabet $X = \{1, \ldots, d\}$.



- X^n denotes the nth level of the tree.

Automorphisms of $\mathcal{T}_d$

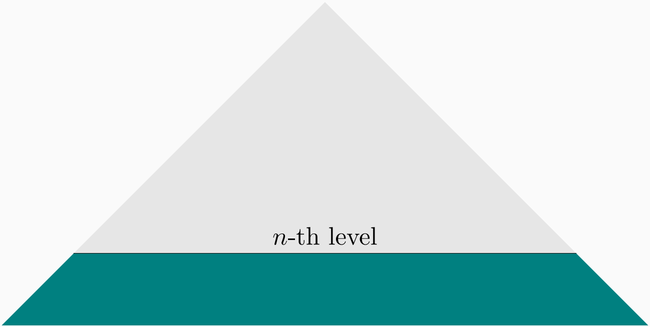
Bijections of the vertices that preserve incidence.



The set Aut $\mathcal{T}_d$ of all automorphisms of $\mathcal{T}_d$ is a group with respect to composition between functions.

Sometimes we write $\mathcal{T}$ for $\mathcal{T}_d$, and, consequently, Aut $\mathcal{T}$ for Aut $\mathcal{T}_d$.

A subgroup of $\operatorname{Aut}\mathcal{T}$: the stabilizer



- The nth level stabilizer $\mathrm{st}(n)$ fixes all vertices up to level n.
- If $H \leq \operatorname{Aut}\mathcal{T}$, we define $\mathrm{st}_H(n) = H \cap \mathrm{st}(n)$.

The stabilizer

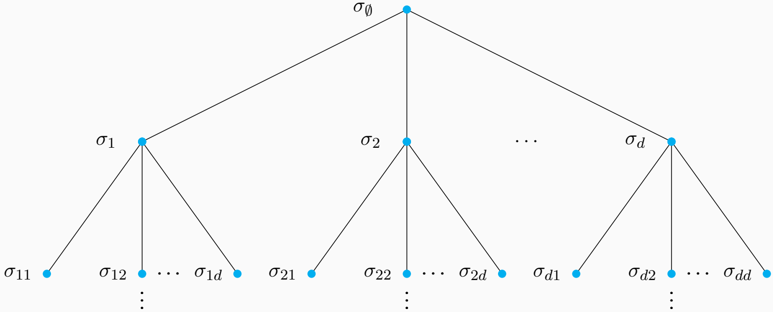
- Stabilizers are normal subgroups of the given group.
- There is a chain of subgroups of $\operatorname{Aut}\mathcal{T}$

$$\operatorname{Aut}\mathcal{T} \supseteq \mathrm{st}(1) \supseteq \mathrm{st}(2) \supseteq \cdots \supseteq \mathrm{st}(n) \supseteq \ldots$$

  where $\bigcap_{n\in\mathbb{N}} \mathrm{st}(n) = 1$.
- Hence $\operatorname{Aut}\mathcal{T}$ is a residually finite group (i.e. a group in which the intersection of all its normal subgroups of finite index is trivial).

Describing elements of $\mathrm{Aut}\,\mathcal{T}$

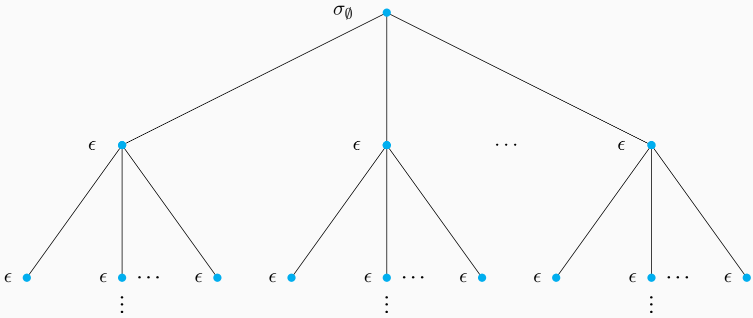
An automorphism $f \in \mathrm{Aut}\,\mathcal{T}_d$ can be represented by writing in each vertex v a permutation $\sigma_v \in \mathrm{Sym}(d)$ which represents the action of f on the descendants of v.



