Knot Invariants and Braids

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Abstract

We consider a knot to be an embedding of the unit circle in $\mathbb{R}^3$. Knots are equivalent if one can be continuously deformed to obtain the other. The classical problem in knot theory is deciding whether two knots are equivalent. This problem requires a knot invariant. A knot invariant is a function which is preserved under equivalence of knots. We aim to define a significant knot invariant known as the Jones Polynomial. To obtain the original definition of the Jones Polynomial we study braids and aim to understand their relation to knots. In the final sections we study some applications of the Jones Polynomial.

Contents

1 Introduction

We consider a knot K to be a knotted loop of string. We can deform the knot without breaking the string such that the resulting knot is equivalent to the original. The classical problem of knot theory is whether two knots are equivalent. To study knots effectively we work with planar projections known as knot diagrams. We present these ideas formally in Section 2. We apply the ideas presented throughout the report to the trefoil knot diagram and it's mirror image in Figure 1.1. Considering these knots there appears to be no way of continuously deforming one into the other. However this is not enough to prove they are distinct. In order to prove two knots are distinct we require the idea of a knot invariant; a function which is preserved under equivalence of knots. It is a difficult problem to define powerful knot invariants which are efficient to compute. The focus of the report is the Jones polynomial; a breakthrough knot invariant discovered using a representation of the braid group $\mathcal{B}_n$.

Figure 1.1: The trefoil knot and the mirror image of the trefoil knot diagrams respectively

Braids can be considered intuitively as n strands woven between n end points. The set of n stranded braids form a group $\mathcal{B}_n$. Section 3 is concerned with understanding the relation between knots and braids. By Alexander's Theorem 3.2.3 it can be shown every knot K gives rise to a braid β via the closure of the braid $\overline{\beta} \cong K$. By Markov's Theorem 3.3.1 there are three modifications which can be applied to a braid under which their closures remain equivalent. Therefore a function on the braid group is a knot invariant if it is invariant under a series of Markov moves. This allows us to use braids to define knot invariants. The aim is to represent the braid as an element of an algebraic structure which is similar but more restricted than the braid group.

In the case of the Jones polynomial we introduce the Temperley-Lieb algebras $\mathcal{TL}_n$. In Section 4 we aim to show there is a diagrammatic interpretation of $\mathcal{TL}_n$ using simple n-diagrams. We can view these diagrams as braids in which the crossings have been resolved.

Section 5 is concerned with constructing the Jones polynomial $V_K(X)$. We construct a representation ρ_n of the braid group on the Temperley-Lieb algebra

$$\rho_n : \mathcal{B}_n \to \mathcal{TL}_n$$

We then compose this with the Markov trace tr

$$tr \circ \rho_n : \mathcal{B}_n \to \mathcal{TL}_n \to \mathbb{C}[X, X^{-1}] \tag{1.1}$$

which assigns to each braid β a Laurent polynomial. We introduce a factor τ^{n-1} as a convention that ensures the Jones polynomial of the unknot is one. We introduce a factor of $(-X^3)^{-w(\beta)}$ where the writhe $w(\beta)$ is the number of crossings of the braid. This ensures that the function is preserved under a series of Markov moves. This results in Theorem 5.4.2 that the Jones Polynomial

$$V_K(X) = (-X^3)^{-w(\beta)} tr(\rho_n(\beta))\tau^{n-1} \quad \text{where} \quad \tau = (-X^2 - X^{-2})$$

is a knot invariant.

Section 6 presents four applications of the Jones polynomial. We will study two alternative methods of computing the Jones polynomial of a knot; via skein relations and via the Kauffman bracket $\langle K\rangle$. In particular we will prove in Theorem 6.2.3 that

$$V_K(X) = (-X^3)^{-w(\beta)}\langle K\rangle$$

We will then consider how effective the Jones polynomial is at solving the problem of whether two knot diagrams are equivalent. It is not a complete invariant which means two knot diagrams having equal Jones polynomials does not imply equivalence. A more specific problem that arises is whether a knot is equivalent to the unknot. Whether the Jones polynomial detects the unknot remains an open problem in knot theory. A strength of the Jones polynomial is the ability to distinguish whether a knot is equivalent to it's mirror image. In particular we aim to apply the Jones polynomial to distinguish the knot diagrams in Figure 1.1.

Section 7 is concerned with the HOMFLY polynomial $P_K(z,m)$. The HOMFLY polynomial extends the Jones polynomial. We construct the HOMFLY polynomial in a similar way to the Jones polynomial. In this case we introduce the Hecke algebras $\mathcal{H}_n$. The Hecke algebras have a structure similar to $\mathcal{B}_n$. This allows us to define a representation φ_m of the braid group on the Hecke algebra and compose this with the Markov trace Tr to give

$$Tr\circ\varphi_m : \mathcal{B}_n \to \mathcal{H}_n \to \mathbb{C}[z^{\pm 1}, m^{\pm 1}]$$

which assigns each braid β a Laurent polynomial in two variables. It remains to find the necessary conditions such that the function is preserved under a series of Markov moves. We will prove in Theorem 7.18 that the Jones polynomial can be obtained from the HOMFLY polynomial via the specific change of variables

$$P_K(z = X^2 - X^{-2}, m = -X^{-4}) = V_K(X)$$

1.1 History of Knot Theory

The mathematical theory of knots was studied originally in the late 18th century by Carl F. Gauss. Gauss' notebooks contain sketches of knots and present a method of coding them using braided strands. The development of this theory since the 19th century is closely associated with advances in chemistry and physics. The most prominent of these was Lord Kelvin during the late 19th century. Kelvin developed the 'vortex atom theory' which proposed that atoms were knotted filaments in a substance called ether. This is detailed in the book [PBI+23].

Kelvin's theory motivated the physicist P. Tait to understand knots. Tait believed that in producing a table which classified knots by their number of crossings he would in fact be creating a table of the elements. In collaboration with T. Kirkman and C. Little by 1900 an extensive table of knots with up to 10 crossings had been produced. In 1973 a duplication in the table was found by Kenneth A. Perko. This demonstrates that deciding whether two knots are equivalent is in general a hard problem.

