Contents of this Booklet

MAS2216/3216 Enumeration and Combinatorics

Semester 1, 2009/2010

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This module is an introduction to Combinatorial Mathematics beginning with counting (enumeration) problems and moving on to graph theory. Graph theory does not refer to the familiar notion of the graph of a function but instead concerns collections of objects that can be visualised as points (vertices) in 3 dimensional space, together with lines (edges) joining them. Such graphs can be very useful in solving many practical problems. Typical example are: in a system of roads joining given towns find the shortest network joining all the towns; find the number of different molecules that have a given chemical formula; determine the maximal flow from source to sink through a network of pipes. Any of these problems can be solved by trying all the possibilities; graph theory looks for economical methods of solution.

Books

1. A Walk Through Combinatorics, M .Bóna (World Scientific).

2. Discrete Mathematics, L. Lovász, J. Pelikan and K. Vestergombi (Springer).

3. Introduction to Graph Theory, R J Wilson (Pearson).

4. **Library §511.1, §511.5, §511.6**

Notes

The printed notes consist of lecture notes, intended to supplement the notes you make during the lectures, exercises and a mock exam with solutions. Material given on slides in the lectures is covered in the printed notes, what is written on the blackboard during lectures may not be. There should be enough space in the printed notes for you to write down the notes you take in lectures. The notes, exercises and other course information can be found on the web at

`www.mas.ncl.ac.uk/~najd2/teaching/mas2216/`

from where they can be viewed or printed out.

AJ Duncan August 2009

Contents

1 Enumeration

Example 1.1. 1. Two students witness burglar Bill making off from a crime scene in his getaway car. The first student tells the police that the number plate began with and R or a P and that the first numerical digit was either a 2 or a 3. The second student recalls that the last letter was an M or an N. Given that all number plates have the same format: two capital letters (between A and Z) followed by 2 digits (between 0 and 9) followed by 3 more letters, how many number plates must the police investigate.

2. There are 7 people to be seated at a round table. How many seating arrangements are possible? How many times must they change places so that everyone sits next to everyone else at least once. What difference does in make if one person always sits in the same place?

3. A lecturer divides a class of 30 students into 5 groups, not necessarily of the same size, and then chooses one representative from each group. In how many ways is this possible? If some of the groups are to be selected to move into another room how many possibilities are there now?

1.1 A basic counting technique

I have shirts of 3 different colours, trousers of 2 different colours and socks of 5 different colours. How many different outfits (colour combinations) are available to me?

The general rule is

Lemma 1.2. *A task is to be carried out in stages. There are n_1 ways of carrying out the first stage. For each of these there are n_2 ways of carrying out second stage. For each of* *these n_2 ways there are n_3 ways of doing the third stage and so on. If there are r stages then there are in total $n_1 n_2 \cdots n_r$ ways of carrying out the entire task.*

1.2 The Pigeonhole Principle

Do any 2 Newcastle students share the same Personal Identification Number (PIN) for their debit cards? Is your PIN number the same as mine?

More generally we have the following lemma.

Lemma 1.3. *If n identical balls are put into k boxes and $n > k$ then some box contains at least 2 balls.*

Returning to the sharing of PINs, is it possible that 3 or more people in Newcastle share the same PIN?

Lemma 1.4. *Suppose n identical balls are placed in k boxes and that $n > kr$, for some positive integer r. Then some box contains at least $r + 1$ balls.*

1.3 Permutations

Example 1.5. Suppose that 7 boys dance with 7 girls, all on the dance floor at once. How many pairings are possible?

Definition 1.6. Some revision: a map $f : X \to Y$ is called

1. an *injection* if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in X$;
2. a *surjection* if, for all $y \in Y$, there is $x \in X$ with $f(x) = y$;
3. a *bijection* if f is an injection and a surjection.

Also *one-one* means the same as injection and *onto* means the same as *surjection*.

Definition 1.7. A bijection from a set X to itself is called a *permutation*.

Theorem 1.8. *The number of permutations of a set of n elements is $n!$.*

1.4 Multisets

In the notation for permutations above each element of the set $\{1, \ldots, n\}$ appears exactly once. By contrast suppose that I drink 5 cups of water, 3 cups of tea and 2 cups of coffee every day. How many different ways can I arrange the order in which I drink all these drinks, (assuming that there is no difference between two cups of the same kind of drink)?

A *multiset* is a collection of elements of a set in which elements may occur more than once.

Theorem 1.9. *Let $a_1, \ldots, a_k$ be positive integers and let $n = a_1 + \cdots + a_k$. If we have a multiset of a_1 elements of type 1, a_2 elements of type 2, ... , a_k elements of type k, then we can arrange these elements in order in*

$$\frac{n!}{a_1! \cdots a_k!}$$

ways.

1.5 Sequences of length k

A database stores information of a certain type as a string of length 10 consisting of capital letters A–Z, lower case letters a–z and numerical digits 0–9: so there are 62 different symbols available. How many different records can be made this way?

Lemma 1.10. *The number of sequences* $a_1, \ldots, a_k$ *of length* k *where all the elements* a_i *belong to a set of size* n *is* n^k.

Example 1.11. I am going to paint each of my fingernails (and thumbnails) a different colour. I have paints of 5 different colours. How many different ways can I do this?

Corollary 1.12. *The number of maps from a set of size k to a set of size n is n^k.*

1.6 Ordered subsets of size k

I'm going to place a bet at Cheltenham races on a race in which there are 15 horses. Only the order of the first 3 horses over the line is recorded and I bet that horses Brave Inca, Straw Bear and Lazy Champion will come in 1st, 2nd and 3rd, respectively. How many outcomes are possible and how likely am I to win my bet?

A *k-subset* is a subset of k elements. A set of n elements as an *n-set*. Note that an ordered subset is a sequence, but *without repetition*. A race outcome above is an ordered 3-set of a 15-set.

Theorem 1.13. *The number of ordered k-subsets of an n-set is*

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

1.7 Subsets

Example 1.14. How many subsets does the set $\{a, b, c\}$ have?

Lemma 1.15. *The number of subsets of a set of size n is 2^n.*

1.8 k-subsets

In a game called "Quickfire Lotto" players buy a ticket and select 4 numbers from a list of the numbers from 1 to 48. Then 4 different winning numbers between 1 and 48 are selected at random. How many tickets would you need to buy to be sure of getting all 4 numbers (and so winning the top prize).

By convention we set $0! = 1$.

Definition 1.16. The *binomial coefficient* for integers $n \geq k \geq 0$ is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

For integers $n < k$ we define

$$\binom{n}{k} = 0.$$

In particular from this definition we have

$$\binom{n}{0} = \binom{n}{n} = \binom{0}{0} = 1 \text{ and } \binom{n}{1} = \binom{n}{n-1} = n.$$

Theorem 1.17. *The number of k-subsets of a set of n elements is*

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

1.9 k-multisets

Example 1.18. It has been decided that classes for module MAS9999 will all be held on a Friday between the hours of 8:00 and 20:00 (so there are 12 hour long slots available). There are to be 5 hours of teaching but no two consecutive hours. In how many ways can the schedule be devised?

A collection of k elements of a set where repetition is allowed is called a *k-multiset*.

Theorem 1.19. *The number of k-multisets of a set of n elements is*

$$\binom{n+k-1}{k}.$$

1.10 The Binomial Theorem

We introduced the binomial coefficients in Definition 1.16 above. These numbers play a central role in enumeration problems so we'll look at them more closely now. Consider expanding

$$(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y).$$

What is the coefficient of say x^3y^4 in the result?

Theorem 1.20 (The Binomial Theorem)**.** *For all positive integers n*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Corollary 1.21.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

It's worth writing the statements of the corollary out in long hand, just to see what they look like. We have

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n} = 2^n$$

and

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{n-1}\binom{n}{n-1}+(-1)^n\binom{n}{n} = 0.$$

Now here are some facts about binomial coefficients. We can prove all of these algebraically or by using counting arguments.

Lemma 1.22. *Let n and k be positive integers.*

(i)

$$\binom{n}{k} = \binom{n}{n-k}.$$

(ii)

$$\binom{n-1}{k}+\binom{n-1}{k-1} = \binom{n}{k}.$$

The Multinomial Theorem

Suppose we wish to compute powers of $(x+y+z)$ instead of $(x+y)$. For example we have

$$(x+y+z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3xy^2 + 3y^2z + 3xz^2 + 3yz^2 + 6xyz.$$

What is the coefficient of $x^a y^b z^c$ in $(x+y+z)^n$?

Definition 1.23. Let $n = a_1 + \cdots + a_k$, where a_i is a non-negative integer, $i = 1, \ldots, k$. Define the *multinomial coefficient*

$$\binom{n}{a_1, \ldots, a_k} = \frac{n!}{a_1! \cdots a_k!}.$$

Theorem 1.24. *Let $x_1, \ldots, x_k$ be real numbers. Then, for all non-negative integers n and positive integers k, we have*

$$(x_1 + \cdots + x_k)^n = \sum_{a_1,\ldots,a_k} \binom{n}{a_1, \ldots, a_k} x_1^{a_1} \cdots x_k^{a_k},$$

where the sum is over all length k sequences $a_1, \ldots, a_k$ of non-negative integers such that $n = a_1 + \cdots + a_k$.

1.11 Inclusion-Exclusion

I have 30 uncles some wicked some virtuous. 12 of them smoke, 12 of them drink and 18 of them gamble. 6 smoke and drink, 9 drink and gamble, 8 smoke and gamble and finally 5 smoke, drink and gamble. How many neither smoke, drink nor gamble?

The general result covering the examples above is the next theorem.

Theorem 1.25. *Let $A_1, \dots, A_k$ be subsets of a set E. Then*

$$\begin{aligned}
|A_1 \cup \cdots \cup A_k| = {} & |A_1| + \cdots + |A_k| \\
& - (|A_1 \cap A_2| + \cdots + |A_{k-1} \cap A_k|) \\
& + (|A_1 \cap A_2 \cap A_3| + \cdots + |A_{k-2} \cap A_{k-1} \cap A_k|) \\
& \vdots \\
& + (-1)^{k-1}|A_1 \cap \cdots \cap A_k|.
\end{aligned}$$

That is

$$|A_1 \cup \cdots \cup A_k| = \sum_{i=1}^{k} (-1)^{i-1} \sum_{s_1,\dots,s_i} |A_{s_1} \cap \cdots \cap A_{s_i}|$$

where, for all i, the subscripts $s_1, \dots, s_i$ run over all i-subsets of $\{1, \dots, k\}$.

1.12 Derangements

Example 1.26. n people come to a party at your house, each wearing a hat. When they leave they are not so sober and they can't remember which hat is which. In the morning each person discovers they have someone else's hat. How many ways can this happen?

A permutation with no fixed points is called a *derangement* of a set and the number of such permutations of an n-set is denoted $D(n)$.

Theorem 1.27. *The number of derangements of an n-set is*

$$D(n) = \sum_{r=0}^{n} (-1)^r \frac{n!}{r!}.$$

1.13 Compositions

Suppose I wish to distribute n toffees to k students. How many ways is it possible to do this, assuming that all the toffees are identical but that no two students are identical?

Now suppose that I feel bad about the possibility that some students may not get any toffees atall. How many ways are there of distributing n toffees amongst k students so that every student gets at least one toffee.

More formally we make the following definition.

Definition 1.28. A sequence $(a_1, \ldots, a_k)$ of k non-negative integers such that $\sum_{i=1}^{k} a_i = n$ is called a *weak composition* of n into k parts. If $a_i > 0$ for all i the sequence is called a *composition*.

Theorem 1.29. *The number of weak compositions of n into k parts is*

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Corollary 1.30. *The number of compositions of n into k parts is*

$$\binom{n-1}{k-1}.$$

Since a_i can be zero in a weak composition there exist weak compositions of n into k parts for all $k > 0$. However, for a composition of n into k parts to exist we must have $k \le n$. Therefore there are finitely many compositions of n: and we have the following Corollary.

Corollary 1.31. *The number of compositions of n is 2^{n-1}.*

1.14 Partitions

Suppose now I have n flowers, each one of a different kind. I wish to arrange them in k identical vases, in such a way that there's at least one flower in each vase. In how many ways can I distribute the flowers among the vases?

Definition 1.32. A *partition* of a set X into k parts is a collection $S_1, \ldots, S_k$ of non-empty subsets of X such that $X = \cup_{i=1}^{k} S_i$ and $S_i \cap S_j = \emptyset$, whenever $i \neq j$.

Example 1.33. List all the partitions of the set $\{1, 2, 3, 4\}$ into 2 non-empty subsets.

Definition 1.34. The number of partitions of $\{1, \ldots, n\}$ into k parts is denoted $S(n, k)$. The numbers $S(n, k)$ are called the *Stirling numbers (of the second kind)*.

From the example above we have $S(4, 2) = 7$.

Example 1.35. Find $S(3, 1)$ and $S(3, 2)$.

Lemma 1.36. *Let* $1 \leq k \leq n$. *Then*

$$S(n,n) = 1 \text{ and } S(n,1) = 1,$$
$$S(n,n-1) = \binom{n}{2} \text{ and}$$
$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Now suppose that I have again n flowers, no two the same, and I wish to distribute them amongst k people, who are all different from each other, so that everyone receives at least one flower. How many ways can this be done? That is, how many ways are there of distributing n different kinds of flower amongst k people, so that each person receives at least one flower?

Theorem 1.37. *The number of surjective functions from a set of size n to a set of size k is $k!S(n,k)$.*

We can now apply the inclusion-exclusion principle to obtain a formula for $S(n,k)$. The formula is not entirely satisfactory as it contains a sum of $k+1$ terms, but it is the best we can do.

Theorem 1.38. *Let k and n be positive numbers. Then*

$$S(n,k) = \frac{1}{k!}\sum_{d=0}^{k}(-1)^d\binom{k}{d}(k-d)^n = \sum_{d=0}^{k}(-1)^d\frac{1}{d!(k-d)!}(k-d)^n.$$

1.15 Integer Partitions

I have a bag of n identical marbles and wish to sort them into k non-empty piles, the order of which does not matter. How many ways can I do this?

Definition 1.39. Let $a_1 \geq \cdots \geq a_k \geq 1$ be integers such that $a_1 + \cdots + a_k = n$. Then $(a_, \ldots, a_k)$ is called an *integer-partition* of n into k parts. The number of integer-partitions of n into k parts is denoted $p_k(n)$ and the number of all integer-partitions of n is denoted $p(n)$.

It can be shown that

$$p(n) \sim \frac{1}{4\sqrt{3}} \exp(\pi\sqrt{\frac{2n}{3}})$$

1.16 Summary

We summarise the counting results found in the following tables.

Permutations	
permutations of an n-set	$n!$
orderings of a_i objects of type i, where $a_1 + \cdots + a_k = n$	$\dfrac{n!}{a_1! \cdots a_k!}$
Sequences	
sequences of length k over an alphabet of size n or functions from a set of size k to a set of size n	n^k
ordered k-subsets of an n-set	$\dfrac{n!}{(n-k)!}$
Subsets	
all subsets of an n-set	2^n
k-subsets of an n-set	$\dbinom{n}{k}$
k-multisets of an n-set	$\dbinom{n+k-1}{k}$

Derangements	
derangements of an n-set	$D(n) = \sum_{r=0}^{n} (-1)^r \frac{n!}{r!}$
Compositions	
weak compositions of n into k parts	$\binom{n+k-1}{k-1}$
compositions of n into k parts	$\binom{n-1}{k-1}$
compositions of n	2^{n-1}
Partitions	
partitions of an n-set into k parts	$S(n,k) = \frac{1}{k!} \sum_{d=0}^{k} (-1)^d \binom{k}{d} (k-d)^n$
surjective functions of an n-set to a k-set	$k!S(n,k)$
integer-partitions of an n-set into k parts	$p_k(n) =??$
integer-partitions of an n-set	$p(n) \sim \frac{1}{4\sqrt{3}} \exp(\pi \sqrt{\frac{2n}{3}})$

2 Graph Theory

2.1 Definitions

Example 2.1. 1. On arrival at a party guests shake hands with some of the people they meet in the hallway, but mostly don't shake hands after that. If we ask each person how many people they shook hands with and then add these numbers we always have an even number. Why?

2. Suppose there are an odd number of people at this party. If we ask each person how many other people they shook hands with then there will be an odd number of people who answer with an even number. Why?

3. Only 6 people make it to the MAS2216 lecture at mid-day on the day after this party. I can guarantee that either 3 of them shook each others hands or 3 of them did not. How can I be sure?

Definition 2.2. A **graph** G consists of

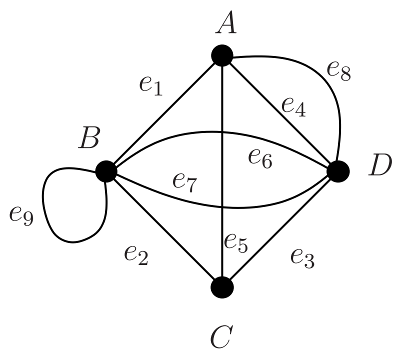
(i) a non-empty set $V(G)$ of **vertices** and

(ii) a set $E(G)$ of **edges**

such that every edge $e \in E(G)$ is a multiset $\{a, b\}$ of two vertices $a, b \in V(G)$.