We say that $\sigma_v \in \mathrm{Sym}(d)$ is the *label* of f at the vertex v. The set of all labels is the *portrait* of f.

Describing elements of $\mathrm{Aut}\,\mathcal{T}$

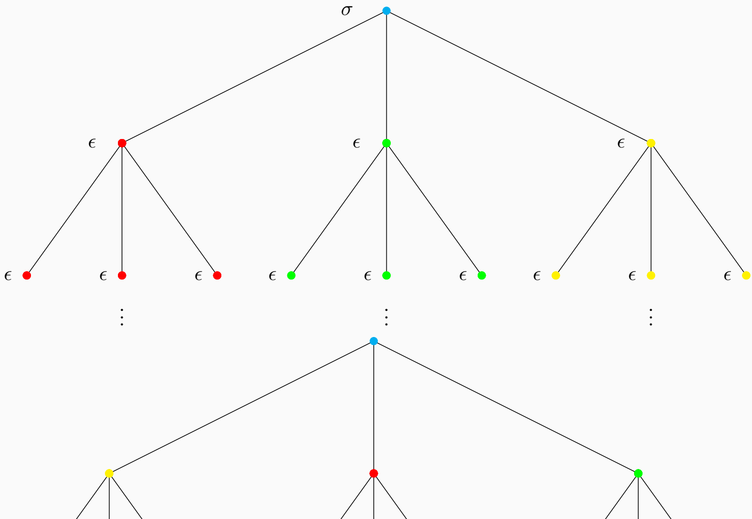
The simplest type are rooted automorphisms: given $\sigma \in \mathrm{Sym}(d)$, they simply permute the d subtrees hanging from the root according to σ.



We denote with ϵ the identity element of $\mathrm{Sym}(d)$.

Example of a rooted automorphism

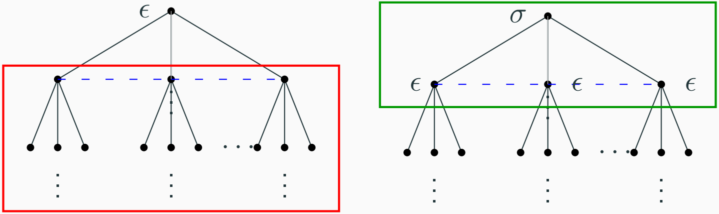
Let $\mathcal{T}_3$ be the ternary tree, and a the rooted automorphism corresponding to the cycle $\sigma = (1\ 2\ 3)$.



Some facts about $\mathrm{Aut}\,\mathcal{T}$: I

We have $\mathrm{Aut}\,\mathcal{T} \cong \mathrm{st}(1) \rtimes \mathrm{Sym}(d)$. Why?

Intuitively: take $f \in \mathrm{st}(1)$, and $\sigma \in \mathrm{Sym}(d)$.



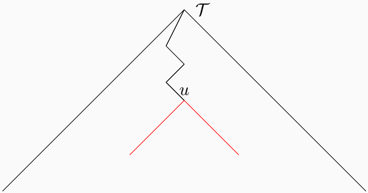
Some facts about Aut $\mathcal{T}$: II

We define the isomorphism

$$\psi : \mathrm{st}(1) \longrightarrow \mathrm{Aut}\,\mathcal{T} \times \overset{d}{\cdots} \times \mathrm{Aut}\,\mathcal{T}$$
$$g \longmapsto (g_1, \ldots, g_d)$$

for every $g \in \mathrm{st}(1)$.

Above, we denoted with g_i the *section* of g at the vertex i, that is the action of g on the subtree $\mathcal{T}_i$ (which is identified with $\mathcal{T}$) that hangs from the vertex i.