Figure 1.2: Perko pair of knots proven to be equivalent [KAP74]

Progress in understanding equivalence of knots was not made until Jules H. Poincaré developed algebraic topology. As a result the first knot invariant polynomial was discovered by topologist James W. Alexander in 1919. This remained the only known knot invariant polynomial until the Spring of 1984 when Vaughan Jones discovered the Jones polynomial. Whilst studying operator algebras Jones unexpectedly noticed the expressions resembled topological relations within the braid group. Jones discussed his ideas with Joan S. Birman, a specialistic in the theory of braids. From these discussions Jones was able to generate the new knot invariant. The Jones polynomial was a more powerful knot invariant and caused a significant amount of interest in knot theory to develop. By the Autumn of 1984 six mathematicians contributed to the discovery of the the HOMFLY polynomial which comprises both the Alexander and Jones polynomials. In 1985, Louis H. Kauffman announced a knot invariant which he discovered to be a recursive formula for the Jones polynomial. The method of constructing the Jones polynomial revealed connections between knot theory and other disciplines such as quantum groups, statistical mechanics and algebraic topology.

2 Knots and Links

To study knots effectively we must ensure we are considering knots which correspond to the intuitive notion of a knotted loop of string of which the ends are joined together. We define these ideas formally in this section using the standard topology on $\mathbb{R}^n$. The material is from Chapters 1-2 of the book [JM19].

A function is an *embedding* if it is a homeomorphism onto its image.

Definition 2.0.1. A *knot* is an embedding of the unit circle $\mathbb{S}^1$ in $\mathbb{R}^3$.

Definition 2.0.2. A *link* is a disjoint embedding of n copies of $\mathbb{S}^1$ into $\mathbb{R}^3$. Each copy of $\mathbb{S}^1$ in the link is called a *component* of the link.

As an abuse of terminology we often use the term *knot* to refer to both knots and links. The study of knot theory also refers to the study of links.

2.1 Equivalence

Intuitively, if we can continuously deform the knotted string without passing it through itself we obtain a knot equivalent to the original. We cannot break the string in this process. Unlike an actual loop of string we are able to extend and shrink the string in order to continuously deform the knot. We consider up to *ambient* isotopy as we are considering also up to isotopy the space in which the knot occupies. All knots in this report will be assumed to be tame.

Definition 2.1.1. Two knots K and K' are *ambient isotopic* if there is a family of homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ for $t \in [0,1]$ sending a point of the knot $x \in K$ to $h_t(x)$ such that $h_o = id_{\mathbb{R}^3}$, $h_1(K) = K'$ and $(x,t) \to (h_t(x), t)$ defines a homeomorphism from $\mathbb{R}^3 \times [0,1]$ to itself.

Definition 2.1.2. A link is said to be *tame* if it is ambient isotopic to a set of simple closed polygons in $\mathbb{R}^3$. Otherwise, the knot is said to be *wild*.

Definition 2.1.3. Two knots K and K' are *equivalent* if they are ambient isotopic. We denote this $K \cong K'$. An *isotopy class* is an equivalence class of the set of knots modulo ambient isotopy.

Imagine the trefoil knot K which is gradually deformed in Figure 2.1 from the left. The top section of the string in the first diagram has been pushed to occupy the bottom space of the knot in the third diagram. The string has then been deformed to give a maximum and minimum in the centre. As these deformations are continuous the knot will remain equivalent to the trefoil knot.

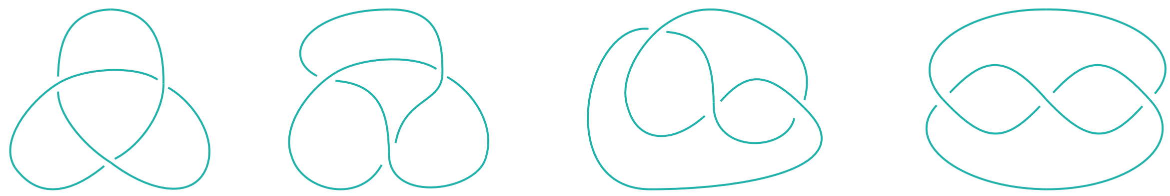


Figure 2.1: The trefoil knot K (left) and deformed trefoil knot K' (right) such that $K \cong K'$

To effectively study knots through diagrams as demonstrated in Figure 2.1 we need to ensure we have captured the entire structure of the knot in the two dimensional diagram. The following projection map does this by ensuring all crossings are visible in the plane.

Definition 2.1.4. A projection $\mathrm{p} : \mathbb{R}^3 \to \mathbb{R}^2$ of a knot K is said to be *regular* if

(i) All multiple points of the projection are double points.

(ii) There are only finitely many double points.

(iii) No double point contains the image of a vertex.

We have the following theorem which allows us to study all knots in such a way.

Theorem 2.1.5. *[CF63, Theorem 3.1] Every knot up to ambient isotopy admits a regular projection.*

We omit the proof which can be found in Section 3 of [CF63]. The idea behind the proof is that if the projection of a knot is non-regular using Definition 2.1.1 we are able to continuously deform the knot in $\mathbb{R}^3$ until the projection becomes regular.

Definition 2.1.6. A *knot diagram* is the image of a regular projection of a knot or a link on which an over or under crossing has been assigned to each double point.

This is demonstrated in Figure 2.1 where the over crossing is a continuous line and the under crossing is implied by a small gap in the diagram to illustrate that this section of the string is no longer visible at this point.

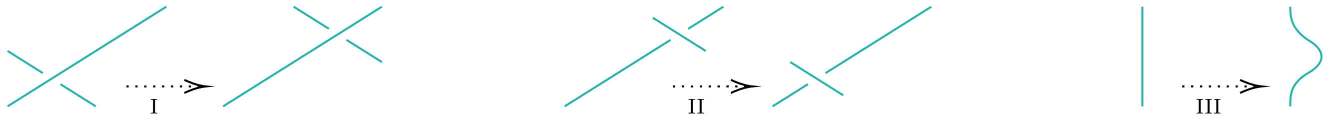
Definition 2.1.7. A knot is equivalent to the *unknot* if it admits a knot diagram with no crossings.

2.2 Reidemeister Moves

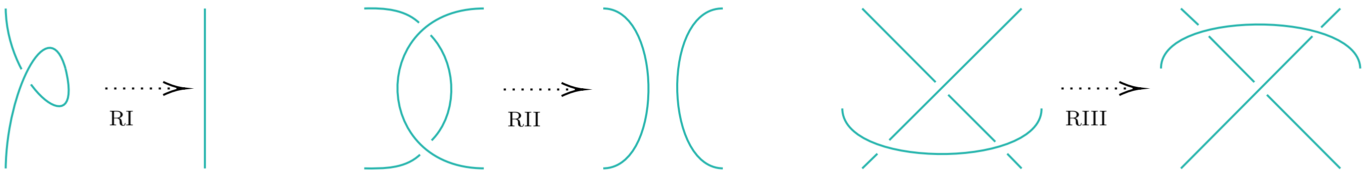
We understand that continuously deforming the knot in three dimensional space preserves equivalence. We would like to capture this process in a knot diagram. There are local changes in a knot diagram which preserve equivalence known as planar isotopy and Reidemeister moves. It is a fundamental theorem of knot theory that two knot diagrams K and K' are equivalent if and only if a series of these changes can be applied locally throughout the knot diagram K to obtain K' and vice versa.