We shall restrict attention to graphs with finite edge and vertex sets in this course. Throughout the remainder of the notes $G = (V, E)$ will denote a graph with (finite) vertex and edge sets V and E.

Example 2.3.



A graph must have at least one vertex but need not have any edges.

Definition 2.4. Let $G = (V, E)$ be a graph.

(i) Vertices a and b are **adjacent** if there exists an edge $e \in E$ with $e = \{a, b\}$.

(ii) Edges e and f are **adjacent** if there exists a vertex $v \in V$ with $e = \{v, a\}$ and $f = \{v, b\}$, for some $a, b \in V$.

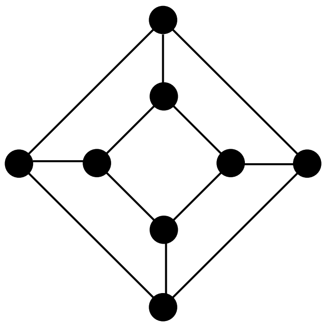
(iii) If $e \in E$ and $e = \{c, d\}$ then e is said to be **incident** to c and to d and to **join** c and d.

(iv) If a and b are vertices joined by edges $e_1, \ldots e_k$, where $k > 1$, then $e_1, \ldots e_k$ are called **multiple** edges.

(v) An edge of the form $\{a, a\}$ is called a **loop**.

(vi) A graph which has no multiple edges and no loops is called a **simple** graph.

Example 2.5. Are these three graphs the same?



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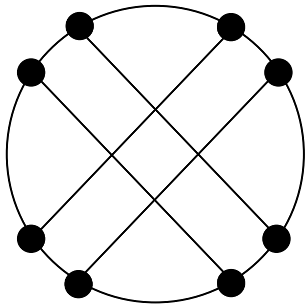

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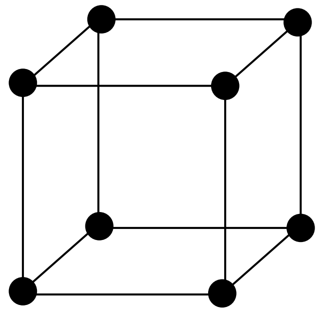
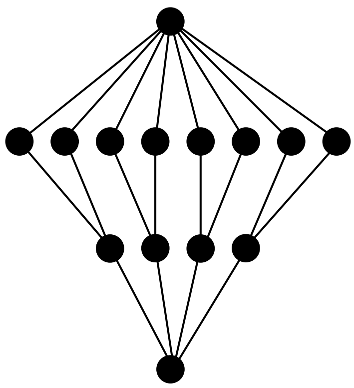


(c)

Example 2.6. What about these three?



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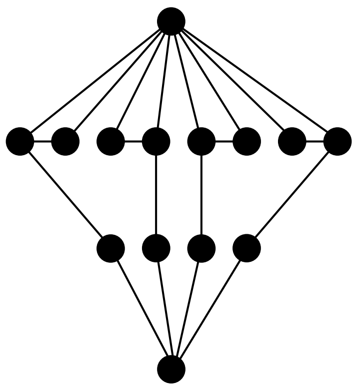

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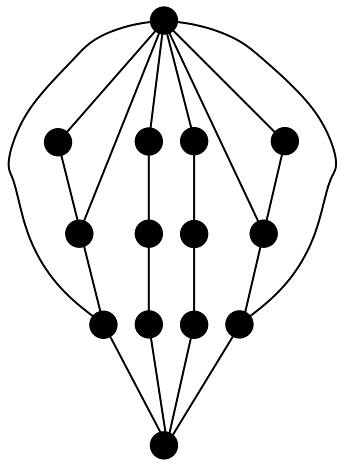


(c)

Definition 2.7. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a bijection $\phi : V_1 \longrightarrow V_2$ such that the number of edges joining u and v in G_1 is the same as the number of edges joining $\phi(u)$ and $\phi(v)$ in G_2, for all vertices $u, v \in V_1$. In this case ϕ is called an **isomorphism** from G_1 to G_2 and we write $G_1 \cong G_2$.

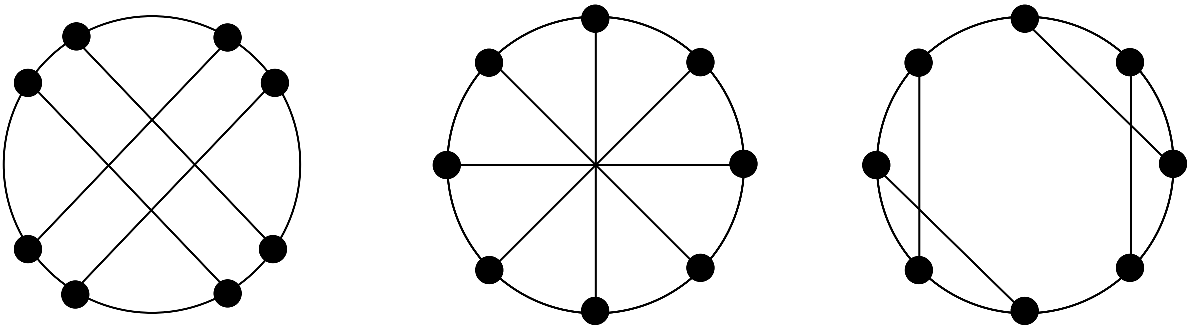
Definition 2.8. The **degree** of a vertex u is the number of ends of edges incident to u and is denoted $\deg(u)$ or $\mathrm{degree}(u)$.

Definition 2.9. Let G be a graph with n vertices. Order the vertices $v_1, \ldots, v_n$ so that $\deg(v_i) \leq \deg(v_{i+1})$. Then G has **degree sequence**

$$\langle \deg(v_1), \ldots, \deg(v_n) \rangle.$$

Definition 2.10. A graph is **regular** if every vertex has degree d, for some fixed $d \in \mathbb{Z}$. In this case we say the graph is regular of **degree** d.

Example 2.11. Graphs which are simple, have 8 vertices, 12 edges and are regular of degree 3. Are any two of these isomorphic? What are their degree sequences?



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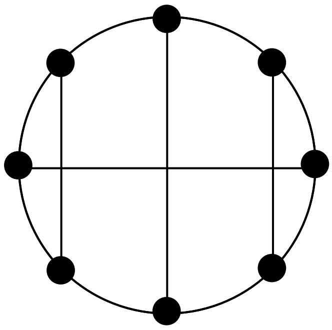

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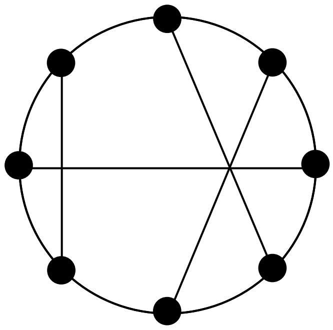

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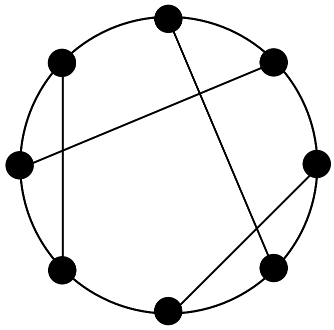

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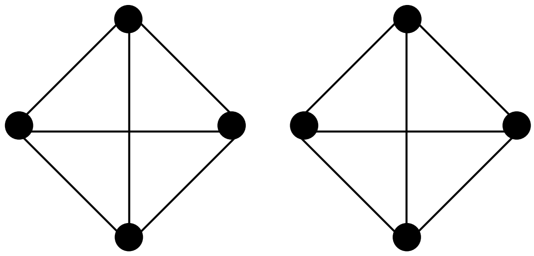

(h)

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2.2 Counting degrees

For this subsection G is a graph with vertices V and edges E, that is $G = (V, E)$.

Lemma 2.12 (The Handshaking Lemma)**.**

$$\sum_{v \in V} \deg(v) = 2|E|.$$

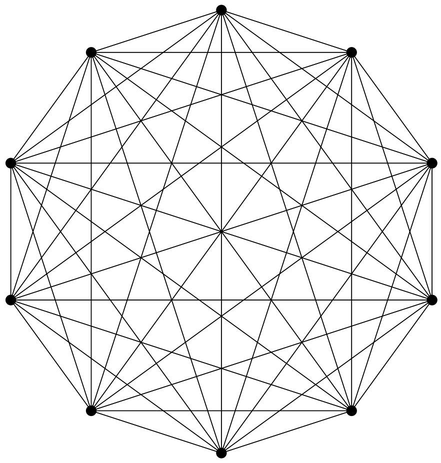
Lemma 2.13. *Suppose that G is a graph with q vertices of odd degree. Then q is even.*

Corollary 2.14. *If G has n vertices and is regular of degree d then G has $nd/2$ edges.*

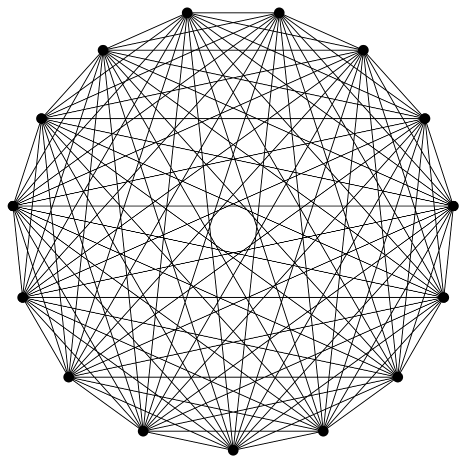
2.3 Some examples

Example 2.15. The **Null** graph N_d, for $d \geq 1$.

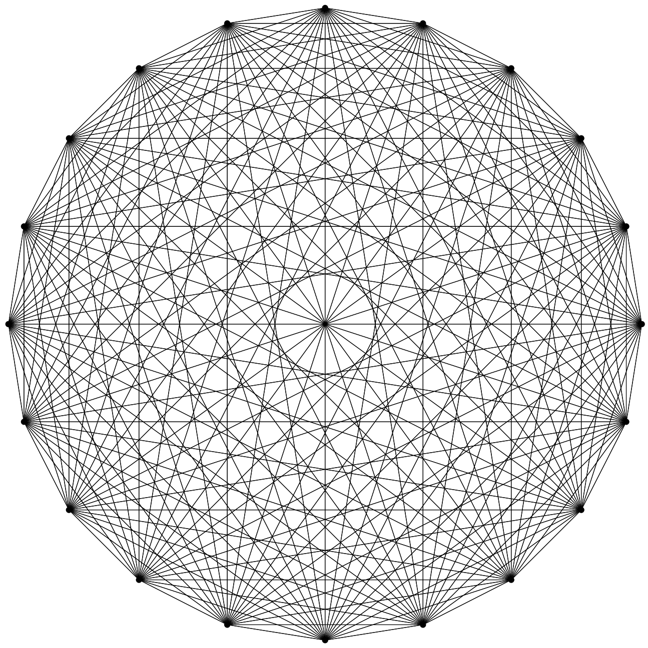
Example 2.16. The **Complete** graph K_d, for $d \geq 1$.



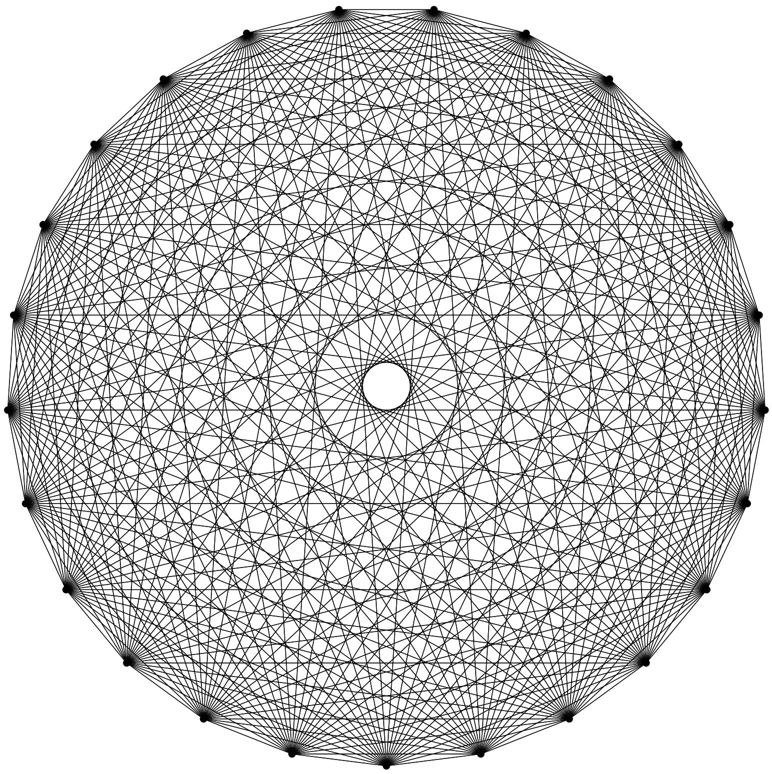
K_{10}



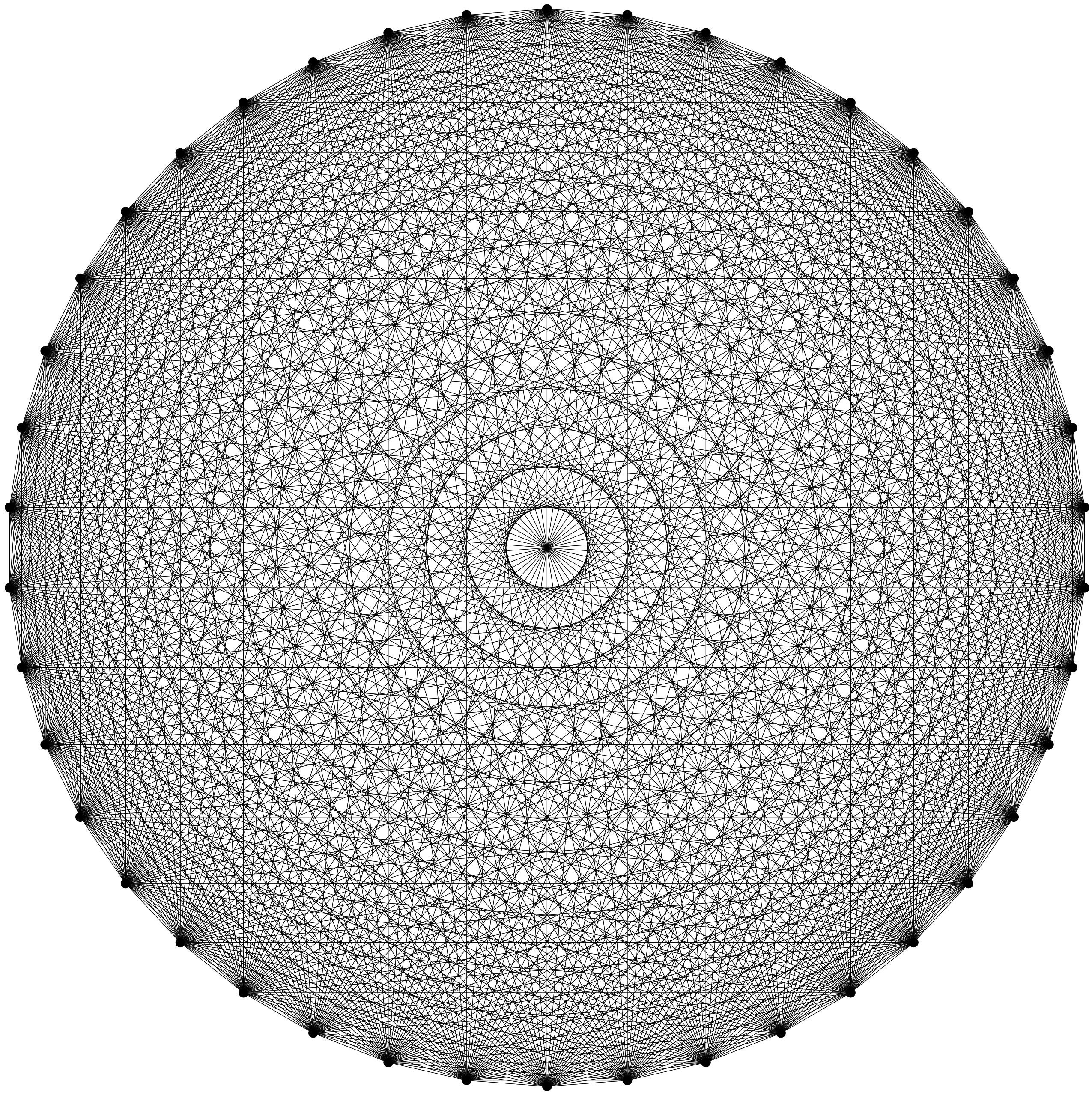
K_{15}



K_{20}



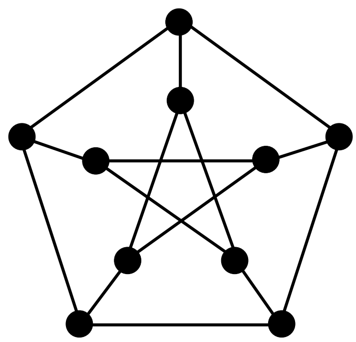
K_{25}



K_{42}

Lemma 2.17. *The complete graph K_d is regular of degree $d-1$ and has $d(d-1)/2$ edges.*

Example 2.18. The **Petersen** graph.