Digression: this implies that Aut $\mathcal{T}$ contains products

I + II = Describing elements of $\operatorname{Aut}\mathcal{T}$

- Any $g \in \operatorname{Aut}\mathcal{T}_d$ can be seen as

$$g = h\sigma, \quad \sigma \in \operatorname{Sym}(d), \quad h \in \operatorname{st}(1) \cong \operatorname{Aut}\mathcal{T}_d \times \overset{d}{\ldots} \times \operatorname{Aut}\mathcal{T}_d$$

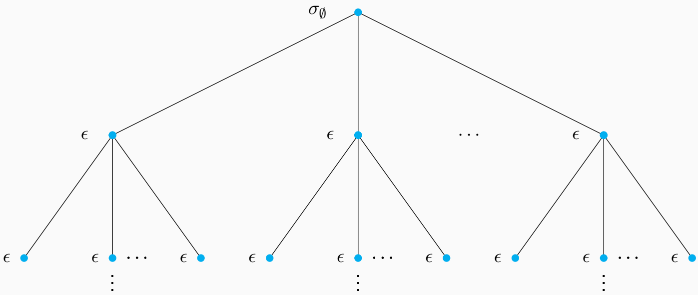
In other words, every $f \in \operatorname{Aut}\mathcal{T}_d$ can be written as

$$f = (f_1, \ldots, f_d)a,$$

where $f_i \in \operatorname{Aut}\mathcal{T}_d$ and a is rooted corresponding to some permutation $\sigma \in \operatorname{Sym}(d)$.

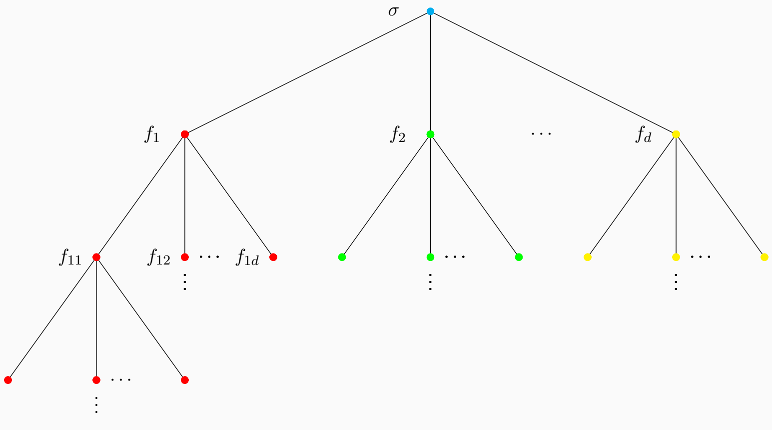
Example

Let $f \in \operatorname{Aut} \mathcal{T}_d$ with $f = (f_1, f_2, \ldots, f_d)a$, where $f_i \in \operatorname{Aut} \mathcal{T}_d$ and a is rooted corresponding to σ. If $f_1 = f_2 = \cdots = f_d = 1$, then f is rooted. Do you remember?

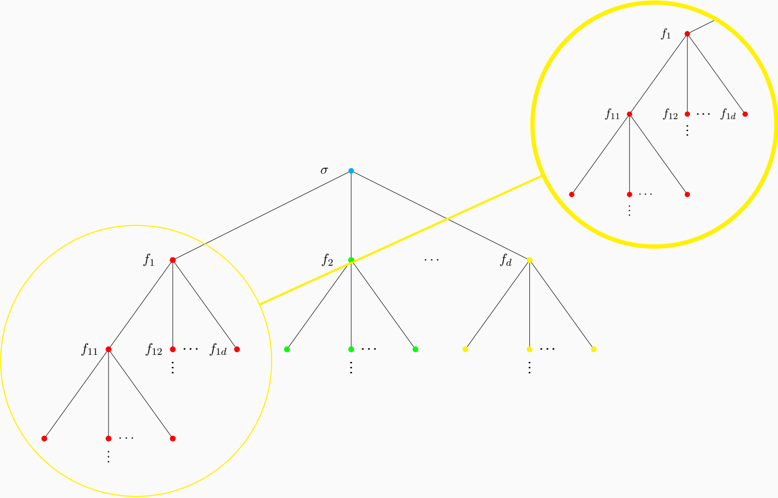


Example: General Case

Let $f \in \operatorname{Aut} \mathcal{T}_d$ with $f = (f_1, f_2, \ldots, f_d)a$, where $f_i \in \operatorname{Aut} \mathcal{T}_d$ and a is rooted corresponding to σ.