Planar isotopy corresponds to the string being stretched and deformed continuously but has no effect on the number of crossings of a knot diagram.

Definition 2.2.1. Planar isotopy consists of the local changes, where the angles and lengths may vary, in a knot diagram given by



Definition 2.2.2. The *Reidemeister moves* consist of the following local changes in a knot diagram



Theorem 2.2.3. *[Reidemeister Theorem] [JM19, Theorem 1.3] Two knots are equivalent if and only if their diagrams are related by a finite sequence of Reidemeister moves and planar isotopy.*

2.3 Knot and Link Invariants

We now understand how we can deform a knot diagram such that equivalence is preserved. We might believe we are unable to obtain one knot diagram from the other through a series of Reidemeister moves. However this is not enough to declare that the knot diagrams are not equivalent as there may exist a series of Reidemeister moves which we have not considered. The problem therefore requires a knot invariant; a function which is preserved under equivalence of knots. The aim is to define a knot invariant such that the codomain is a set in which it is more accessible for understanding whether two elements are equivalent.

Definition 2.3.1. A *knot invariant* f is a function on the set of isotopy classes of knots. That is for a set S

$$f : \left\{ \frac{knots}{isotopy} \right\} \rightarrow S \tag{2.1}$$

In other words a knot invariant has the property that if two knots K and K' are equivalent then $f(K) = f(K')$. Alternatively, stated as the contrapositive,

$$f(K) \neq f(K') \Rightarrow K \ncong K'$$

However a knot invariant does *not* necessarily recognise equivalence of knots. If a knot invariant g detects equivalence for all knots

$$g(K) = g(K') \Rightarrow K \cong K'$$

then g is known as a *complete* knot invariant. Although complete invariants are known they can not be efficiently calculated.

3 Braids

The aim of the report is to derive a significant knot invariant known as the Jones polynomial. In order to do this we turn to the study of braids. The set of n stranded braids form a group $\mathcal{B}_n$. This allows us to present braids algebraically. In particular braids allow us to study the structure of a knot through an ordered sequence of generators which correspond to crossings. The aim in this section is to understand the relationship between knots and braids. This will allow us to use braids to define knot invariants.

An intuitive way to consider a braid is as a woven structure of n strands. The strands are anchored at n start positions and n end positions. A strand i is woven without going back on itself such that it travels from the starting position i to some end position $s(i)$. The material is from Section 1 of the book [KT08].

Definition 3.0.1. An *n-stranded braid* is a set $\beta \subset [0,1] \times \mathbb{R}^2$ formed by n disjoint topological intervals called the strands of β such that the projection $[0,1] \times \mathbb{R}^2 \to [0,1]$ maps each string homeomorphically onto [0,1] and

$$\beta \cap (\{0\} \times \mathbb{R}^2) = \{(0,1,0),(0,2,0),...,(0,n,0)\}$$

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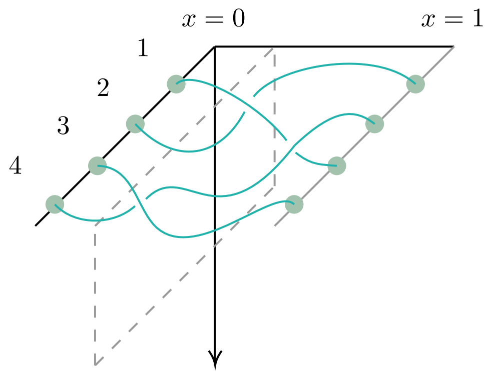


Figure 3.1: A 4-stranded braid

Figure 3.1 highlights that each of the n strands must intersect the boundaries $x \times \mathbb{R}^2$ for $x \in [0,1]$ at one point. This corresponds to the restriction that the strand cannot go back on itself.

We consider braids β and β' to be ambient isotopic if the strands of β can be continuously deformed to yield β'. This implies that the endpoints of each strand must remain the same. We define this formally as follows.

Definition 3.0.2. Two braids β and β' are *isotopic* if there is a family of homeomorphisms $h_s : \beta \to [0,1] \times \mathbb{R}^2$ for $s \in [0,1]$ sending a point of the braid $b \in \beta$ to $h_s(b)$ such that $h_0 = id_\beta$, $h_1(\beta) = \beta'$ and $(b,s) \to (h_s(b), s)$ defines a homeomorphism from $[0,1] \times \mathbb{R}^2 \times [0,1]$ to itself.

We aim to study braids as diagrams. The following axioms ensure that the entire structure of the braid is captured in the braid diagram. To construct the braid diagram in Figure 3.1 the strands are projected onto the plane $[0,1] \times \mathbb{R}$. By continuously deforming the strands to satisfy Definition 3.0.3 it can be seen that every braid up to isotopy produces a braid diagram.

Definition 3.0.3. A *braid diagram* on an n stranded braid β is a set $\beta \subset [0,1] \times \mathbb{R}$ consisting of n *strands* which are topological intervals such that

(i) The projection $[0,1] \times \mathbb{R} \to [0,1]$ maps each strand homeomorphically onto [0,1].

(ii) Every point of $\{0,1\} \times \{1,2,...,n\}$ is the endpoint of a unique strand.

(iii) Every point of $[0,1] \times \mathbb{R}$ belongs to at most two strands.

(iv) At every intersection of strands one strand is defined as an over and one an under crossing.

Theorem 3.0.4. *[KT08, Theorem 1.6] Two braids are equivalent if and only if their braid diagrams are related by a finite sequence of Reidemeister moves RII, $RIII$ and planar isotopy.*

Due to the restriction of the braid strand not going back on itself the Reidemeister move RI is not included in Theorem 3.0.4.

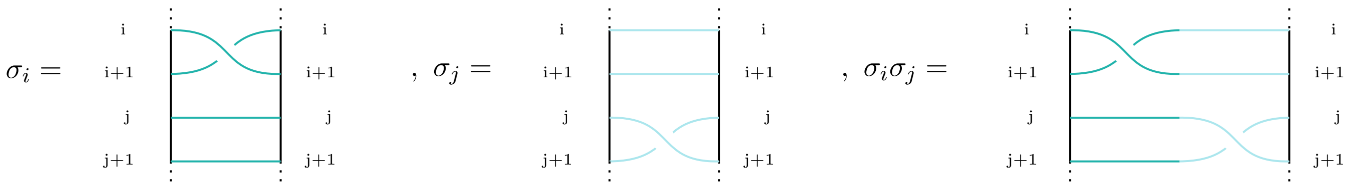
3.1 Artin Braid Group

We are able to consider the braid as a sequence of crossings. Let σ_i corresponds to the i^{th} strand passing *over* the $(i+1)^{th}$ strand. Let ${\sigma_i}^{-1}$ correspond to the i^{th} strand passing *under* the $(i+1)^{th}$ strand. These crossings become the generators of the Artin Braid Group $\mathcal{B}_n$.