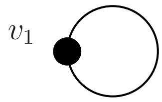


2.4 Subgraphs

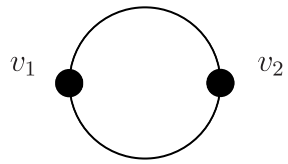
Definition 2.19. A **subgraph** of a graph $G = (V, E)$ is a graph $H = (V', E')$ such that $V' \subset V$ and $E' \subset E$.

Example 2.20.

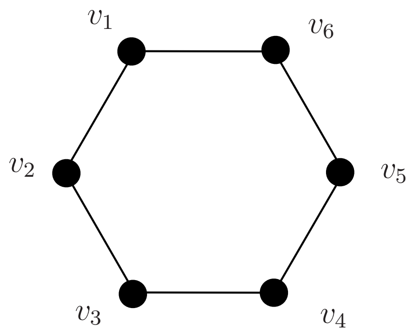
1. For $d \geq 1$ we define the **cycle graph** C_d to be the graph with d vertices $v_1, \ldots, v_d$ and d edges $\{v_1, v_2\}, \ldots, \{v_{d-1}, v_d\}, \{v_d, v_1\}$. ($C_1$ has one vertex v_1 and one edge $\{v_1, v_1\}$.) The cycle graph is regular of degree 2 and simple if $d \geq 3$.



C_1

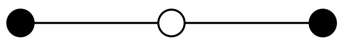


C_2

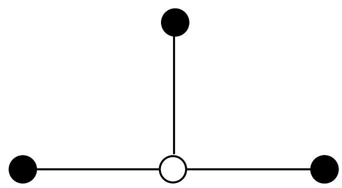


C_6

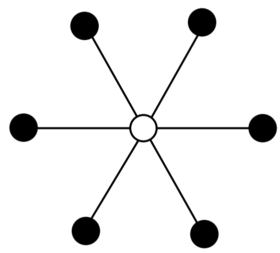
2. The **star** graphs are the graphs $K_{1,s}$, $s \geq 1$:



$K_{1,2}$

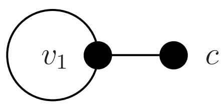


$K_{1,3}$

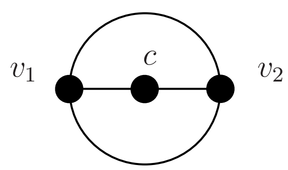


$K_{1,6}$

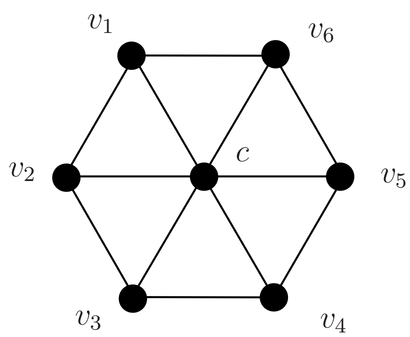
3. For $d \geq 1$ we define the **wheel graph** W_d to be the graph with $d+1$ vertices $c, v_1, \ldots, v_d$ and $2d$ edges $\{v_1, v_2\}, \ldots, \{v_{d-1}, v_d\}, \{v_d, v_1\}, \{c, v_1\}, \ldots \{c, v_d\}$.



W_1



W_2



W_6

2.5 Walks, paths and connectedness

Definition 2.21. A sequence

$$v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n,$$

where

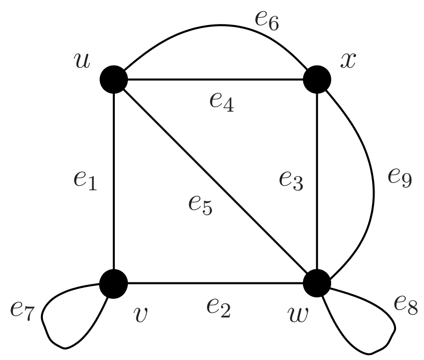
(i) $n \geq 0$ and

(ii) $v_i \in V$ and $e_i \in E$ and

(iii) $e_i = \{v_{i-1}, v_i\}$, for $i = 1, \ldots, n$,

is called a **walk** of **length** n. The walk is **from** its **initial** vertex v_0 and **to** its **terminal** vertex v_n.

Example 2.22.



Walks in simple graphs

If G is a simple graph then, to simplify notation, we may describe a walk by writing only the subsequence of vertices, which we call the **vertex sequence** of the walk. For example the sequence

$$v_1, c, v_5, v_4, c, v_2$$

is the vertex sequence of a unique walk in the wheel graph W_6 shown above.

Open and closed walks and paths

Definition 2.23. Let $W = v_0, e_1, v_1, \ldots, e_n, v_n$ be a walk in a graph.

(i) If $v_0 = v_n$ then W is a **closed** walk. A walk which is not closed ($v_0 \neq v_n$) is called **open**.

(ii) If $v_i \neq v_j$ when $i \neq j$, with the possible exception of $v_0 = v_n$, then W is called a **path**. (If $v_0 \neq v_n$ the path is said to be **open** and if $v_0 = v_n$ it is **closed**.)

Connected graphs

Definition 2.24. A graph is **connected** if, for any two vertices a and b there is an path from a to b. A graph which is not connected is called **disconnected**.

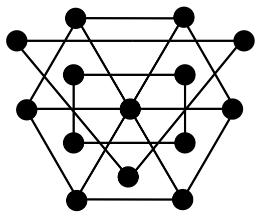
Lemma 2.25. *Let a and b be vertices of a graph. There is a path from a to b if and only if there is a walk from a to b.*

Definition 2.26. A **connected component** of a graph G is a subgraph H of G such that

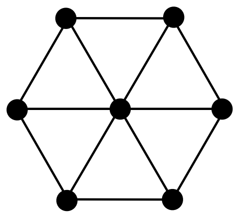
1. H is a connected subgraph of G and

2. H is not contained in any larger connected subgraph of G.

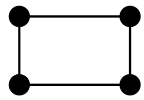
The graph G on the left has 3 connected components A, B and C, as shown.



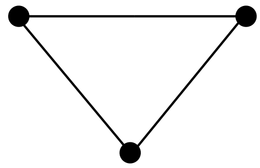
G



A



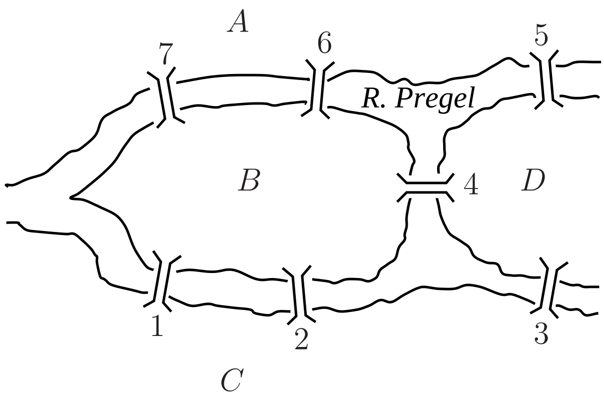
B



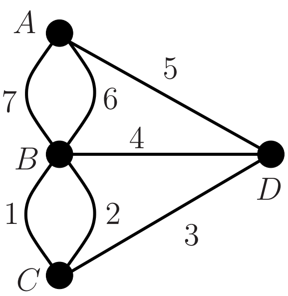
C

2.6 Eulerian graphs

The Königsberg bridge problem



The River Pregel in Königsberg



A graph of the Königsberg bridges

Definition 2.27. A walk in which no edges are repeated is called a **trail**. A closed walk which is a trail is called a **circuit**.

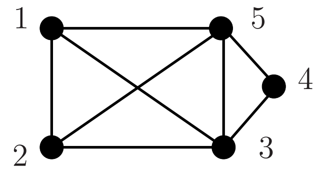
Definition 2.28. 1. A trail containing every edge of a graph is called an **Eulerian trail**.

2. A circuit containing every edge of a graph is called an **Eulerian circuit**.

3. A graph is called **semi-Eulerian** if it is connected and has an Eulerian trail.

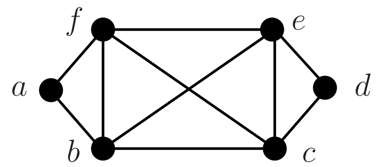
4. A graph is called **Eulerian** if it is connected and has an Eulerian circuit.

Example 2.29. 1. The walk $1, 2, 3, 1, 5, 4, 3, 5, 2$ is a semi-Eulerian trail in the graph G_1 below. Therefore G_1 is semi-Eulerian. Does G_1 have an Eulerian circuit?



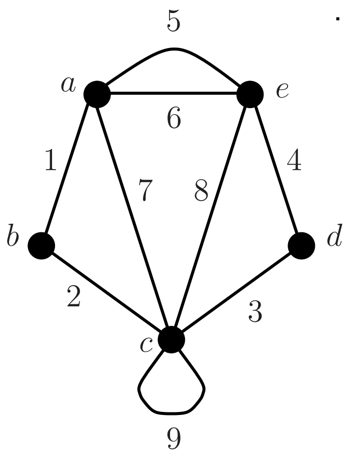
G_1

2. The walk $a, b, c, d, e, f, b, e, c, f, a$ is an Eulerian circuit in the graph G_2 below. Therefore G_2 is Eulerian.



G_2

3. The walk $a, 1, b, 2, c, 3, d, 4, e, 5, a, 7, c, 9, c, 8, e, 6, a$ is an Eulerian circuit in the graph G_3 below.



G_3

Theorem 2.30 (Euler, 1736)**.** *If G is an Eulerian graph then every vertex of G has even degree.*

Example 2.31. 1. There is no Eulerian circuit for the graph of Example 2.29.1 above: this graph has vertices of odd degree.

2. The graph of the Königsberg bridge problem has vertices of odd degree so is not Eulerian.

3. The graph K_d is not Eulerian if d is even. (Why not?)

Lemma 2.32. *Let G be a graph such that every vertex of G has even degree. If $v \in V(G)$ with $\deg(v) > 0$ then v lies in a circuit of positive length.*

Note. In this lemma G is not necessarily connected.

Theorem 2.33. *Let G be a connected graph. Then G is Eulerian if and only if every vertex of G has even degree.*

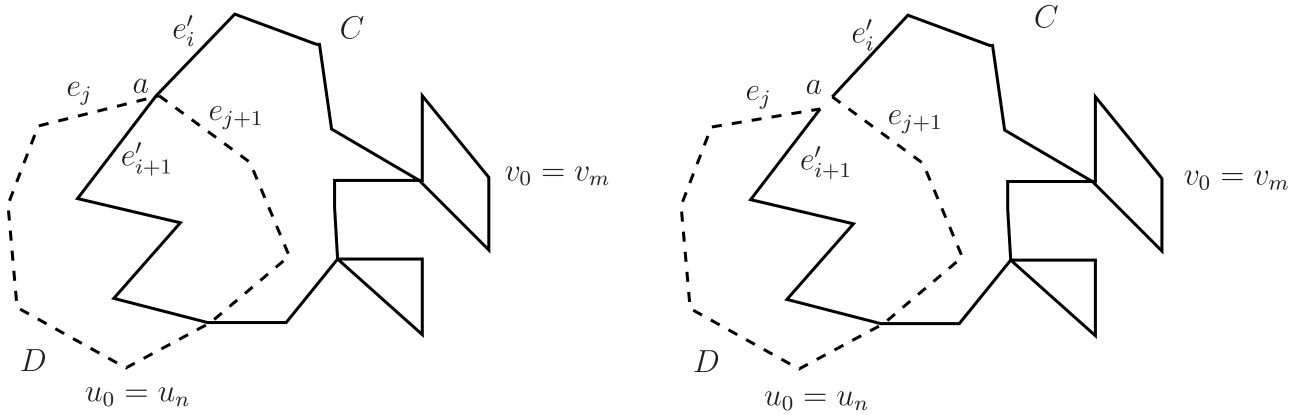
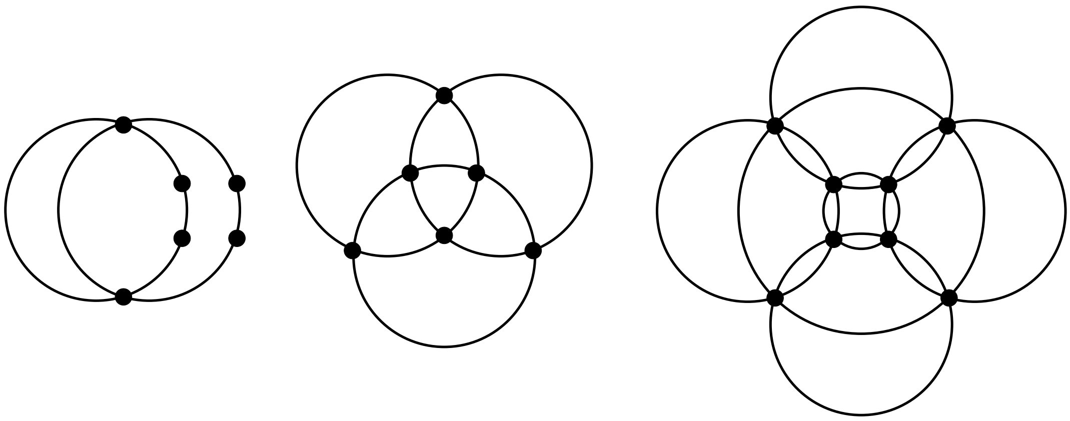


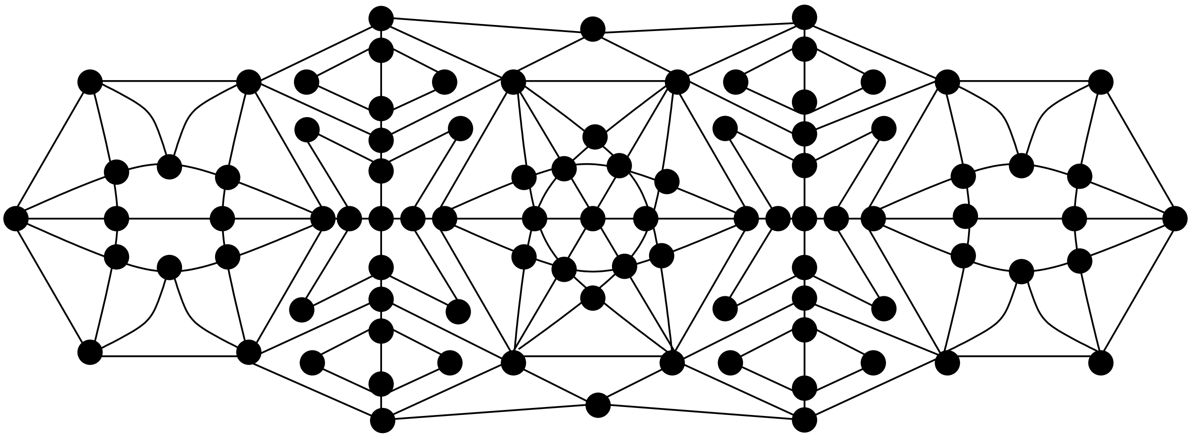
Figure 2.1: Construction of a circuit in Theorem 2.33

Example 2.34.



Theorem 2.35. *A connected graph is semi-Eulerian but not Eulerian if and only if precisely* 2 *of its vertices have odd degree.*

Example 2.36. The following graph has exactly 2 vertices of odd degree and is therefore semi-Eulerian.



2.7 Hamiltonian graphs

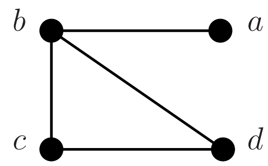
People at a party are to be seated at a circular table. Is it possible to arrange the seating so that everyone sits next to two people they know?

Definition 2.37.

1. A path containing every vertex of a graph is called a **Hamiltonian path**.
2. A closed path containing every vertex of a graph is called a **Hamiltonian closed path**.
3. A graph is called **semi-Hamiltonian** if it has a Hamiltonian path and **Hamiltonian** if it has a Hamiltonian closed path.

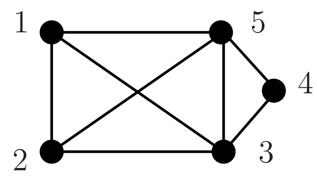
Example 2.38.