Example: General Case

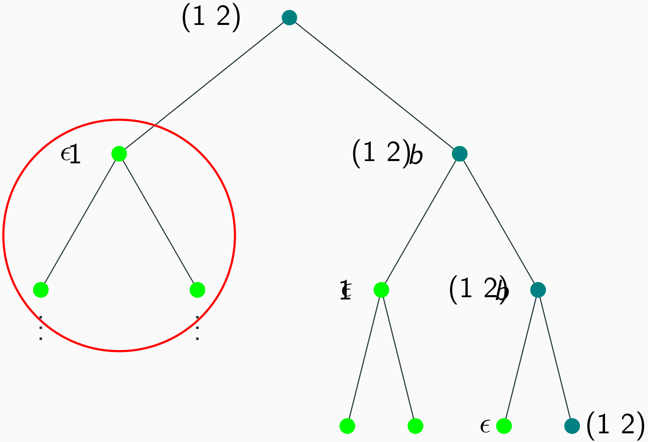


Example (another!)

If $\mathcal{T}_2$ is the binary tree and a is rooted corresponding to $(1\ 2)$, let

$$b = (1, b)a.$$

How does b act on $\mathcal{T}_2$?



Exercise

If $\mathcal{T}_7$ is the 7-adic tree and a is rooted corresponding to $(1\ 2\ 3\ 4\ 5\ 6\ 7)$, let

$$b = (a, a^{-1}, a^2, 1, 1, 1, b)a.$$

How does b act on $\mathcal{T}_7$?

Branch groups

Introduction

- Branch groups were introduced by Grigorchuk in 1997.
- Recall that in the full group of automorphisms we have

$$\mathrm{st}(n) \simeq \mathrm{Aut}\,\mathcal{T} \times \overset{d^n}{\cdots} \times \mathrm{Aut}\,\mathcal{T},$$

  since $\psi_n : \mathrm{st}(n) \longrightarrow \mathrm{Aut}\,\mathcal{T} \times \overset{d^n}{\cdots} \times \mathrm{Aut}\,\mathcal{T}$ is an isomorphism.
- If $G \leq \mathrm{Aut}\,\mathcal{T}$, we have

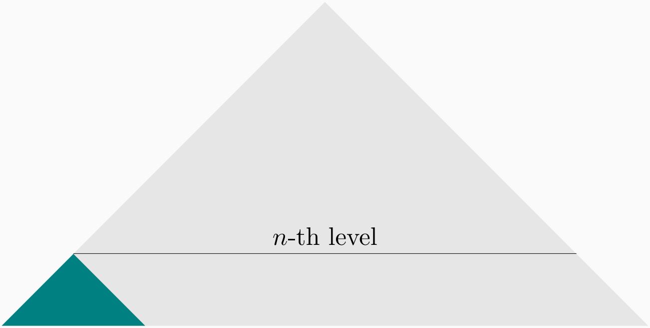
$$\psi_n : \mathrm{st}_G(n) \longrightarrow \psi_n(\mathrm{st}_G(n)),$$

  where $\psi_n(\mathrm{st}_G(n))$ need not be a direct product.
- The question is: given $G \leq \mathrm{Aut}\,\mathcal{T}$, can we find for every $n \in \mathbb{N}$ a subgroup (eventually of finite index) of $\mathrm{st}_G(n)$ which is a direct product?

Rigid stabilizers

The *rigid stabilizer* of the vertex u is

$$\mathrm{rst}_G(u) = \{g \in G : g \text{ fixes all vertices outside } \mathcal{T}_u\}$$



The *rigid stabilizer* of the nth level is $\mathrm{rst}_G(n) = \prod_{u \in X^n} \mathrm{rst}_G(u)$.

About the "question"

- If G is the whole $\operatorname{Aut}\mathcal{T}$ then the rigid stabilizer coincides with the nth level stabilizer.
- And if $G \leq \operatorname{Aut}\mathcal{T}$?
- Bad news: this is not usually the case for arbitrary subgroups of $\operatorname{Aut}\mathcal{T}$.
- Good news: in some cases, there exist "nice" rigid stabilizers.
- Informally speaking: the subgroup $\psi_n(\mathrm{rst}_G(n))$ is the largest subgroup of $\psi_n(\mathrm{st}_G(n))$ which is a "geometric" direct product.