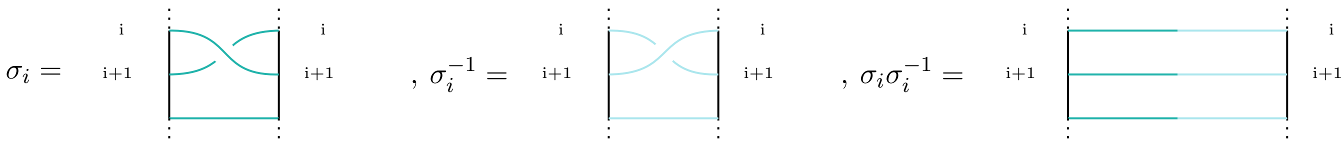
Definition 3.1.1. The *Artin Braid Group* $\mathcal{B}_n$ is defined

$$\mathcal{B}_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ , \ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1 \rangle \tag{3.1}$$

with composition of braids $\sigma_i \sigma_j$ defined horizontally with σ_i on the left and σ_j on the right such that the strands meeting at positions $\{1, ..., i, ..., j, ...n\}$ are glued as follows



The inverse of a generator σ_i^{-1} such that $\sigma_i {\sigma_i}^{-1} = I_{\mathcal{B}_n} = \sigma_i^{-1} \sigma_i$ for all $i \in \{1, ..., n-1\}$ is given by the mirror image as follows



The elementary braids are those having just one crossing σ_i. Any braid can be viewed as a composition of elementary braids. With manipulation of strands as required it may be assumed each crossing of a braid happens at a different horizontal level. An example using the braid given in Figure 3.1 is given in Figure 3.2. The braid is written in the order of which the crossings are composed;

$$\beta \cong \sigma_{i_1}^{\alpha_1} \sigma_{i_2}^{\alpha_2} \cdots \sigma_{i_m}^{\alpha_m} \tag{3.2}$$

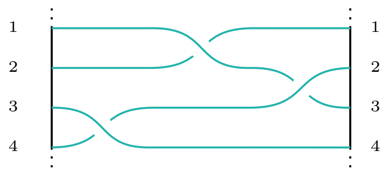


Figure 3.2: A braid diagram as the composition of elementary braids $\sigma_3\sigma_1\sigma_2^{-1}$

The Symmetric Group $\mathcal{S}_n$ admits a presentation

$$\mathcal{S}_n = \langle \delta, ..., \delta_{n-1} \mid \delta_1^2 = I,\ \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1},\ \delta_i\delta_j = \delta_j\delta_i \text{ for } |i-j| > 1\rangle \tag{3.3}$$

where δ_i is the transposition $(i, i+1)$ and group composition is defined as the usual permutation composition. If ${\sigma_i}^2 = 1$ is added to the presentation in Definition 3.1.1 then the presentation of $\mathcal{S}_n$ is achieved. The difference is that the elements of the braid group $\mathcal{B}_n$ have over and under paths specified at each crossing. This is shown in Figure 3.3.

Every n-stranded braid determines a permutation on n elements. As shown in Figure 3.1 each strand of β connects a point $(i, 0, 0)$ to a point $(s(i), 0, 1)$ for $\{i, s(i)\} \subset \{1, 2, ..., n\}$. The sequence

$$(s(1), s(2), ..., s(n))$$

is the *underlying permutation* of the braid.

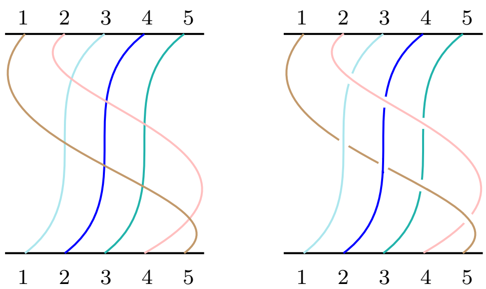


Figure 3.3: The underlying permutation diagram and a corresponding braid diagram respectively

The map $\phi : \mathcal{B}_n \to \mathcal{S}_n$ is a therefore a surjective group homomorphism given by

$$\phi(\sigma_i) = \delta_i$$

Figure 3.4 illustrates an elementary braid in $\mathcal{B}_2$ composed with itself. Despite each strand returning to the starting position the strands are woven in place and therefore cannot be stretched out to the identity. The identity would not be achieved through further compositions therefore the braid has infinite order. Figure 3.4 illustrates a comparison to the underlying permutation element in $\mathcal{S}_2$. Composing the permutation with itself results in the identity (1)(2). The difference is that we have removed the over or under restriction at each crossing.

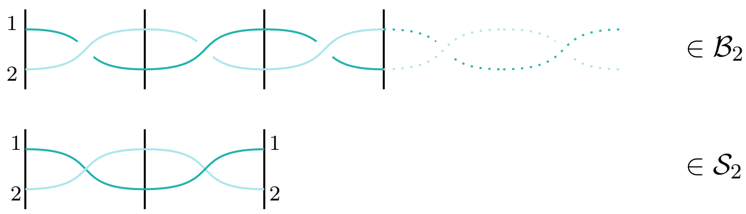


Figure 3.4: A braid $\beta \in \mathcal{B}_2$ of infinite order compared to the underlying permutation of β

3.2 Alexander's Theorem

In order to use braids to define knot invariants we must understand the relation between knots and braids. Extending each strand of the braid from the ith end position and connecting it to the ith starting position forms the closure of the braid $\overline{\beta}$. Therefore every braid gives rise to a knot via

$$\overline{\beta} \cong K$$

The reverse direction is less clear to see as knots do not usually present themselves in the form of the closure of a braid. As we are considering knot diagrams up to equivalence we are able to apply a series of Reidemeister moves to achieve this form. Slicing the closure of the braid at the endpoints obtains the corresponding braid. It is Alexander's Theorem 3.2.3 that this is possible for every knot. Notice there are many different sequences of Reidemeister moves which can be applied to the knot to form the closure of a braid. This implies for each knot K there are non-equivalent braids $\overline{\beta_{j_1}}, \overline{\beta_{j_2}}, ..., \overline{\beta_{j_m}}$ such that

$$K \cong \overline{\beta_{j_1}} \cong \overline{\beta_{j_2}} \cong ... \cong \overline{\beta_{j_m}}$$

Definition 3.2.1. Let $\beta \in \mathcal{B}_n$. The *closure* $\overline{\beta}$ of a braid β is formed by connecting each starting point i with the endpoint i by non-crossing strands for all $i \in \{1, ..., n\}$.