1. The walk a, b, c, d is a Hamiltonian path in the graph G_1 below. Therefore G_1 is semi-Hamiltonian.



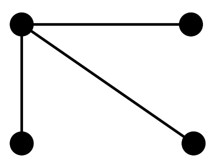
G_1

2. The walk $1, 2, 3, 4, 5, 1$ is a Hamiltonian closed path in the graph G_2 below. Therefore G_2 is Hamiltonian.



G_2

3. The graph G_3 below is not semi-Hamiltonian (and therefore not Hamiltonian).



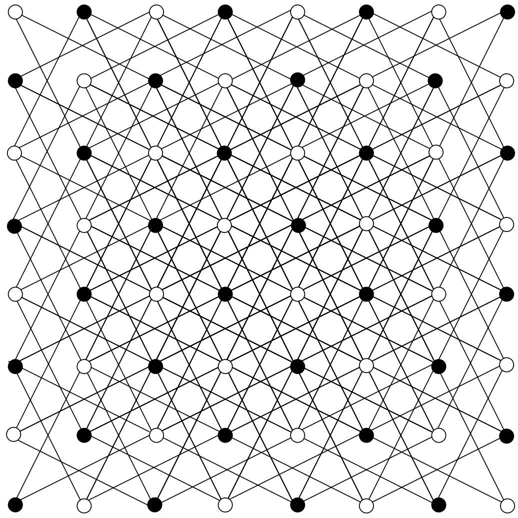
G_3

4. The complete graph K_2 is semi-Hamiltonian but not Hamiltonian. For $d \neq 2$ the graphs K_d are Hamiltonian.

5. The cycle graphs are Hamiltonian for $d \geq 1$.

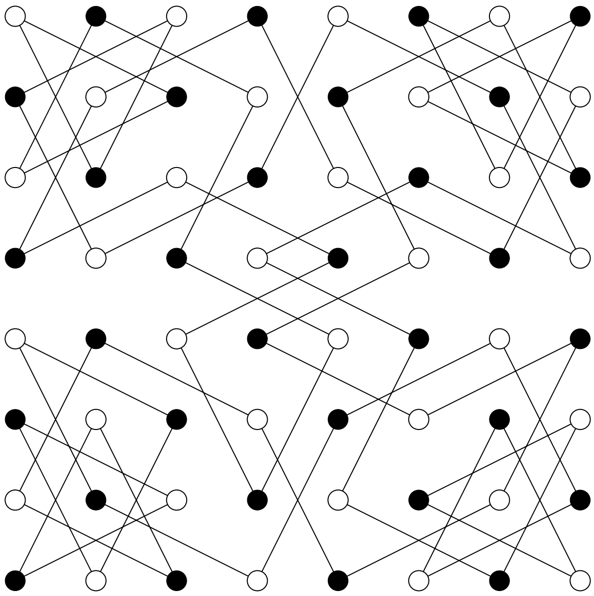
6. The wheel graph W_d is Hamiltonian for $d \geq 2$. The wheel graph W_1 is semi-Hamiltonian but not Hamiltonian.

7. Construct a graph with one vertex corresponding to each square of a chess-board and an edge joining two vertices if a knight can move from one to the other. We call this the **knight's move graph**.

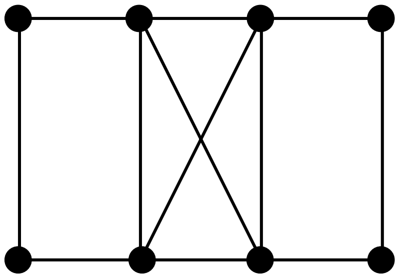


The knight's move graph

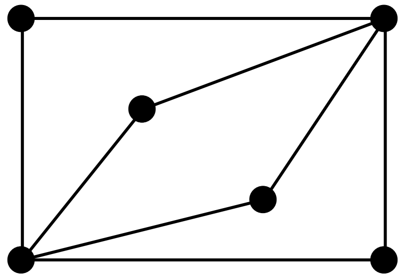
Here is a Hamiltonian closed path for the knight's move graph.



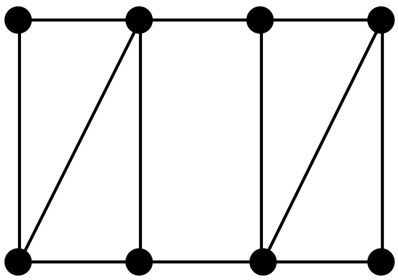
Example 2.39. Note that although there are theorems relating Eulerian and Hamiltonian graphs there do exist graphs with any combination of these properties:



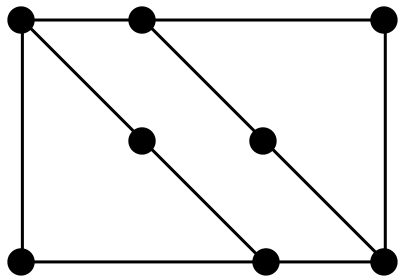
A graph which is Hamiltonian and Eulerian



A graph which is Eulerian and non-Hamiltonian



A graph which is Hamiltonian and non-Eulerian



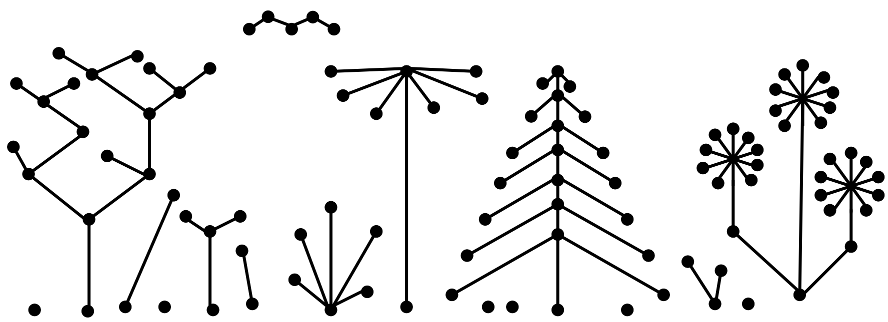
A graph which is non-Eulerian and non-Hamiltonian

2.8 Trees

Definition 2.40. A closed path of length at least 1 is called a **cycle**.

1. A **forest** is a graph with no cycle.
2. A **tree** is a connected graph with no cycle.

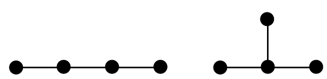
Example 2.41. A forest:



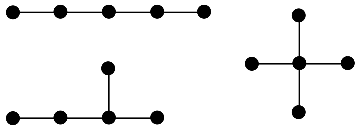
Example 2.42.

1. There is only one tree with one vertex, N_1. There is only one tree with 2 vertices, K_2. There is only one tree with 3 vertices:

2. There are 2 trees with 4 vertices:

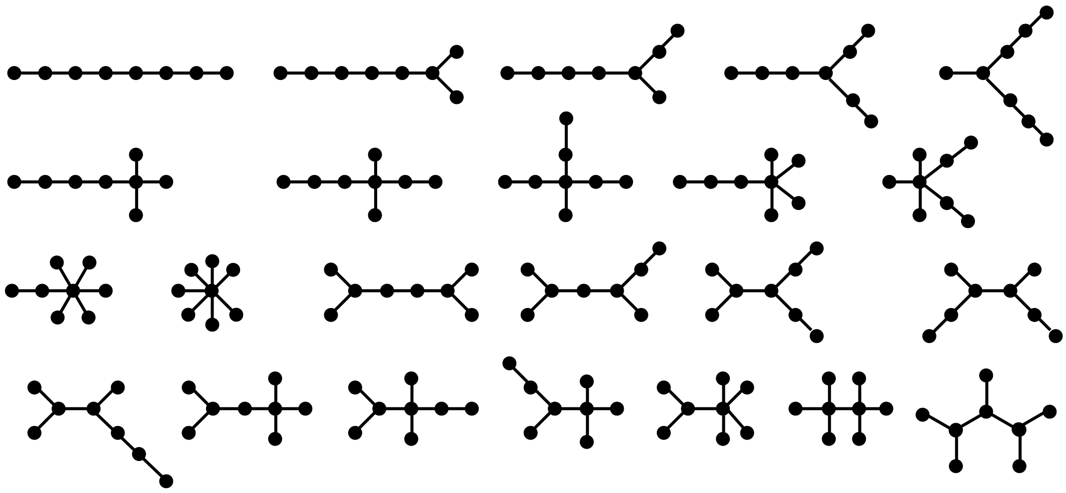


3. There are 3 trees with 5 vertices.



4. There are 6 trees with 6 vertices and 11 trees with 7 vertices (see the Exercises).

5. There are 23 trees with 8 vertices:



There are several possible ways of formulating the definition of a tree. Starting from the above definition we can prove the next theorem, which could have been used as the definition.

Lemma 2.43. *If a graph G contains two distinct paths from vertices u to v then G contains a cycle.*

Theorem 2.44. *A graph G is a tree if and only if there is exactly one path from u to v, for all pairs u, v of vertices of G.*

Here is yet another characterising property of trees.

Theorem 2.45. *Let G be a graph with n vertices. Then G is a tree if and only if it is connected and has $n-1$ edges.*

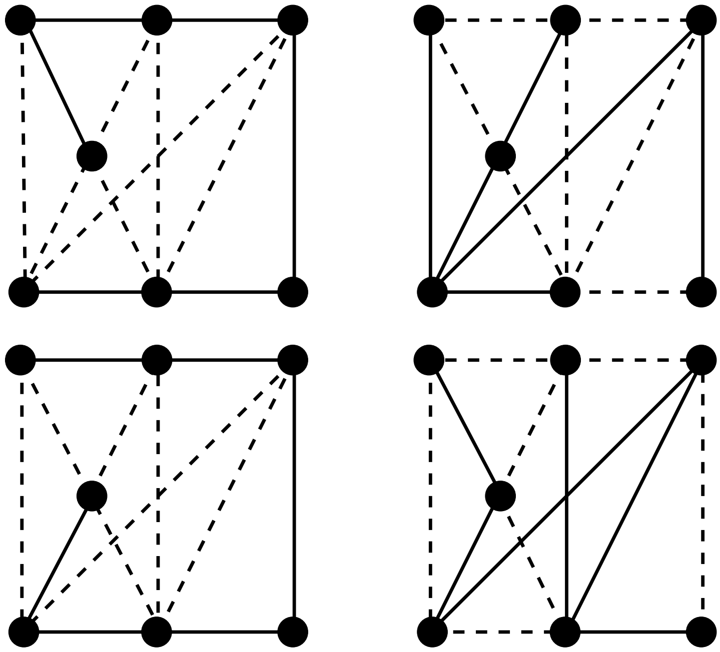
2.9 Spanning Trees

Definition 2.46. Let G be a graph. A **spanning tree** for G is a subgraph of G which

1. is a tree and

2. contains every vertex of G.

Example 2.47. In the diagrams below the solid lines indicate some of the spanning trees of the graph shown: there are many more.



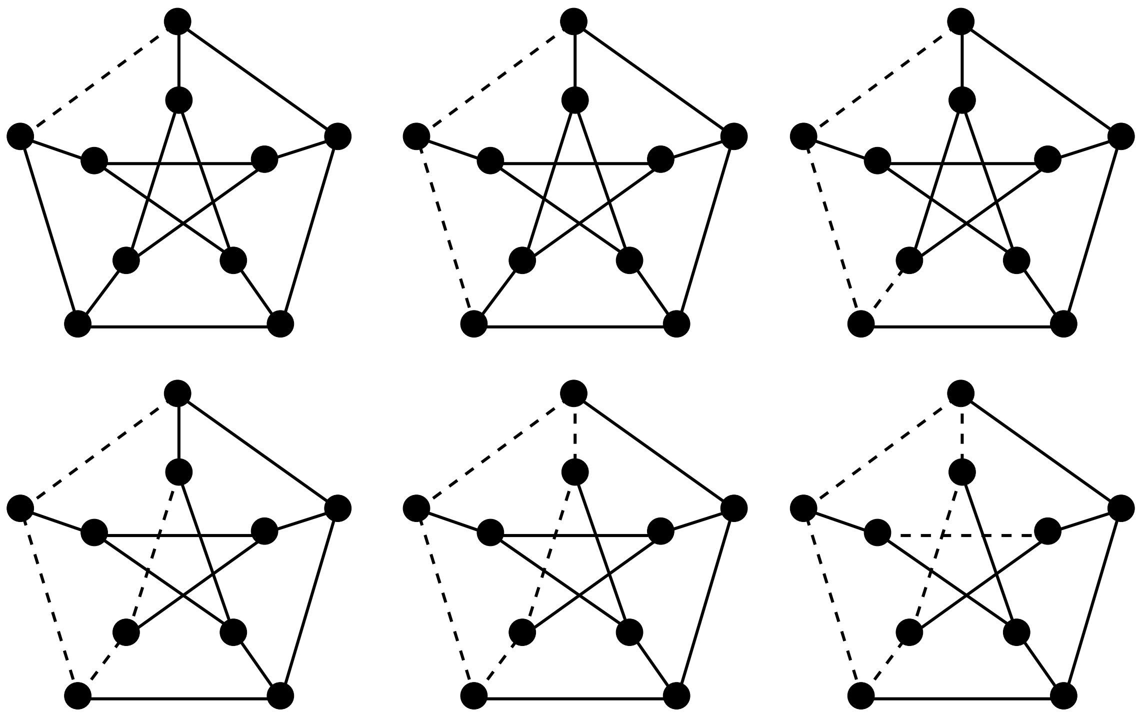
Theorem 2.48. *Every connected graph has a spanning tree.*

The above proof suggests an algorithm for construction of a spanning tree of a graph.

The cut-down algorithm
Given a connected graph G to construct a spanning tree:

1. If G is a tree stop.
2. Choose an edge e from a cycle and replace G with $G - e$. Repeat from 1.

Example 2.49. Starting with the Petersen graph:

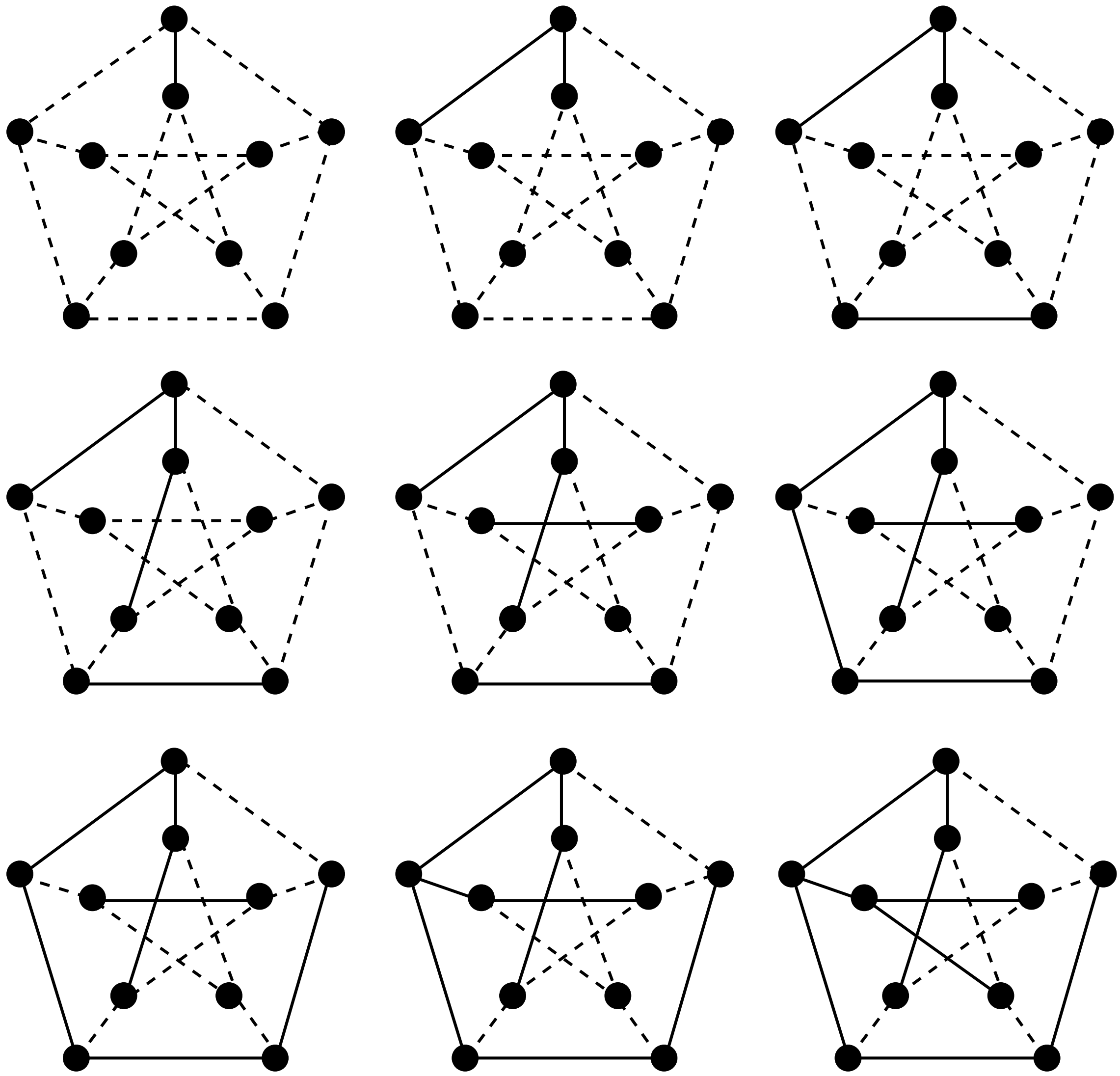


Another approach is the following.

The build-up algorithm
Given a connected graph G to construct a spanning tree:

1. Start with a graph T consisting of the vertices of G and no edges.
2. If T is connected stop.
3. Add an edge e of G to T which does not form a cycle in T. Repeat from 2.