Branch groups

Let $G \leq \operatorname{Aut} \mathcal{T}$ a *spherically transitive* group (a group that acts transitively on each level of $\mathcal{T}$). Digression: It is true that a spherically transitive group cannot be finite? Think about it :)

- We say that G is a branch group if for all $n \geq 1$, the index of the rigid nth level stabilizer in G is finite. In other words, for all $n \geq 1$,

$$|G : \mathrm{rst}_G(n)| < \infty.$$

- We say that G is a weakly branch group if all of its rigid vertex stabilizers are nontrivial for every vertex of the tree.
- Branch $\longrightarrow$ weakly branch.
- These groups try to approximate the behaviour of the full group $\operatorname{Aut} \mathcal{T}$, where $\mathrm{rst}(n) = \mathrm{st}(n)$ is as large as possible.
- The most important families of subgroups of $\operatorname{Aut} \mathcal{T}$ consist almost entirely of (weakly) branch groups.
- The first Grigorchuk group is a branch group.

More definitions: Self-similar groups

Let $G \leq \operatorname{Aut} \mathcal{T}$.

- A group G is said to be *self-similar* if taken $g = (g_1, \ldots, g_d)\sigma \in G$ we have $g_i \in G$ for any $i = \{1, \ldots, d\}$.
- Example: $\operatorname{Aut} \mathcal{T}$ is self-similar, the first Grigorchuk is self-similar.
- Non-example: The group $G = \langle a, b \rangle$, where $a = (b, c)\sigma$ and $c \notin G$, then G is not self-similar.

Regular branch groups

Let G be a self-similar group. We say that G is a *regular branch* if there exists a subgroup K of $\mathrm{st}_G(1)$ of finite index such that

$$\psi(K) \supseteq K \times \overset{d}{\ldots} \times K.$$

More precisely we have this situation:

$$\begin{array}{ccc}
G & & G \times \overset{d}{\ldots} \times G \\
\Big| & & \Big| \\
\mathrm{st}_G(1) & \xrightarrow{\psi} & \psi(\mathrm{st}_G(1)) \\
\Big| & & \Big| \\
K & \xrightarrow{\psi} & \psi(K) \\
\Big| & & \Big| \\
L & \xrightarrow{\psi} & K \times \overset{d}{\ldots} \times K
\end{array}$$

Regular branch groups

- We say that G is a weakly regular branch group if K has infinite index in G.
- If we want to emphasize the subgroup K, we say that G *is (weakly) regular branch over* K.
- Regular branch $\longrightarrow$ branch.

Next lecture

Examples of (weakly) branch groups

Next week we will present the following groups of automorphisms of rooted trees together with their main properties:

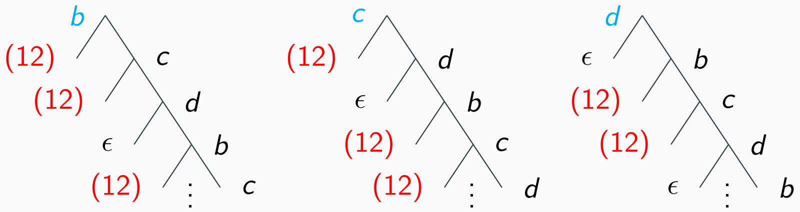
- The Grigorchuk groups
- The GGS-groups
- The Basilica group
- The Hanoi Tower group

I am sure I was too quick, so there is still time . . . .

The first Grigorchuk group (finally!)

$$\Gamma = \langle a, b, c, d \rangle$$

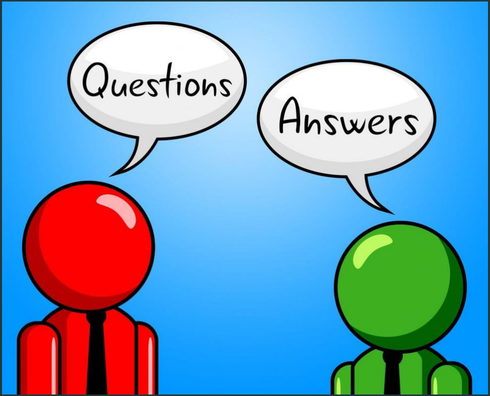
$$a = (1,1)(12) \quad b = (a, c) \quad c = (a, d) \quad d = (1, b)$$



Some properties of Γ

29th of October

$\ddot\smile$


Questions
Answers

Thank you :)