Proposition 3.2.2. *Every braid β gives rise to a knot K.*

Proof. This follows immediately from taking the closure $\overline{\beta}$ of a braid β which produces a knot K such that $K \cong \overline{\beta}$. ■

Theorem 3.2.3 (Alexander's Theorem). *[JM19, Theorem 4.12] Every knot is ambient isotopic to the closure of a braid.*

Proof. Let K be a knot. We aim to construct the knot diagram of K surrounding a point • such that the string travels clockwise around this point. We can then push all the crossings to one side using Reidemeister's Theorem 2.2.3.

Suppose a is a section of the string travelling anticlockwise around the point. This is illustrated in Figure 3.5 where c denotes an arbitrary part of K. We aim to continuously deform this section of the string so that it travels clockwise around the point. It is sufficient to consider the following two cases.

Suppose a contains no crossings. Then we can apply a series of Reidemeister moves to extend the arc to surround • travelling now in a clockwise direction. This is shown in I of Figure 3.5.

Suppose a contains one crossing. Then extend the arc to surround • as shown in II or III of Figure 3.5 depending on whether the crossing is over or under respectively.

If a contains more than one crossing then we can consider a as a series of individual crossings and apply the previous cases to each crossing. It is now possible to apply planar isotopy to push all the crossings to occupy one side of the knot to give the closure of a braid. ■

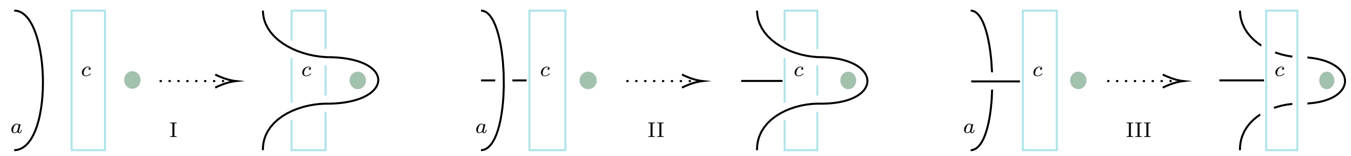


Figure 3.5: Resolving the anticlockwise arc a to travel clockwise around the origin •

Example 3.2.4. Consider the trefoil knot K in Figure 2.1 which is equivalent to the knot K'. As we are considering up to equivalence of knots we can apply a series of Reidemeister moves to K' such that the bottom section of the knot now occupies the space above the knot. This is the knot diagram appearing in Figure 3.6. Slicing this knot to remove the dotted sections of string forms the corresponding braid β.

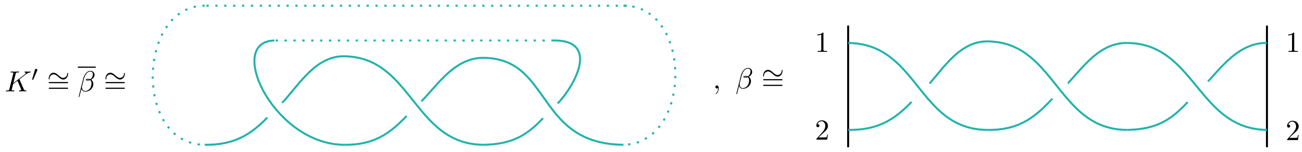


Figure 3.6: The closure of a braid $\overline{\beta} \cong \overline{\sigma_1^3}$ equivalent to the trefoil knot K

3.3 Markov's Theorem

By Alexander's Theorem 3.2.3 we now have a surjective map from braids to knots. We would like to understand the kernel of this map. That is we would like to understand how non-equivalent braids $\beta_{j_1}, ..., \beta_{j_m}$ are related when $\overline{\beta_{j_1}} \cong ... \cong \overline{\beta_{j_m}} \cong K$. In order to do this we need to understand how we can modify our braids such that the closures correspond to equivalent knot diagrams.

Theorem 3.3.1 (Markov's Theorem). *[Fas05, Theorem 2] Let $\beta_n \in \mathcal{B}_n$ and $\beta'_m \in \mathcal{B}_m$ be two braids in the respective braid groups. Then the knots $K \cong \overline{\beta_n}$ and $K' \cong \overline{\beta'_m}$ are ambient isotopic if and only if β_n and β'_m are related by the following moves:*

(MI) *Equivalence in a given braid group.*

(MII) *Conjugation: Let $\alpha, \beta \in \mathcal{B}_n$. Suppose we compose $\alpha\beta\alpha^{-1}$ to give the conjugate braid. When we we take the closure $\overline{\alpha\beta\alpha^{-1}}$ the braids α and α^{-1} compose through the closure strands to obtain the identity braid of no crossings.*

$(MIII)$ *Embedding: Let $\beta \in \mathcal{B}_n$. Suppose we add a strand to β to make an $(n+1)$-strand braid. Suppose we then cross the nth strand with the $(n+1)$th strand denoted $\beta\sigma_n^{\pm 1}$. When we take the closure $\overline{\beta\sigma_n{}^{\pm 1}}$ by applying RI from Definition 2.2.2 the $(n+1)$th strand is incorporated into the non crossing nth closure strand.*

We omit the proof which can be found in Section 2 of the book [Bir75].

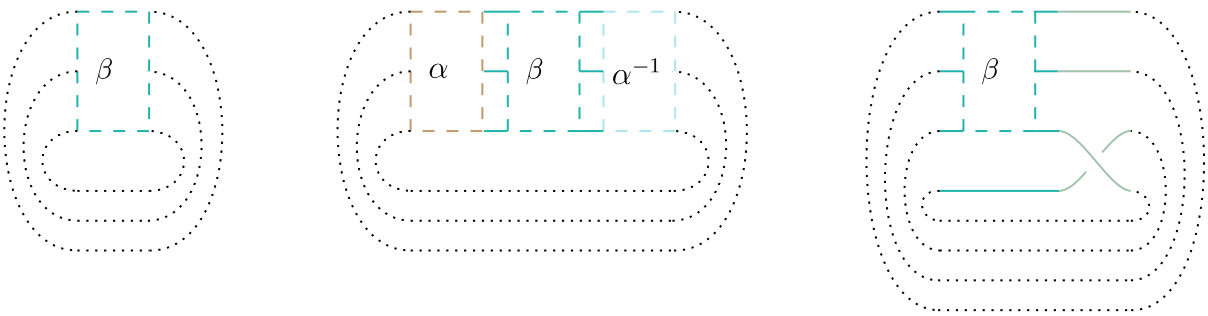


Figure 3.7: Closure of braid $\overline{\beta}$, a conjugation $\overline{\alpha\beta\alpha^{-1}}$ and an embedding $\overline{\beta\sigma_n}$ respectively

In Figure 3.7 by Markov Theorem 3.3.1 we have

$$\overline{\beta} \cong \overline{\alpha\beta\alpha^{-1}} \cong \overline{\beta\sigma_n^{\pm 1}}$$

A specific problem in knot theory is to determine if a given knot is equivalent to the unknot given in Definition 2.1.7. We use Markov's Theorem 3.3.1 to simplify the knot in the following example.