Example 2.50. This time starting with the Petersen graph:



2.10 Weighted graphs

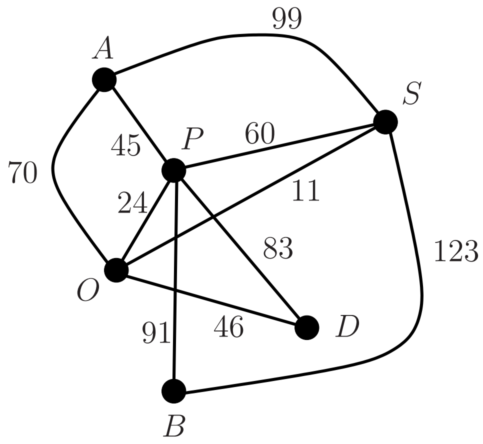
It is often useful to associate further information to the edges and vertices of a graph. For example the edges of a graph may represent roads, in which case we may wish to associate a distance, cost of travel or speed restriction to each edge. If the vertices represent places we may require them to carry additional information about population, temperature or cost of living. We concentrate here on graphs in which additional information is associated to edges. We assume that the required information is encoded as a number.

Definition 2.51. Let G be a connected graph with edge set E. To each edge $e \in E$ assign a non-negative real number $w(e)$. Then G is called a **weighted** graph and the number $w(e)$ is called the **weight** of e. The sum

$$W(G) = \sum_{e \in E} w(e)$$

is called the **weight** of G.

Example 2.52. The following drawing shows a weighted graph G. The weight of edge $\{A, S\}$ is $w(\{A, S\}) = 99$ and the weight of edge $\{O, P\}$ is $w(\{O, P\}) = 24$. The graph has weight $W(G) = 652$.



The Minimum Connector Problem

A subgraph of a connected graph G which contains all the vertices of G is called a **spanning subgraph**. We have seen several examples of spanning trees and obviously every spanning graph must contain a spanning tree.

In a connected, weighted graph the problem of finding a spanning subgraph of minimal weight is called the **minimal connector** problem. A spanning subgraph of minimal weight is always a spanning tree, so the problem is to find a spanning tree of minimal weight. The following algorithm does so. Again we leave aside the problem of testing for a cycle.

The Greedy Algorithm (also known as Kruskal's Algorithm)

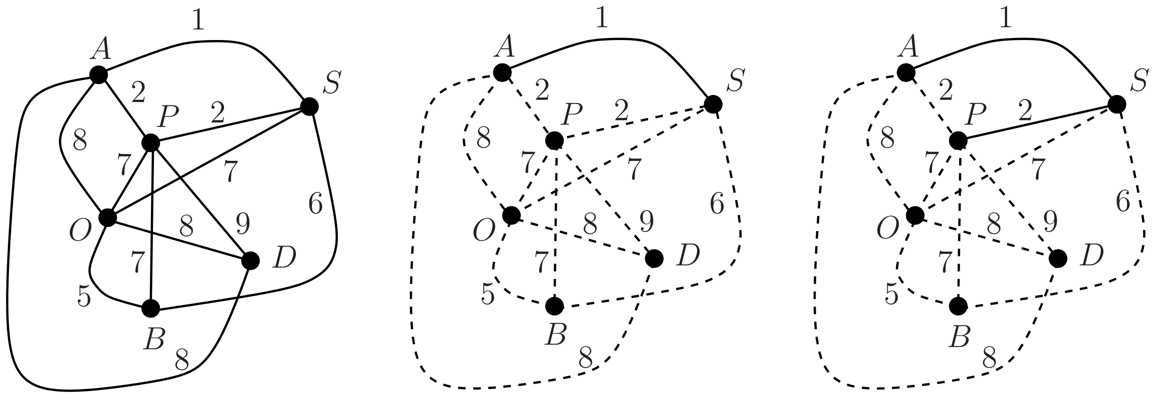
Let G be a connected weighted graph. To find a spanning tree T for G of minimal weight:

Step 1

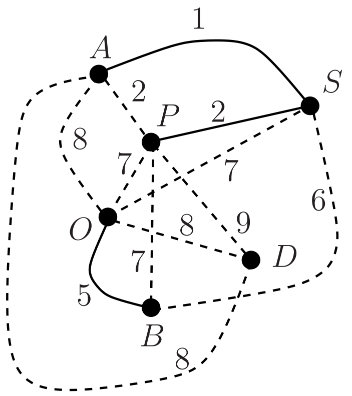
Step 2

Step 3

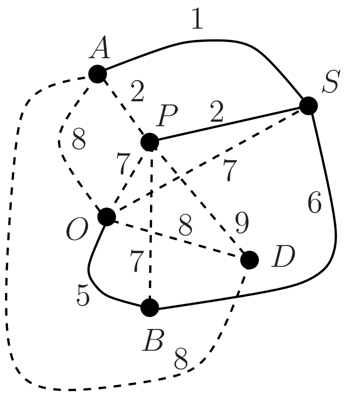
Example 2.53. The algorithm proceeds as shown on the weighted graph G below, producing forests $T_1, \ldots, T_5$ the last of which, T_5, is a minimal weight spanning tree. Note that there are some choices that have to be made in the running of the algorithm on this graph. For instance, either of the edges of weight 2 could have been included in T. A different choice results in a different minimal weight spanning tree, of which there may be many.



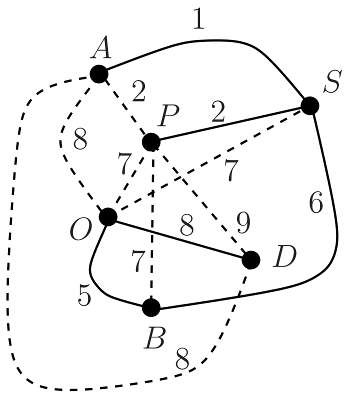
G T_1 T_2



T_3



T_4



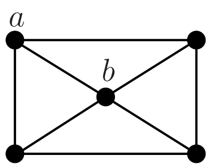
T_5

The Travelling Salesman Problem

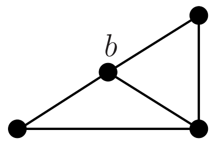
A problem which arises in many applications is: "Given a connected weighted graph G, find a closed walk in G containing all vertices of G and of minimal weight amongst all such closed walks." This problem proves to be very difficult to solve in general. An easier problem, which we shall call the **Travelling Salesman** problem is: "Given a connected weighted graph G, find a minimal weight Hamiltonian closed path in G." The Travelling Salesman problem is easier in the sense that there are fewer possible solutions, so the search has fewer items to consider. However it is still very difficult to solve. We show here how the algorithm for the Minimum Connector problem can be used to find a lower bound for the Travelling Salesman problem. First however we establish some useful notation.

Definition 2.54. Let G be a graph and let v be a vertex of G. The graph $G - v$ obtained from G by **deleting** v is defined to be the graph formed by removing v and all its incident edges from G.

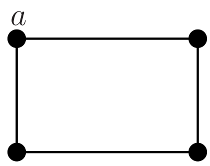
Example 2.55.



G



$G - a$



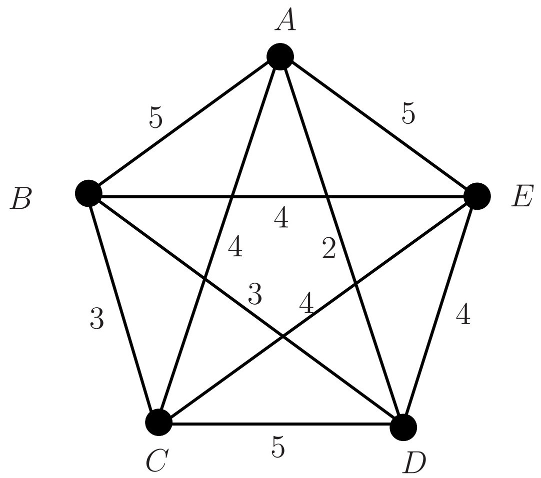
$G - b$

Theorem 2.56. *If G is a weighted graph, C is a minimal weight Hamiltonian closed path in G and v is a vertex of G then*

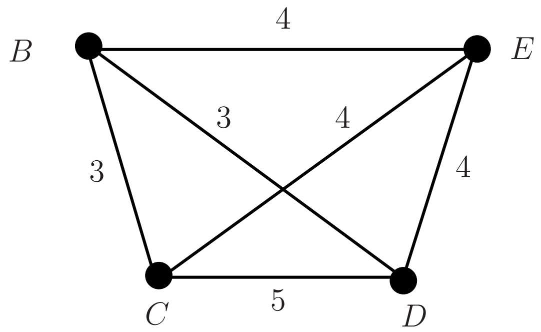
$$w(C) \geq M + m_1 + m_2,$$

where M is the weight of a minimal weight spanning tree for $G - v$ and m_1 and m_2 are the weights of two edges of least weight incident to v.

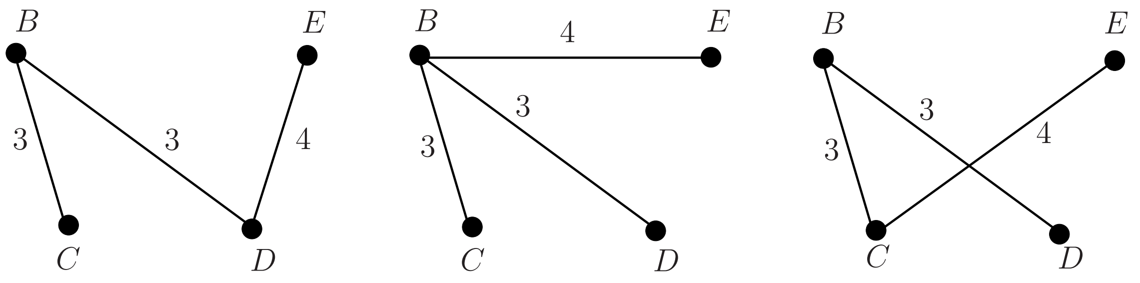
Example 2.57. We shall find a lower bound for the Travelling salesman problem in the weighted graph G below by removing vertex A.



Removing A we obtain the weighted graph $G - A$:

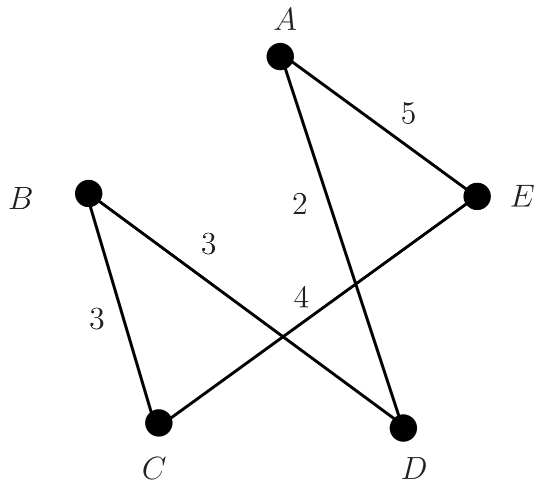


Running the Greedy Algorithm on $G - A$ we might obtain any of the minimal weight spanning trees below. We show all three only for purposes of illustration: any one will suffice. In this case we have $M = 10$.



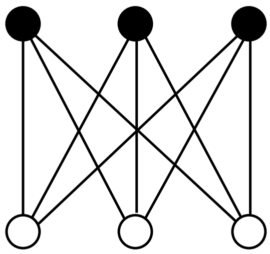
Spanning Tree 1 Spanning Tree 2 Spanning Tree 3

The edges of minimal weight incident to A are $\{A, C\}$ and $\{A, D\}$ which have weights $m_1 = 2$ and $m_2 = 4$. Combining this information we have a lower bound of $10+2+4 = 16$.



2.11 Planar Graphs

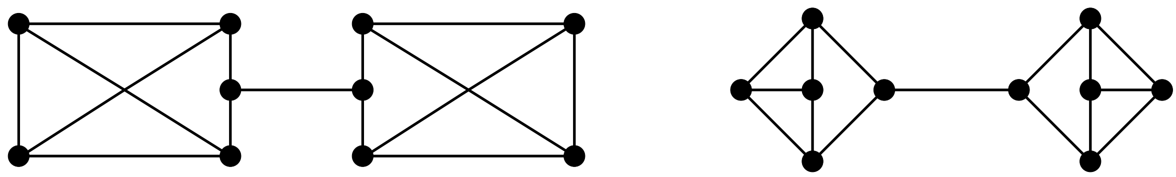
An electronics engineer wishes to make a board on which there are 3 input terminals and 3 output terminals. Each input terminal is to be connected to all output terminals. Connections are to be made by lines of solder laid on the board (not necessarily straight). Two different lines of solder must not cross. Is this possible?



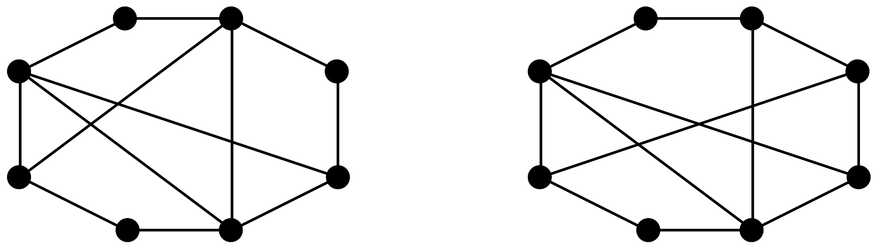
The complete bipartite graph $K_{3,3}$

Definition 2.58. A graph is **planar** if it can be drawn in the plane without edges crossing.

Example 2.59. 1.



2.

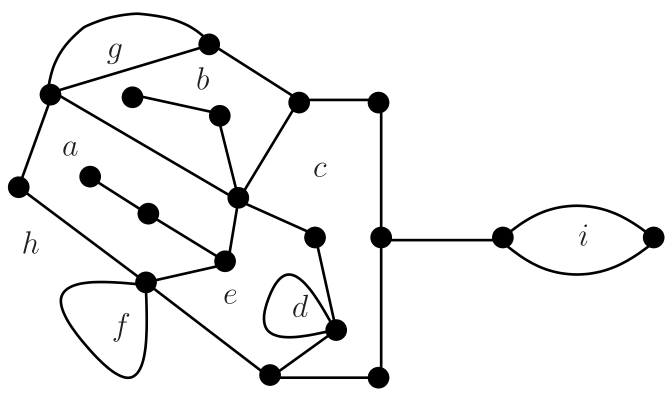


Definition 2.60. Let D be a planar graph (drawn on the plane). If x is a point of the plane not lying on D then the set of all points of the plane that can be reached from x without crossing D is called a **face** of D. One face is always unbounded and is called the **exterior face**.

(To make a rigorous definition of *face* requires the Jordan Curve theorem, which says that a simple closed curve in the plane divides the plane into two parts, one inside and one outside the curve. This theorem is beyond the scope of this course.)

Example 2.61. 1. All trees are planar with one face (which is exterior).

2. The graph below has 9 faces labelled $a, \ldots, i$. Face h is the exterior face.



Euler noticed that for a plane drawing of a platonic graph with n vertices m edges and r faces the sum $n - m + r = 2$. He went on to prove the following theorem.

Theorem 2.62 (Euler's Formula)**.** *Let G be a connected plane graph (i.e. a plane drawing of a connected graph) with n vertices, m edges and r faces. Then $n - m + r = 2$.*

Definition 2.63. Let F be a face of a planar graph. The **degree** of F, denoted $\deg(F)$ is the number of edges in the boundary of F, where edges lying in no face except F count twice. (To compute $\deg(F)$ walk once round the boundary of F, counting each edge on the way.)

The degree of a face has much in common with the degree of a vertex. Compare the following to Lemma 3.1.

Lemma 2.64. *If G is a planar graph with m edges and r faces $F_1, \ldots, F_r$ then*

$$\sum_{i=1}^{r} \deg(F_i) = 2m.$$

Corollary 2.65. *If G is a simple connected planar graph with $n \geq 3$ vertices and m edges then $m \leq 3n - 6$.*

Corollary 2.66. *If G is a connected simple planar graph with $n \geq 3$ vertices, m edges and no cycle of length 3 then $m \leq 2n - 4$.*

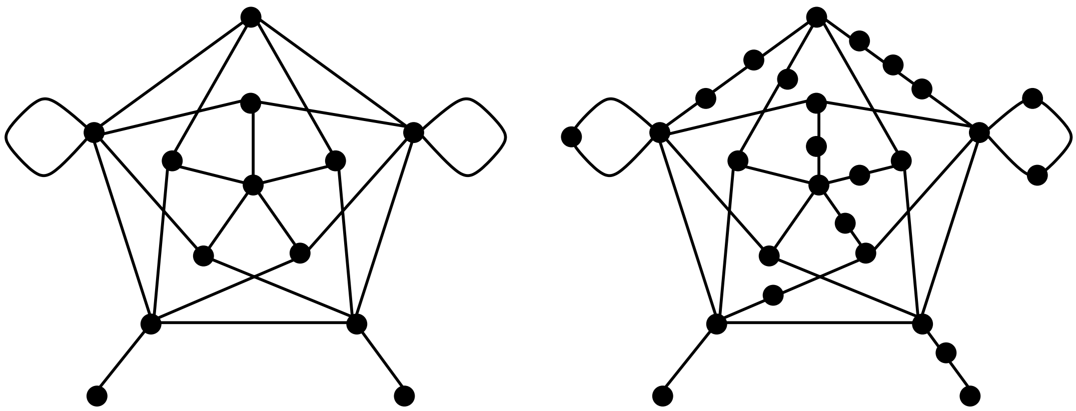
Theorem 2.67. *The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are both non-planar.*

If a graph G is non-planar then any graph which contains G as a subgraph is also non-planar. It follows that if a graph contains K_5 or $K_{3,3}$ as a subgraph it must be non-planar. We can however prove a stronger result. First some terminology.