Example 3.3.2. Consider the knot diagram K given in Figure 3.8. We achieve the closure of a braid as described in the proof of Alexander's Theorem 3.2.3 such that $K \cong \overline{\beta}$. The result is shown in Figure 3.8. Notice the far right strands of β are the mirror image of the far left strands. Therefore we consider the the braid β as a composition of braids $\alpha\beta'\alpha^{-1}$. By Markov's Theorem 3.3.1 we have $\overline{\beta} \cong \overline{\beta'}$.

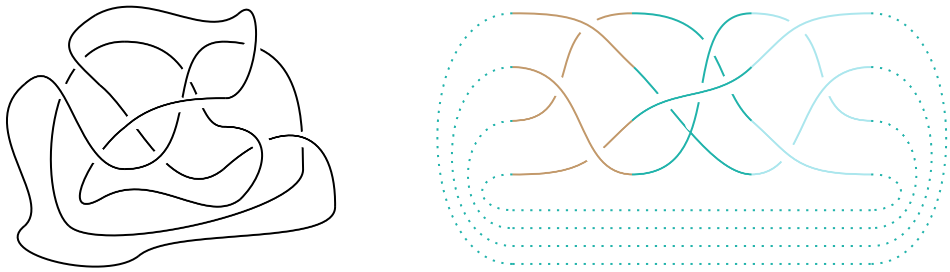


Figure 3.8: The knot diagram K equivalent to the closure of the braid $\overline{\beta} \cong \overline{\alpha\beta'\alpha^{-1}}$

By Reidemeister's Theorem 2.2.3 we can apply a series of planar isotopy and Reidemeister moves to $\overline{\beta'}$. This is demonstrated in Figure 3.9. This implies $K \cong \overline{\beta} \cong \overline{\beta'} \cong K_\circ$ where $K_\circ$ is the unknot. Therefore K is equivalent to the unknot.

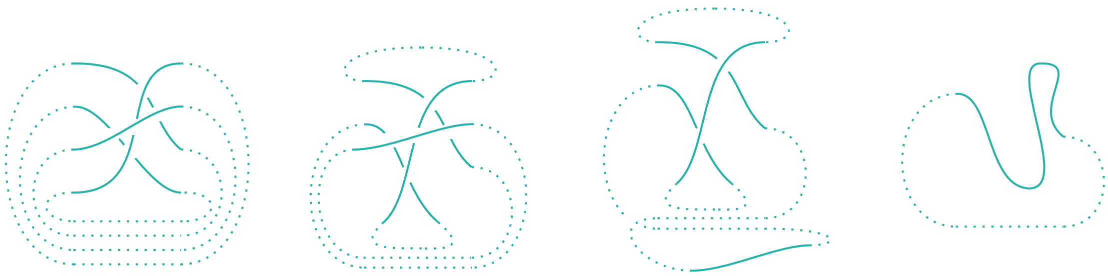


Figure 3.9: The Reidemeister moves $RIII$, RII, RI and planar isotopy applied to the knot K'

We now further understand the relation between knots and braids. Suppose f is a function on the braid group

$$f : \mathcal{B}_n \to S$$

for some set S. Suppose $\beta_{j_1}, ..., \beta_{j_m}$ are braids which may not be equivalent. For f to be a knot invariant the following must hold;

$$\overline{\beta_{j_1}} \cong ... \cong \overline{\beta_{j_m}} \cong K \implies f(\beta_{j_1}) = ... = f(\beta_{j_m})$$

This holds by Markov's Theorem 3.3.1 if and only if $\beta_{j_1}, ..., \beta_{j_m}$ are related by a series of moves MI, MII and $MIII$. Therefore in order to prove that f is a knot invariant it is required to prove f remains invariant under a series of Markov moves.

4 Temperley-Lieb Algebra

To construct a knot invariant using the braid group we aim to represent the braids as elements of an algebraic structure. The structure be must similar but more restricted than $\mathcal{B}_n$ to gain a further insight into knots. In the case of the Jones polynomial we introduce the Temperley-Lieb algebra $\mathcal{TL}_n(\tau)$. There is a similarity between the relations in the Definition 4.0.1 of $\mathcal{TL}_n$ and the Definition 3.1 of $\mathcal{B}_n$. This similarity can be captured diagrammatically. Simple n-diagrams can be viewed as braids in which the crossings are resolved. The aim in this section is to prove that the simple n-diagrams give a diagrammatic interpretation of the elements of $\mathcal{TL}_n$. In order to do this we need to further understand the structure of $\mathcal{TL}_n(\tau)$; specifically the dimension. This is the purpose of the following lemmas. The material follows Chapter 5.7 of the book [KT08].

Definition 4.0.1. Let R be the commutative ring of Laurent polynomials $R = \mathbb{C}[X, X^{-1}]$ such that X is a formal variable, let $\tau \in R$ be a fixed parameter. The *Temperley-Lieb Algebra* $\mathcal{TL}_n(\tau)$ is defined as the unital associative R-algebra with generators $e_1, ..., e_{n-1}$ satisfying the relations:

$$e_i^2 = \tau e_i \tag{4.1}$$

$$e_i e_j e_i = e_i \quad \text{for } |i-j| = 1 \tag{4.2}$$

$$e_i e_j = e_j e_i \quad \text{for } |i-j| > 1 \tag{4.3}$$

Definition 4.0.2. A *word*, w, in the alphabet $\{e_1, ..., e_{n-1}\}$ is an element of $\mathcal{TL}_n(\tau)$. The empty word is the identity element of $\mathcal{TL}_n(\tau)$ denoted $I_{\mathcal{TL}_n}$.

Definition 4.0.3. The *index* of w is defined to be the maximum of all indices $i_1, ...i_r$ appearing in w. We denote the maximal index of w as m; this implies e_m is the *maximal generator* of w.