Definition 2.68. A graph H is a **subdivision** of a graph G if H is obtained from G by the addition of a finite number of vertices of degree 2 to edges of G.

Note that in this definition it is possible to add no vertices and so a graph is a subdivision of itself.

Example 2.69. The graph H below right is a subdivision of the graph G below left.



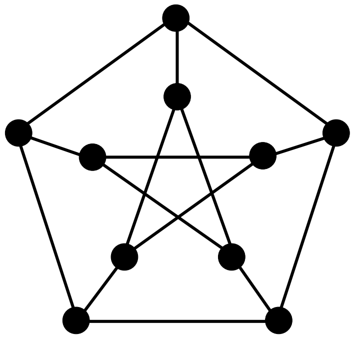
G H

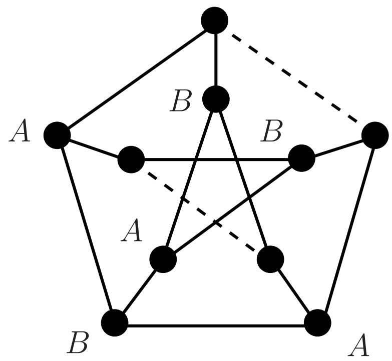
The following theorem is an easy consequence of Theorem 2.67.

Theorem 2.70. *If G is a graph containing a subgraph which is a subdivision of K_5 or $K_{3,3}$ then G is non-planar.*

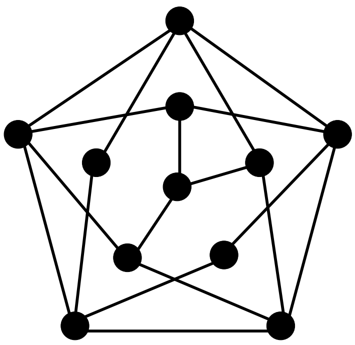
Example 2.71. Neither Corollary 2.65 nor Corollary 2.66 are sufficient to show that the graphs of this example are non-planar.

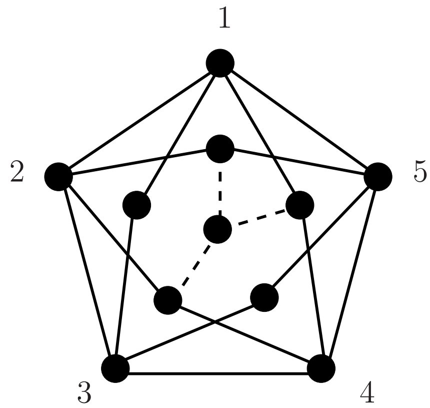
1. The Petersen graph shown below has 10 vertices and 15 edges. The diagram on the right shows a subgraph which is a subdivision of $K_{3,3}$. Therefore the graph is non-planar. (Vertices which are not labelled A or B are those added in the subdivision.





2. The graph shown below has 11 vertices and 18 edges. The diagram on the right shows a subgraph which is a subdivision of K_5. Therefore the graph is non-planar.





A more surprising theorem, which we shall not prove here, is known as Kuratowski's theorem:

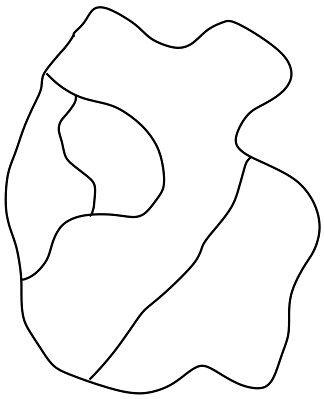
Theorem 2.72 (Kuratowski). *If G is a non-planar graph then G contains a subgraph which is a subdivision of K_5 or $K_{3,3}$.*

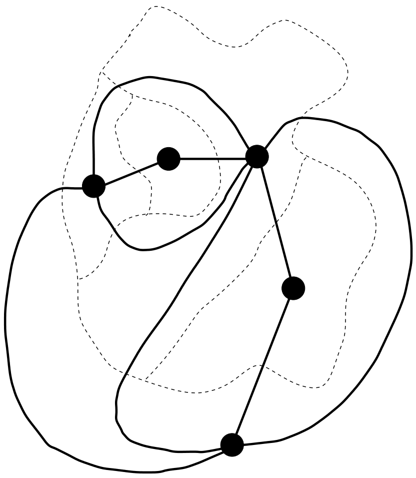
2.12 The Four-Colour Problem

In 1852 De Morgan made the conjecture that any map of countries could be coloured using only 4 colours, in such a way that countries with a common border would have different colours.

We can interpret this question in terms of graph theory: given a map of countries we construct a planar drawing of a graph as follows. Place one vertex in each country (the "capital" of the country). Join two vertices with an edge whenever their countries have a common border.

Example 2.73. The map of countries on the left gives rise to the planar graph on the right.





Now, colouring a map with four colours, so no two neighbouring countries are the same colour, corresponds to colouring the vertices of the graph with four colours so that no two adjacent vertices are the same colour. Such a colouring is called a **4-colouring** of the graph. If it could be shown that any planar graph without loops is 4–colourable then it would follow that every map of countries can be coloured as required by De Morgan. The graph theoretic version of the conjecture is therefore:

Conjecture 2.74 (The 4–colour problem)**.**
Every simple planar graph is 4*–colourable.*

The problem has a long and chequered history.

1852 De Morgan proposes the 4–colour conjecture.

1873 Cayley presents a proof to the London Mathematical Society. The proof is fatally flawed.

1879 Kempe publishes a proof; which collapses.

1880 Tait gives a proof which turns out to be incomplete.

1976 Appel & Haken at the University of Illinois prove the 4–colour conjecture: using thousands of hours of CPU time on a Cray computer.

A problem with Appel & Haken's proof is that the program runs for so long that it is impossible to verify manually. We cannot even to be sure that the hardware performed well enough, over such an extended period, to give a reliable result.

By contrast a 6–colour theorem is easy to prove.

Theorem 2.75.
Every simple planar graph G is 6*–colourable.*

A proof of a 5–colour theorem, although somewhat harder, can be found in most introductory texts on graph theory.

We finish with a result which links vertex and edge colouring. The map of countries shown above does itself constitute a graph: put a vertex at each point where two borders meet. The resulting graph is plane, connected, regular of degree three and has no bridges or loops. Furthermore any "reasonable" map of countries constitutes a plane drawing of a graph with all these properties. The 4–colour conjecture states that the faces of such a plane graph can be coloured using 4 colours, where **colouring** means that no edge meets two faces of the same colour. In 1880 Tait made the following connection between 4–colouring of faces and edge–colouring.

Theorem 2.76. *Let G be a plane drawing of a graph which is connected, regular of degree three and has no bridges or loops. Then the faces of G can be coloured using 4 colours if and only if G has a proper edge–colouring using 3 colours.*

MAS2216/MAS3216 Enumeration and Combinatorics

Assignment Exercises

Your answers to questions should show your working and reasoning. Marks are not awarded for correct answers but for comprehensible, well-reasoned arguments leading to these correct answers, written clearly and legibly.

The list of questions to be handed in for assesment, and their due dates, will be circulated by email.

1 Enumeration

1.1 I have a drawer full of socks, 14 red and 8 blue. If I get dressed in the dark, so I can't tell which colour socks I'm getting out, how many socks do I need to get out to be sure I have a pair the same colour. How many do I need to get out to guarantee I get a blue pair and how many do I need to get out to guarantee a red pair?

1.2 27 boys and 15 girls are to be lined up in a row. How many ways can this be done? How many ways can this be done so that all the girls are together? Answer both parts of the question first under the assumption that all the boys and girls look different from each other and second under the assumption that all boys look the same and all girls look the same.

1.3 How many words can be made from the letters A, A, A, A, A, A, L, L, M, M, M, N, N, N. How many can be made with all N's following each other (that is containing subword NNN)?

1.4 How many sequences of length n can be made from the 36 symbols 0–9 and A–Z with no symbol repeated twice, where n is (a) 36; (b) 14; (c) > 36?

1.5 I wish to seat 16 people be around 2 tables each with 8 places. I want to do this on as many consecutive nights as possible. However there are some conditions that may be set as follows.

(a) If a person sits in the same chair on 2 separate occasions then I will be thrown into a snake pit.

(b) If the same two people sit next to each other and at the same table on two separate occasions I shall have to drink Newcastle Brown Ale for the rest of the evening.

(c) If the same two people sit next to each other (at either table) on two separate occasions I shall have to watch England playing cricket.

In case (5a) how many nights can I survive without being thrown to the snakes. In case (5b) how many nights can I survive without having to drink a bottle of brown. In case (5c) how many nights can I survive before the final punishment.

1.6 How many subsets does the set $\{a_1, \ldots, a_n\}$ of n elements have? How many subsets are there containing the element a_1? Prove your claim.

1.7 As shop keeps 6 styles of shoes and you want to give each of your 24 nieces one pair of shoes. It turns out the shop has in stock exactly 4 pairs in each style. How many ways are there of distributing shoe styles to nieces. (All nieces are different.)

1.8 How many ordered 5-subsets of $\{0, \ldots, 9\}$ are there? How many of these start with 0? How many positive integers of exactly 5 digits, but with no two digits the same, are there? (In base 10.)

1.9 A competition is to be organised during a 45 day period. The organiser must schedule 8 match days and a 2-day final (the 2 finals days being consecutive). Between each of the match days there must be at least 3 rest days with no matches. After the eighth match day and before the finals there must be at least 5 days of rest with no matches. How many ways are there to schedule the competition?

1.10 Let k and n be positive integers such that $k \leq (n-1)/2$. Show, using an algebraic argument, that

$$\binom{n}{k} \leq \binom{n}{k+1},$$

with equality if only if $k = (n-1)/2$.

1.11 Prove that

$$\sum_{a_1+a_2+a_3+a_4+a_5=n} \binom{n}{a_1, a_2, a_3, a_4, a_5} = 5^n \text{ and}$$

$$\sum_{a_1+a_2+a_3+a_4+a_5=n} \binom{n}{a_1, a_2, a_3, a_4, a_5} (-1)^{a_1+a_3+a_5} = (-1)^n,$$

where the a_i are non-negative integers.

1.12 Use Inclusion-Exclusion to find how many integers n there are with $1 \leq n \leq 100$ which are not divisible by any of the numbers 3, 5 or 7.

1.13 A class of 8 students take a test and then mark each others work. The scripts must be distributed to the students for marking so that noone marks their own script (and each student marks exactly one script). How many ways can this be done? (Compute this number explicitly.)

1.14 Find the number of compositions of $2m$ into parts of even size, where m is a non-negative integer. How many ways are there of putting 24 identical socks into (labelled) piles each of which has an even (positive) number of socks in it. (The number of piles is not fixed; but must clearly be at least 1 and at most ....)

1.15 Use Lemma 1.36 and examples from the notes to find $S(4, k)$, for $k = 1, 2, 3, 4$. (You can quote the value of $S(4, 2)$ from the notes.)

Now use Lemma 1.36 again to find $S(5, k)$, for $k = 1, \dots, 5$. (Don't quote the value of $S(5, 2)$ found in the problem class questions.)

Next compute $S(5, 3)$ again, this time using Theorem 2.19.

How many ways are there of dividing a class of 5 people into 3 non-empty groups?

1.16 Write down all integer partitions of 12 into 5 parts and so find $p_5(12)$.

2 Graph Theory

2.1 There are 11 non-isomorphic simple graphs with 4 vertices. Draw all those with (a) no edges; (b) 1 edge; (c) 2 edges; (d) 3 edges; (e) 4 edges (f) 5 edges; (g) 6 edges; (h) more than 6 edges (if any).

2.2 If v is a vertex of a graph G then we define the graph $G - v$ to the graph obtained from G by deleting v and all edges incident to v. If e is an edge of G then $G - e$ is the graph obtained by deleting the edge e from G. Let G be a graph with n vertices and m edges. Let v be a vertex of G of degree d and let e be an edge of G. How many vertices and edges have $G - v$ and $G - e$? Justify your answers.

2.3 In the graph G of Figure 1 below find

(a) an open trail of length 7, which is not a path;

(b) an open path of length 11;

(c) cycles of length 6 and 10. (A *cycle* is a closed path of length at least 1.)

Does the graph G have any circuits, of positive length, which are not cycles? If not why not?

2.4 Which of the following graphs are Eulerian, which are semi-Eulerian but not Eulerian and which are not semi-Eulerian? Wheel graphs W_1, W_2 and W_n, where $n \geq 3$. Complete graphs K_5, K_{2n}, K_{2n+1}, where $n \geq 1$.

2.5 Secret service agents A, B, C, D, E and F must meet in pairs in an underground bunker to exchange vital information. For security reasons no two pairs may meet at once. To preserve the integrity of the information it is preferred that one of the participants at each meeting (except the last) is present at the next. The following

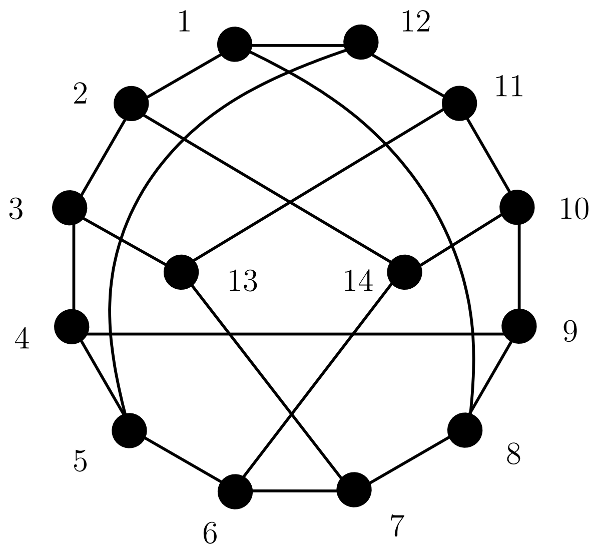
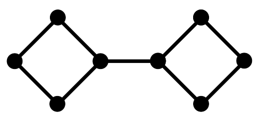


Figure 1: G

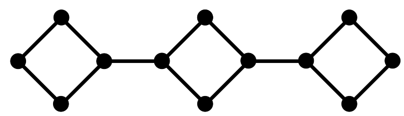
pairs must meet (in some order).
A must meet B and F.
B must meet C, D and E.
C must meet E and F.
D must meet F.
E must meet F.

How can this be done so that only these pairs meet, in the shortest possible time. Justify your answer and give an appropriate ordering if possible. [**Hint:** Construct a graph with vertices corresponding to agents and two vertices joined if their agents meet. The requirement is then to list all edges in such a way that for each edge in the list one of its vertices appears in the next edge on the list. This means you need an ________ _____.]

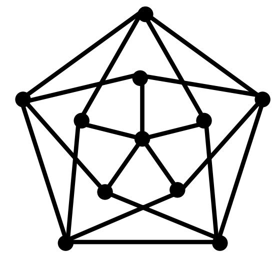
2.6 Which of the following are Hamiltonian and which are semi-Hamiltonian. Give your reasons. In particular find Hamiltonian closed paths for those which are Hamiltonian and Hamiltonian paths for those which are semi-Hamiltonian but not Hamiltonian.



(a)



(b)



(c)

2.7 Draw all trees with 7 vertices (there are 11).

2.8 Let T_1 and T_2 be spanning trees of a connected graph G.

(a) Show that if e is an edge of T_1 then there is an edge f of T_2 such that $(T_1 - e) \cup f$ is a spanning tree for G. [**Hint:** Suppose $e = \{u, v\}$. Then $T_1 - e$ is the disjoint union of two components with u and v in different components. There is a path from u to v in T_2 some edge of which must join one component to the other.]

(b) Deduce that T_1 can be transformed into T_2 by replacing edges of T_1 with edges of T_2, one at time in such a way that a spanning tree for G is obtained at each stage.

2.9 Let G be a simple connected planar graph with $n \geq 3$ vertices, m edges and r faces. Assume that every vertex of G has degree at least 3.

(a) Show that $2m \geq 3n$ and use Euler's formula to deduce that

$$2 \leq r - (m/3).$$

(b) Assume now that all faces of G have degree either 5 or 6. Let p and h be the number of faces of degree 5 and 6, respectively, so $r = p + h$. Show that $2m = 5p + 6h$. Use this and the previous part of the question to show that $p \geq 12$ in this case.

(c) Under the conditions of (9b), show that if G is also regular of degree 3 then $p = 12$.

(d) Draw a (planar) diagram of a simple connected planar graph which has 7 faces altogether and

- an exterior face of degree 10,
- 6 faces of degree 5, precisely one of which does not meet the exterior face,
- 5 vertices of degree 2, all meeting the exterior face, and all other vertices of degree 3.

Explain how this can be used to construct a simple planar connected graph which is regular of degree 3 and has 12 faces of degree 5.

(e) Find, and draw diagrams, of all simple connected planar graphs such that every face is of degree 3 and

- all vertices have degree 3; and then
- all vertices have degree 4.