Lemma 4.0.4. *[KT08, Lemma 5.25.] A non-empty word w is equal in $\mathcal{TL}_n(\tau)$ to a scalar multiple of a word in which the maximal generator appears exactly once.*

Proof. We will proceed by induction on the index m of the word w.

If $m = 1$ then w is a positive power of e_1. This can be seen by applying Relation 4.1 recursively.

$$e_i^2 = \tau e_i, \; e_i^3 = \tau(\tau(e_i)) = \tau^2 e_i, \; \ldots, \; e_i^p = \tau^{p-1} e_i, \; \ldots$$

for all $p > 1$. As this is a scalar multiple of the maximum generator the lemma holds for $m = 1$.

Assume the claim holds for all w such that the index is less than m.

We will show the claim holds for a word of index m.

Consider a non-empty word $w = e_{i_1} \cdots e_{i_r}$ of index m. Suppose e_m appears in w at least twice. Then w is a composition of words

$$w = w_1 e_m w' e_m w_2$$

for w' of index $l < m$. We define w purposefully so w' is restricted to not contain e_m. If w' did contain e_m by Relation 4.2 this would potentially cancel; however we need to guarantee e_m appears at least twice. This results in the following cases for the index l of w'.

Case 1: If $l < m-1$ then by 4.3 w' commutes with e_m. This allows us to shift e_m through w'. Therefore by 4.1,

$$w = w_1 e_m w' e_m w_2 = w_1 w' e_m^2 w_2 = \tau w_1 w' e_m w_2$$

We have reduced the number of occurrences of e_m in w by one. Thus applying this method for the words w_1 and w_2 by repetition we will reduce the occurrences of e_m in w to one.

Case 2: If $l = m-1$ then by induction hypothesis as $l < m$ we may assume that e_l appears only once in w'. So now w' is of the form $w' = w_3 e_{m-1} w_4$ for which w_3, w_4 are words of index strictly less than $m-1$. Therefore w_3 and w_4 commute with e_m by Relation 4.3. Using 4.2 we obtain

$$w = w_1 e_m w' e_m w_2 = w_1 e_m w_3 e_{m-1} w_4 e_m w_2 = w_1 w_3 e_m e_{m-1} e_m w_4 w_2 = w_1 w_3 e_m w_4 w_2$$

We have reduced the number of occurrences of e_m in w by one. Thus applying this method recursively we we will reduce the occurrences of e_m in w to one.

Therefore in each case we are able to transform the word into a scalar multiple of a word in which the maximal generator appears exactly once. ■

We now aim to define a spanning set of $\mathcal{TL}_n(\tau)$. For this we write the words in $\mathcal{TL}_n(\tau)$ as follows.

Definition 4.0.5. For $1 \leq k \leq n-1$, let $T_{n,k}$ be the set of $2k$-tuples $(i_1, ..., i_k, j_1, ..., j_k)$ of integers $i_1, j_1, ..., i_k, j_k$ satisfying

$0 < i_1 < i_2 < \cdots < i_k < n$,

$0 < j_1 < j_2 < \cdots < j_k < n$,

$j_1 \leq i_1, j_2 \leq i_2, \cdots, j_k \leq i_k$.

For such a tuple $\underline{s} = (i_1, ..., i_k, j_1, ..., j_k)$ set

$$e_{\underline{s}} = (e_{i_1} e_{i_1 - 1} \cdots e_{j_1})(e_{i_2} e_{i_2 - 1} \cdots e_{j_2}) \cdots (e_{i_k} e_{i_k - 1} \cdots e_{j_k}) \tag{4.4}$$

Observe in the expression for $e_{\underline{s}}$ the indices are decreasing from the left to right in each parenthesis. Therefore the index in $e_{\underline{s}}$ is i_k. A word of the form $e_{\underline{s}}$ is a *reduced word* in $\mathcal{TL}_n(\tau)$ for all

$$\underline{s} \in T_n := T_{n,1} \sqcup T_{n,2} \cdots \sqcup T_{n,n-1}$$

Lemma 4.0.6. *[KT08, Lemma 5.26] The set $\{e_{\underline{s}}\}_{\underline{s} \in T_n}$ of reduced words spans $\mathcal{TL}_n(\tau)$.*

Proof. To prove the set of reduced words spans the algebra it is sufficient to prove any word w is a scalar multiple of a reduced word. We will proceed by induction on the index m of the word w.

If $m = 1$ then by the proof of Lemma 4.0.4 w is a scalar multiple of e_1 which is a reduced word.

Assume the claim holds for any word of index less than m.

We will show the claim holds for a word w of index m. By Lemma 4.0.4 w is a scalar multiple of some $w_0 = w_1 e_m w_2$ where w_1 and w_2 have index strictly less than m. By the induction hypothesis we may assume that w_2 is reduced.

Suppose $w_2 = e_{\underline{s}} = (e_{i_1}e_{i_1-1}\cdots e_{j_1})(e_{i_2}e_{i_2-1}\cdots e_{j_2})\cdots(e_{i_k}e_{i_k-1}\cdots e_{j_k})$ for $\underline{s} \in T_{n,k}$ with $i_k < m$.

If $i_k < m-1$ then by Relation 4.3 w_2 commutes with e_m implying

$$e_m w_2 = (e_{i_1}e_{i_1-1}\cdots e_{j_1})(e_{i_2}e_{i_2-1}\cdots e_{j_2})\cdots(e_{i_k}e_{i_k-1}\cdots e_{j_k})(e_m)$$

If $i_k = m-1$ then e_m will shift through w_2 until it reaches e_{i_k} with which it cannot commute by Relation 4.3 resulting in

$$e_m w_2 = (e_{i_1}e_{i_1-1}\cdots e_{j_1})(e_{i_2}e_{i_2-1}\cdots e_{j_2})\cdots(e_m e_{i_k}e_{i_k-1}\cdots e_{j_k})$$

In both cases we have written w_0 as a word of the form $w'(e_m e_{m-1}\cdots e_l)$ such that;

w' is a word of index $m' < m:$ this follows from w_0 being defined to contain e_m exactly once,

$l \leq m:$ this follows from w_0 being defined to contain e_m exactly once.

By the induction hypothesis we can assume

$$w' = e_{\underline{s'}} = (e_{i'_1}e_{i_1-1'}\cdots e_{j'_1})(e_{i'_2}e_{i_2-1'}\cdots e_{j'_2})\cdots(e_{i'_k}e_{i_k-1'}\cdots e_{j'_k}) \text{ for } \underline{s'} = (i'_1,...i'_p,j'_1,...j'_p) \in T_{n,p}$$

The index for w' is $i'_p = m' < m$. To compare notation set $j'_p = l'$. We are now considering w_0 in the form

$$w_0 = (e_{i'_1}\cdots e_{m'}e_{j'_1}\cdots e_{l'})(e_m e_{m-1}\cdots e_l)$$

The value of l' will effect whether w_0 in this form is a reduced word.