Show all your working and before drawing any diagram write down the number of vertices, edges and faces of the graph you are going to draw.

MAS2216/MAS3216 Enumeration and Combinatorics

Problem Class Exercises

1 Enumeration

1.1 At a canteen for lunch I can either have a sandwich or a pizza. The sandwiches come with 4 different types of bread and 7 different fillings. The pizzas have 3 different types of base and 6 different toppings. How many choices do I have?

1.2 A backgammon competition is organised for n players. During the competition each player will play all other $n-1$ players. Show that at any given point during the competition there are at least two players who have completed the same number of their matches as each other.

1.3 How many different words can be made from the letters U, U, U, U, V, V, V, W, W, X, X? How many are there such that the two X's do not occur next to each other?

1.4 How many injective functions are there from a set of size k to a set of size n? Compare the number of functions from a set X of size k to a set Y of size 365 to the number of injections from X to Y. Use this to find the probability that amongst k people two of them have the same birthday. (Ignore leap years.)

1.5 Suppose 12 people are to be seated around a table but that one person, Cleopatra, always sits at the head of the table. How many seating arrangements are possible, if two arrangements are the same only if people sit in the same chairs?

1.6 How many ways can 12 people be seated at a round table if two seating arrangements are said to be the same when the same people sit next to each other.

1.7 How many different ways are there of seating 16 people at 2 circular tables, one of 9 and one of 7. First assume that the chairs are labelled and two seating plans are the same only if the same people sit in the same chairs. Then answer the question under the assumption that two seating plans are the same if the same people sit next to each other.

1.8 There are 30 students in a class, 19 female and 11 male. How many ways are there to choose a group of 7 containing at least one male and one female.

1.9 In spelling test every question is worth 1 mark. The test is taken by 10 children and a total of 32 answers are correct. How many different mark distributions are possible.

1.10 A group of 6 families from different countries swap houses for a 1 week holiday. Nobody can go to their own house so how many ways can this be arranged?

1.11 A student has to take 15 hours of classes every week. (Classes run Mon-Fri.) At least 3 classes must be taken on Monday, at least 1 on Tuesday, at least 3 on Wednesday and at least 2 on Friday.

(a) How many different possibilities are there for the numbers of classes that can be taken on each day?

(b) What's the answer if exactly one class must be taken on a Tuesday?

1.12 Find the number of weak compositions of 27 into 7 parts each of which is odd. Generalise this to find the number of weak compositions of $2n + 1$ into $2k + 1$ odd parts, for integers $0 \leq k \leq n$.

1.13 List all the partitions of the set $\{1, 2, 3, 4, 5\}$ into 2 non-empty subsets and so find the Stirling number $S(5, 2)$.

1.14 Find formulas for

(a) $S(n, 2)$, and

(b) $S(n, n - 2)$, for all $n \geq 2$.

2 Graph Theory

2.1 Where possible draw the graphs below. If you can't draw the graph say why not.

(a) A simple graph with 1 edge and 2 vertices.

(b) A simple graph with 2 edges and 2 vertices.

(c) A non-simple graph with no loops.

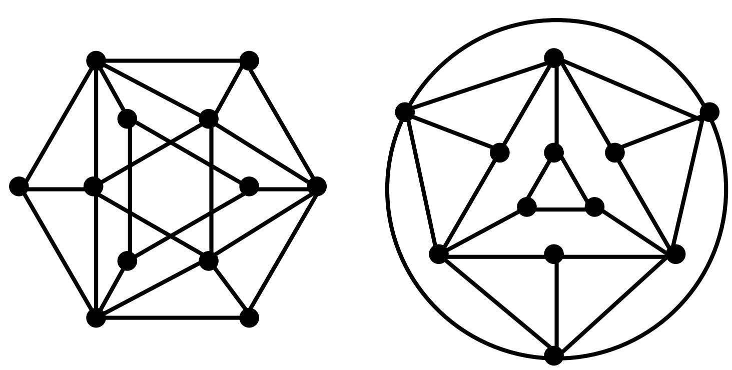
(d) A non-simple graph with no multiple edges.

(e) A graph with 6 vertices and degree sequence $\langle 1, 2, 3, 4, 5, 5 \rangle$.

(f) A simple graph with 6 vertices and degree sequence $\langle 1, 2, 3, 4, 5, 5 \rangle$.

(g) A simple graph with 6 vertices and degree sequence $\langle 2, 3, 3, 4, 5, 5 \rangle$.

2.2 Show, by labelling the vertices, that the graphs below are isomorphic.



2.3 (a) Is it true that any two isomorphic graphs have the same number of vertices? If so why?

(b) Let G_1 and G_2 be graphs and let ϕ be an isomorphism from G_1 to G_2. If v is a vertex of G_1 is it true that $\deg(v) = \deg(\phi(v))$ and if so why? Show, using the Handshaking Lemma, that G_1 and G_2 have the same number of edges.

(c) Show that isomorphic graphs have the same degree sequence.

(d) If two graphs have the same degree sequence, need they be isomorphic?

2.4 Let G be a simple graph with $n \geq 2$ vertices. Can G have a vertex of degree 0? Can G have a vertex of degree $n-1$? Can G have a vertex of degree n or more?

(a) Show that G cannot have degree sequence $\langle 0, 1, \ldots, n-1 \rangle$. [*Hint:* Consider the vertex of degree 0 and that of $n-1$.]

(b) Show that G must have 2 vertices of the same degree.

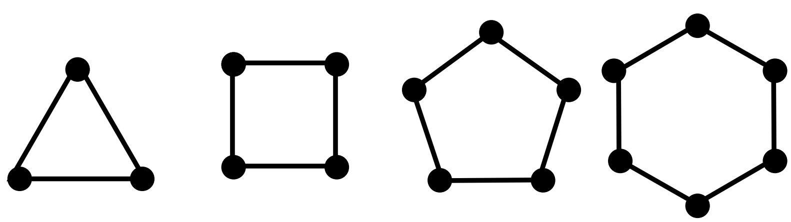
2.5 Which of the graphs (a), (b), (c) and (d) are isomorphic to subgraphs of the graph G where G is of

(i) the complete graph K_n, where $n \geq 1$ (the answer may not be the same for all n);

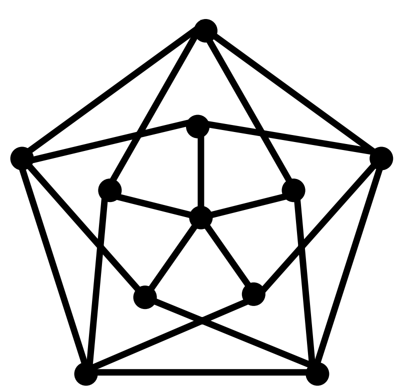
(ii) the Petersen graph;

(iii) the graph of Example 3.6 in the notes;

(iv) the graph G below?



(a) (b) (c) (d)



AJD August 2009

G

2.6 In the Petersen graph find

(a) a trail of length 5;

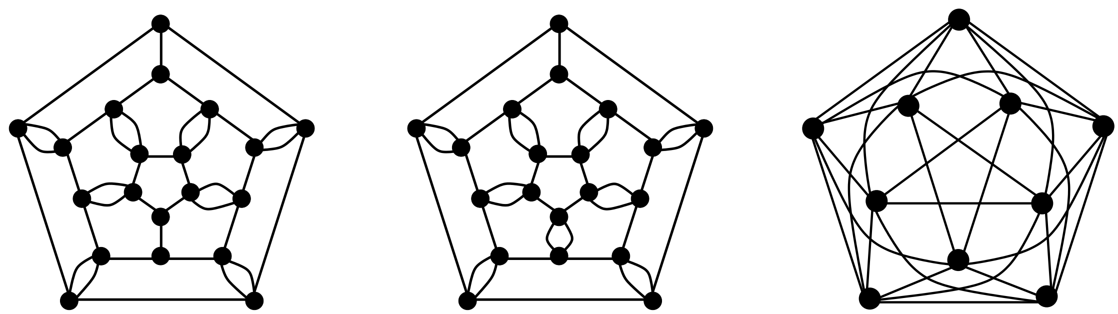
(b) a path of length 9;

(c) cycles of length 5 and 9.

Do you think the Petersen graph has any circuits, of positive length, which are not cycles? If not why not?

2.7 Let G be a graph and let u and v be vertices of G (which may or may not be the same). Suppose that G contains two distinct paths P and P' from u to v. ("Distinct" means "not equal".) Show that G contains a cycle.

2.8 Which of the following graphs are Eulerian, which are semi-Eulerian but not Eulerian and which are neither?

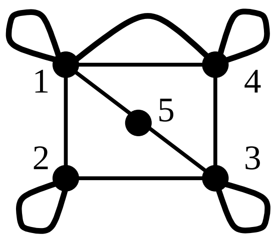


(a) (b) (c)

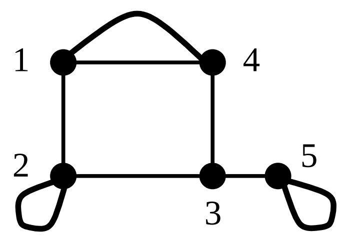
2.9 A collection $C_1, \ldots, C_n$ of Hamiltonian closed paths of a graph G is called a *decomposition into Hamiltonian closed paths* if every edge of G belongs to one and only one of the closed paths C_i. It is a fact that for all $d \geq 1$ the complete graph K_{2d+1} has a decomposition into d Hamiltonian closed paths. Find a decomposition into 3 Hamiltonian closed paths of the complete graph K_7.

Use the above fact to solve the following problem. King Arthur and his knights wish to sit at the round table every evening in such a way that each person has different neighbours on each occasion. If there are 10 knights (and 1 king) for how long can they do this? If King Arthur wishes to make sure this can be done for 7 evenings in a row how many knights must he have?

2.10 Draw all spanning trees for the graphs below.



(a)



(b)

2.11 Let G be a connected graph and let e be an edge of G.

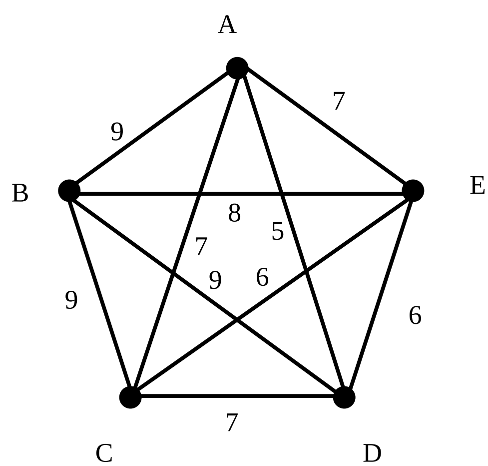
(a) Show that if e appears in every spanning tree for G that e is a bridge.

(b) Show that if e appears in no spanning tree for G then e is a loop.

2.12 (a) Find lower bounds for the travelling salesman problem corresponding to the graph below

i. by removing vertex B and

ii. by removing vertex E.



(b) Find the solution to the travelling salesman problem for this graph by inspection.

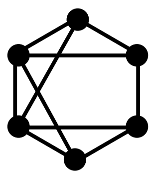
2.13 The *girth* of a graph is the length of its shortest cycle. Let G be a simple planar connected graph with $n \geq 3$ vertices and m edges. Show that if the girth of G is 5 then

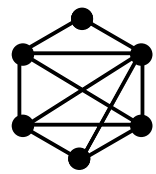
(a) $m \leq \frac{5}{3}(n-2)$.

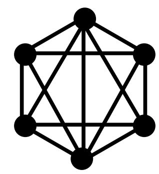
(b) Find the girth of the Petersen graph and use the inequality above to show that the Petersen graph is non-planar.

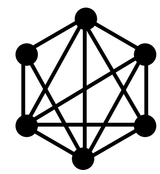
(c) Generalize 2.13a) to planar graphs of girth g.

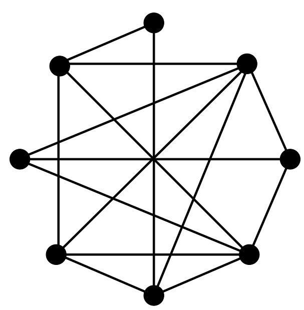
2.14 Which of the following graphs are planar? For those that are planar give a drawing in the plane. For those that are non-planar find subgraphs which are subdivisions of K_5 or $K_{3,3}$.

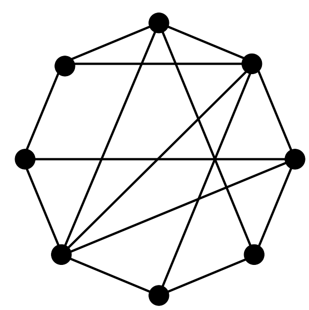

(a)


(b)


(c)


(d)


(e)


(f)

2.15 Let G be a graph with n vertices and m edges.

(a) Show that if every vertex of a graph has degree at least 6 then $m \geq 3n$.

(b) Use the previous part of the question and Corollary 2.65 to show that if G is a simple connected planar graph then G has at least one vertex of degree $d \leq 5$.

Student Number: **Degree Programme:**

NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 1 Mock

MAS2216

Enumeration and Combinatorics

Time allowed: 1 hour 30 minutes

Credit will be given for ALL answers to questions in Section A, and for the best TWO answers to questions in Section B. No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.

Marks allocated to each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are FIVE questions in Section A and THREE questions in Section B.

Answers to questions in Section A should be entered directly on this question paper in the spaces provided. Rough work related to Section A and solutions to the questions of Section B should be done in the answer book. The rough work will not be marked. This question paper should be inserted in the answer book which is handed in at the end of the examination.
You may submit part of your answer to question B8 on the diagrams on the question paper.

A1	A2	A3	A4	A5	**A**	B6	B7	B8	**B**	**Total**

SECTION A

A1. 25 boys and 16 girls are to be lined up in a row. How many ways can this be done? How many ways can this be done so that all the girls are together? Answer both parts of the question first under the assumption that all the boys and girls look different from each other and second under the assumption that all boys look the same and all girls look the same.

Question A1 continued on next page

[10 marks]

A2. Prove that

$$\sum_{a_1+a_2+a_3+a_4=n} \binom{n}{a_1, a_2, a_3, a_4} = 4^n \text{ and}$$

$$\sum_{a_1+a_2+a_3+a_4=n} \binom{n}{a_1, a_2, a_3, a_4} (-1)^{a_2+a_4} = 0,$$

where the a_i are non-negative integers.

[4 marks]

A3. Use Inclusion-Exclusion to find how many integers n there are with $1 \leq n \leq 100$ which are not divisible by any of the numbers 3, 5 or 7.

Question A3 continued on next page

[10 marks]

A4. State the Handshaking Lemma and use it to calculate the number of edges of the graph K_{24}.

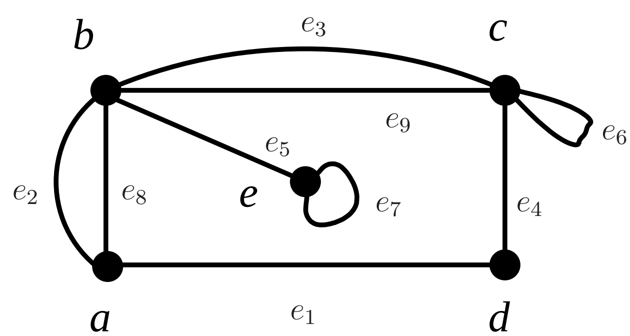
[4 marks]

A5. In the graph below, which has vertices a, b, c, d, e and edges $e_1, \cdots, e_9$, give an example of

(a) A walk which is not closed and is not a trail;

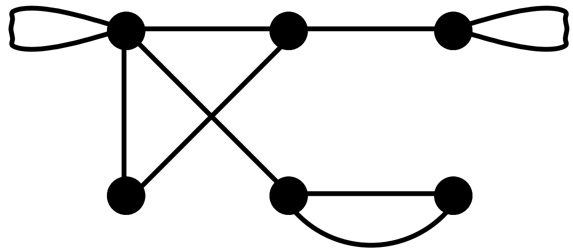
(b) A trail which is not closed and is not a path;

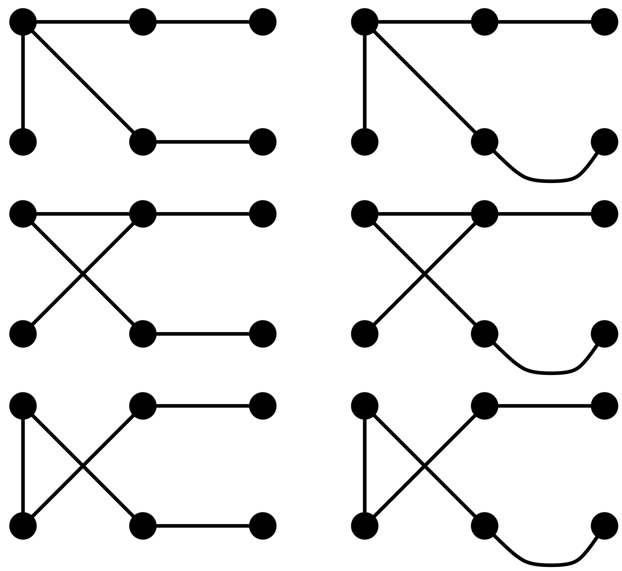
(c) A closed trail which has positive length and is not a cycle.



[6 marks]

A6. Draw all 6 spanning trees for the graph below.





[6 marks]

SECTION B

B7. (a) Give the definition of a *partition* of a set X into k parts and of the *Stirling number* $S(n, k)$.

(b) The nth *Bell number* $B(n)$ is defined to be the number of **all** partitions of $\{1, \ldots, n\}$ into nonempty parts. That is

$$B(n) = \sum_{k=1}^{n} S(n, k).$$

(i) Consider the set $X = \{1, \ldots, n, n+1\}$.Show that there are

$$\binom{n}{i} B(i)$$

partitions of X into nonempty parts such that $n+1$ lies in a block of size $n - i + 1$.

(ii) Use the previous part of the question to show that

$$B(n+1) = \sum_{i=1}^{n} \binom{n}{i} B(i).$$

(iii) Prove by induction that if $n \geq 3$ then $B(n) < n!$.

[30 marks]

B8. (a) State a theorem relating the number of vertices of a tree to the number of its edges.

(b) A *bridge* in a connected graph G is an edge e of G such that the graph $G - e$, formed from G by removing e, is not connected. Show that every edge of a tree is a bridge.

(c) (i) Let T be a tree. Show that addition of a new edge to T creates a cycle.

(ii) Let G be a graph and suppose that u and v are vertices of G lying in separate connected components. Show that there is no path from u to v in G.

(iii) Let G be a graph with no cycles such that the addition of any new edge to G creates a cycle. Show that G is a tree.

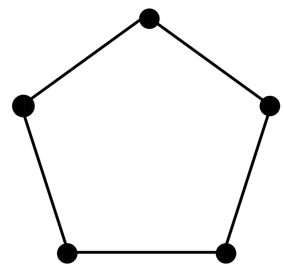
(d) (i) Let F be a forest with k connected components and n vertices. Show that F has $n - k$ edges.

(ii) Let H be a graph with n vertices, $n - k$ edges and k connected components. Show that H is a forest.

[30 marks]

B9. (a) Let n be an integer $n \geq 3$. Show that a simple graph G with $n-2$ vertices has at most $\frac{1}{2}(n^2 - 5n + 6)$ edges.

(b) A *plane triangulation* is a drawing of a graph D in the plane, so that edges do not cross and such that every face of D has degree 3. Draw a plane triangulation D which has the graph Q below as a subgraph. (Don't forget the exterior face.)



Q

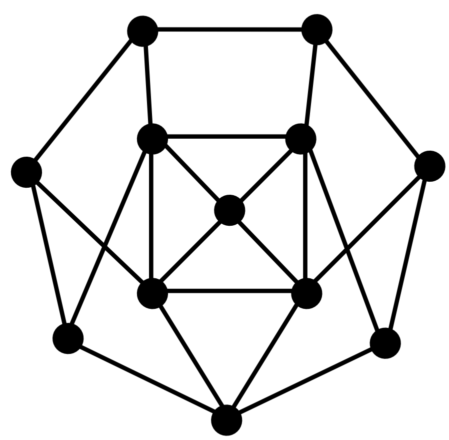
Let P be a planar graph.

(i) Let F be a face of P of degree $d > 3$. Show that by adding a new vertex inside F a new plane drawing can be made in which F is replaced by d faces of degree 3.

(ii) Explain how to construct a plane triangulation D which has P as a subgraph.

(iii) Let D be a plane triangulation with n vertices and m edges. Use Euler's Theorem to prove that $m = 3n - 6$.

(c) Find a subgraph of the graph below which is a subdivision of K_5 or $K_{3,3}$. Explain (in no more than one line) why this means that the graph is non-planar.



[30 marks]

THE END

SECTION A

A1. 25 boys and 16 girls are to be lined up in a row. How many ways can this be done? How many ways can this be done so that all the girls are together? Answer both parts of the question first under the assumption that all the boys and girls look different from each other and second under the assumption that all boys look the same and all girls look the same.

First assume we can tell everyone apart. There are 41 people altogether so they can be ordered in 41! different ways. If all the girls must be kept together first order the 25 boys: there are 25! ways of doing this. Fix an ordering of the boys. The line of girls can come before the first boy, between the ith and $(i+1)$th boy, for $i = 1, \ldots, 25$, or after the 25th boy. That is, in any one of 26 positions. Furthermore if we fix a position for the line of girls there are then 16! ways of ordering this line. Thus the total number of ways of lining them up with girls together is $25! \times 26 \times 16! = 26! \times 16!$. (These numbers are big enough that it's not worth writing them down.)

If we can't tell boys apart or girls apart then the collection of boys and girls constitutes a multiset of 41 elements. Therefore there are altogether

$$\frac{41!}{25!16!} = 103077446706$$

ways of lining them up. If the girls must all be kept together then there are 26 possibilities, one for each of the 26 positions in which we can place the girls.

[10 marks]

A2. Prove that

$$\sum_{a_1+a_2+a_3+a_4=n} \binom{n}{a_1, a_2, a_3, a_4} = 4^n \text{ and}$$

$$\sum_{a_1+a_2+a_3+a_4=n} \binom{n}{a_1, a_2, a_3, a_4} (-1)^{a_2+a_4} = 0,$$

where the a_i are non-negative integers.

These both follow from the multinomial theorem, the first by setting $x_i = 1$, for $i = 1, \ldots, 4$, and the second by setting $x_1 = x_3 = 1$ and $x_2 = x_4 = -1$.

[4 marks]

A3. Use Inclusion-Exclusion to find how many integers n there are with $1 \le n \le 100$ which are not divisible by any of the numbers 3, 5 or 7.

First we compute the number of integers which are divisible by 3, 5 or 7. Let A_i be the set of positive integers between 1 and 100 which are divisible by i. The largest multiple of 3 no greater than 100 is $99 = 3 \times 33$, so $|A_3| = 33$. As $5 \times 20 = 100$ we have $|A_5| = 20$. Similarly $7 \times 14 = 98$, so $|A_7| = 14$. Integers which are divisible by both 3 and 5 are precisely those divisible by 15, so $A_3 \cap A_5 = A_{15}$ and $6 \times 15 = 90$ so $|A_3 \cap A_5| = |A_{15}| = 6$. Similarly, $A_3 \cap A_7 = A_{21}$ and $4 \times 21 = 84$ so $|A_3 \cap A_7| = |A_{21}| = 4$. Also

Question A3 continued on next page

$A_5 \times A_7 = A_{35}$ and $2 \times 35 = 70$ so $|A_5 \times A_7| = 2$. Finally integers divisible by 3, 5 and 7 are precisely those divisible by 105 so there are none in the given range and $|A_3 \cap A_5 \cap A_7| = \emptyset$. Using the inclusion-exclusion theorem we have

$$\begin{aligned}
|A_3 \cup A_5 \cup A_7| &= |A_3| + |A_5| + |A_7| \\
&\quad - |A_3 \cap A_5| - |A_3 \cap A_7| - |A_5 \cap A_7| \\
&\quad + |A_3 \cap A_5 \cap A_7| \\
&= 33 + 20 + 14 - 6 - 4 - 2 + 0 = 55.
\end{aligned}$$

There are therefore 45 integers between 1 and 100 which are not divisible by any of 3, 5 or 7.

[**10 marks**]

A4. State the Handshaking Lemma and use it to calculate the number of edges of the graph K_{24}.

The Handshaking Lemma states that the sum of degrees of vertices of a graph is equal to twice the number of edges, that is:

$$\sum_{v \in V(G)} \text{degree}(v) = 2 \times \text{no. of edges of } G.$$

K_{24} has 24 vertices all of degree 23 so it has $(24 \times 23)/2 = 254$ edges.

[**4 marks**]

A5. In the graph below, which has vertices a, b, c, d, e and edges $e_1, \cdots, e_9$, give an example of

(a) A walk which is not closed and is not a trail;

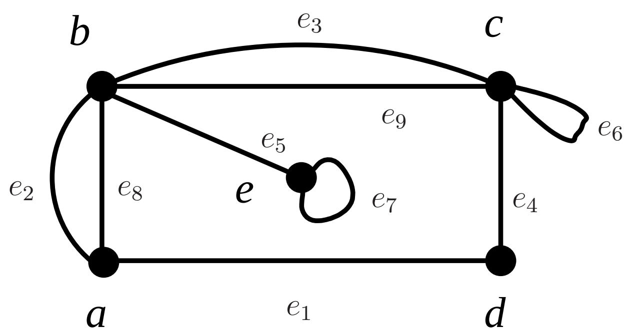
$$a, e_1, d, e_4, c, e_4, d.$$

(b) A trail which is not closed and is not a path;

$$a, e_8, b, e_9, c, e_3, b.$$

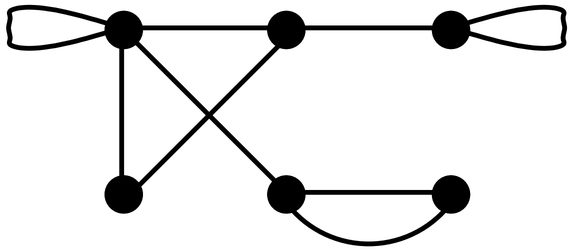
(c) A closed trail which has positive length and is not a cycle.

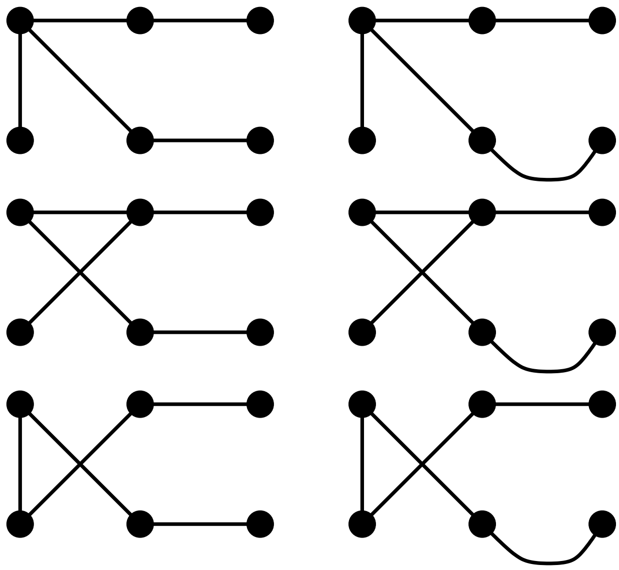
$$a, e_8, b, e_9, c, e_3, b, e_2, a.$$



[6 marks]

A6. Draw all 6 spanning trees for the graph below.





[6 marks]

Solutions to Section B

B6 (a) A *partition* of a set X into k parts is a collection $S_1, \dots, S_k$ of non-empty subsets of X such that $X = \cup_{i=1}^k S_i$ and $S_i \cap S_j = \emptyset$, whenever $i \neq j$.

The number of partitions of $\{1, \dots, n\}$ into k parts is denoted $S(n, k)$ and called a *Stirling number* (of the second kind).

(b) First consider the number of non-empty subsets of $\{1, \dots, n+1\}$ which contain $n+1$. These are the same as the subsets of $\{1, \dots, n\}$. The subsets of size $n-i+1$ containing $n+1$ are the same as the subsets of size $n-i$ of $\{1, \dots, n\}$. There are therefore

$$\binom{n}{n-i} = \binom{n}{i}$$

of them. Now given a block of size $n-i+1$ containing $n+1$ there are a further $(n+1)-(n-i+1) = i$ elements of $\{1, \dots, n\}$ to be partitioned into non-empty subsets. By definition there are $B(i)$ partitions of the latter set. Thus the total number of partitions of $\{1, \dots, n+1\}$ into non-empty parts such that $n+1$ is in a block of size $n-i+1$ is $B(i)\binom{n}{i}$ as claimed.

(c) Given any partition of $\{1, \dots, n+1\}$ into non-empty parts the element $n+1$ lies in a block of size n_i+1, for some i between 0 and n. From the previous part of the question therefore the number of such partitions is

$$B(n+1) = \sum_{i=0}^{n} B(i)\binom{n}{i}.$$

(d) When $n = 3$ we have $n! = 6$. $B(3)$ is the number of partitions of $\{1, 2, 3\}$ into non-empty parts. Since $S(3, 1) = 1 = S(3, 3)$ and there are 3 partitions into 2 parts: $\{1, 2\}$ and $\{3\}$,
$\{1, 3\}$ and $\{2\}$, and
$\{2, 3\}$ and $\{1\}$, we have $S(3, 2) = 3$, so $B(3) = 5$. Hence the result holds when $n = 3$.

Assume that for some $m \geq 3$ we have $B(k) \leq k!$, whenever $k \leq m$. We wish to show the same holds of $m+1$. we have

$$\begin{aligned}
B(m+1) &= \sum_{i=0}^{m} B(i)\binom{m}{i} \\
&< \sum_{i=0}^{m} i!\binom{m}{i}, \text{ using the inductive assumption,} \\
&= \sum_{i=0}^{m} \frac{i!m!}{i!(m-i)!} \\
&= \sum_{i=0}^{m} \frac{m!}{(m-i)!} \\
&= m! \sum_{i=0}^{m} \frac{1}{(m-i)!} < m!(m+1) = (m+1)!.
\end{aligned}$$

Hence $B(m+1) < (m+1)!$ as required. By induction the result therefore holds for all $n \geq 3$.

B7 (a) A connected graph with n vertices is a tree if and only if it has $n-1$ edges.

(b) Suppose T is a tree with n vertices. Then T is connected and has $n-1$ edges. Moreover T has no cycle. Now remove an edge e to form the graph $T-e$. This graph still has no cycles. If it is connected then it must be a tree. However T has n vertices and $n-2$ edges so cannot be a tree: therefore it is not connected and e is a bridge.

(c) (i) As T is a tree it has n vertices and $n-1$ edges, for some integer n. Adding a new edge we obtain a connected graph with n vertices and n edges. From the result above this graph cannot be a tree. As it is connected it must, therefore, contain a cycle.

(ii) If there is a path P from u to v then connected component of G containing u contains all the edges and vertices of P and so contains v. Hence u and v are in the same connected component, a contradiction.

(iii) As G contains no cycle we need only show that G is connected. Suppose that G is not connected and that u and v are vertices of G lying in separate connected components. Then there is no path from u to v in G. Add the new edge $e=\{u,v\}$ to G. Now, by assumption this new graph contains a cycle C. There is no cycle in G so e must be an edge of C. Now removing e we leave a path in G from u to v, a contradiction. Hence G is connected, as required.

(d) (i) Let F have m edges and connected components $T_1,\dots,T_k$. Then T_i is a tree, for all i. Suppose T_i has n_i vertices and m_i edges. Then $m_i=n_i-1$, for all i. Now $n=\sum_{i=1}^k n_i$ and

$$m=\sum_{i=1}^k m_i=\sum_{i=1}^k (n_i-1)=(\sum_{i=1}^k n_i)-k=n-k,$$

as required.

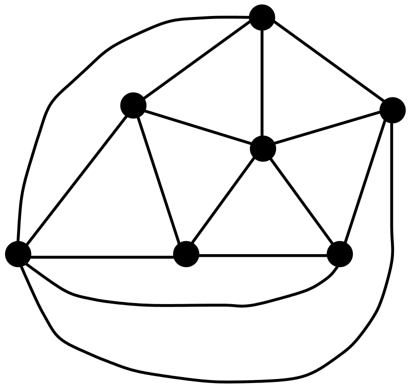
(ii) We must show that H contains no cycle. Suppose on the contrary that H does have a cycle and let e be an edge of this cycle. Let H' be the connected component of H containing e. Then H' is connected, using the given fact. Hence $H-e$ has k connected components n vertices and $n-k-1$ edges. If $H-e$ has a cycle continue by removing an edge of a cycle and repeat this process untill there are no cycles left and c edges have been removed from H. At this stage we have a graph G with no cycles and so G is a forest. Moreover G has n vertices k connected components and $n-k-c$ edges. From the previous part of the question we must have $c=0$, so in fact H had no cycle and was a forest.

B8 (a) As G is simple with $n-2$ vertices, a vertex of G can have degree at most $n-3$. Thus

$$|E(G)| = \frac{1}{2}\sum_{v\in V(G)} \deg(v) \leq \frac{1}{2}(n-2)(n-3) = \frac{1}{2}(n^2-5n+6),$$

as required.

(b) A plane triangulation having the graph Q as a subgraph:



(i) Let F be a face of P of degree d. Place a new vertex v_F inside F and add an edge joining v_F to each vertex of P which occurs in the boundary of F. This cuts the face F up into d faces of degree 3.

(ii) Repeating the process described in the previous part of the solution, for each face of P, we obtain a plane triangulation D which has P as a subgraph.

(iii) As G is a plane triangulation we have

$$2m = \sum_{f \text{ a face}} \deg(f) = 3r,$$

where r is the number of faces of G. From Euler's Theorem

$$2 = n - m + r = n - m + \frac{2}{3}m = n - \frac{m}{3}.$$

Therefore $6 = 3n - m$, so $m = 3n - 6$.

(c) The following is a subdivison of K_5: where the vertices of K_5 are $1, 2, 3, 4$ and 5. A graph which has a subgraph isomporphic to a subdivision of K_5 cannot be planar, so this graph is non-planar.