Case 1: If $l' < l$ then $w_o = w'(e_m e_{m-1}\cdots e_l) = (e_{i'_1}\cdots e_{m'}e_{j'_1}\cdots e_{l'})(e_m e_{m-1}\cdots e_l)$ is reduced.

Case 2: If $l' \geq l$ then $w' = w''(e_{m'}e_{m'-1}\cdots e_{l'})$, where $l \leq l' \leq m' < m$ and w'' has index $< l'$.

We proceed with Case 2. We aim to write $w_0 = w''(e_{m'}e_{m'-1}\cdots e_{l'})(e_m e_{m-1}\cdots e_l)$ as a reduced word.

If $l' < m-1$ we can apply Relation 4.3. Then $e_{l'}$ will commute with all the elements until $e_{l'+1}$ as $|l'+1-l'| = 1$ implies these elements will not commute. At this point we apply Relation 4.2 and the bracket reduces to $e_{l'}$. We can commute the remaining elements. As $l'+2 \leq m$ we achieve the reduced form.

$$e_{l'}(e_m e_{m-1}\cdots e_l) = e_m e_{m-1}\cdots e_{l'+2}(e_{l'}e_{l'+1}e_{l'})e_{l'-1}\cdots e_l = (e_{l'}e_{l'-1}\cdots e_l)(e_m e_{m-1}\cdots e_{l'+2})$$

Therefore

$$w_0 = w''(e_{l'}e_{l'-1}\cdots e_l)(e_m e_{m-1}\cdots e_{l'+2})$$

By assumption $w''(e_{l'}e_{l'-1}\cdots e_l)$ has index $< m$ so the word w_0 is of the form considered in Case 1.

If $l' = m-1$ then $m-1 \leq m' < m$ implies $m' = l' = m-1$ and by 4.2,

$$e_{l'}(e_m e_{m-1}\cdots e_l) = (e_{m-1}e_m e_{m-1})\cdots e_l = e_{m-1}\cdots e_l$$

Therefore

$$w_0 = w''(e_{l'}e_{l'-1}\cdots e_l)$$

where w'' has an index $< l' = m' = m-1$. Thus w_0 has index $m-1$ and the result that w_0 is a reduced word follows from the induction hypothesis.

In both cases we have that w_0 can be written as a scalar as a reduced word as required. ■

Lemma 4.0.7. *[KT08, Lemma 5.27] The cardinality of T_n is equal to the nth Catalan Number,*

$$\text{card } T_n = \frac{1}{n+1}\binom{2n}{n}$$

.

Proof. We define an *admissible path* associated with any element $(i_1, ..., i_k, j_1, ..., j_k) \in T_{n,k}$ as a path

$$(0,0) \to (i_1, 0) \to (i_1, j_1) \to (i_2, j_1) \to (i_2, j_2) \to \cdots \to (i_k, j_{k-1}) \to (i_k, j_k) \to (n, j_k) \to (n, n)$$

By Definition 4.0.5 each vertex is a coordinate of integers. This is a path of oriented polygonal lines alternating between horizontal edges directed to the right and vertical edges directed upward. The lines are each of unit lengths.

For $e_{\underline{s}} \in T_n$ the admissible path lies under the diagonal in $\{(x, y) \in \mathbb{R} \mid 0 \leq y \leq x\}$. Equally any admissible path from $(0, 0)$ to (n, n) lying under the diagonal can be obtained from a unique element of T_n. Therefore it is enough to count such paths as follows.

We translate the admissible path lying under the diagonal along the vector $(1, 0)$ to obtain an admissible path from $(1, 0)$ to $(n + 1, n)$. The path no longer intersects the diagonal.

Let γ be any admissible path from $(1, 0)$ to $(n + 1, n)$. To calculate the number of γ's which do not intersect the diagonal we subtract the number of γ's intersecting the diagonal from the total number of γ's.

The admissible path always has n unit horizontal edges and n unit vertical edges. The number of ways of arranging these unit edges is the binomial coefficient;

$$\binom{2n}{n}$$

which is therefore the total number of γ's.

Associate to each γ an admissible path γ' from $(0, 1)$ to $(n + 1, n)$ as follows: Let (i, i) be the smallest point of which γ intersects the diagonal. Replace the section of γ from $(1, 0)$ to (i, i) by its reflection in the diagonal. The path γ' is constructed by joining the reflected section and the section of γ from (i, i) to $(n + 1, n)$.

An example is given in Figure 4.1 where γ' is the path issued from $(0, 1)$.

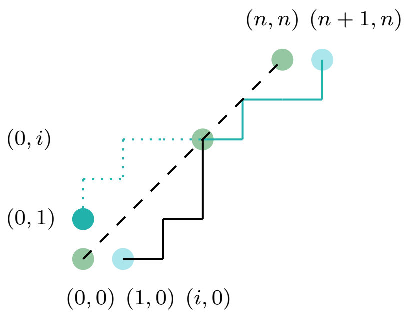


Figure 4.1: An example of the path γ' associated to γ

As shown in Figure 4.1 at point (i, i) the path intersects the diagonal. Any such path from (0,1) must intersect the diagonal to cross over to $(n+1, n)$. Therefore any path γ' can be obtained from a unique admissible path from (1,0) to $(n+1, n)$ by this construction.

Any admissible path from (0,1) to $(n+1, n)$ has $n+1$ unit horizontal edges and $n-1$ unit vertical edges, therefore the total number of such paths is equal to

$$\binom{2n}{n+1}$$

which is therefore the number of γ's intersecting the diagonal.

Therefore counting total admissible paths lying under the diagonal we have

$$\dim T_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = (1 - \frac{n}{n+1})\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}$$

. ■

Lemma 4.0.8. *The dimension of $\mathcal{TL}_n(\tau)$ is less than or equal to the nth Catalan Number*

$$\dim \mathcal{TL}_n(\tau) \leq \frac{1}{n+1}\binom{2n}{n}$$

The proof follows from Lemmas 4.0.6 and 4.0.7. The set of reduced words which will have the cardinality of the nth Catalan number spans $\mathcal{TL}_n(\tau)$. This means the dimension of $\mathcal{TL}_n(\tau)$ cannot exceed the nth Catalan number.

4.1 Kauffman's Definition of the Temperley-Lieb Algebra

We now introduce the simple n-diagrams as tangles with no crossings. Tangles can be considered as sliced regions of a knot. Therefore simple n-diagrams can be considered as regions of a knot in which the crossings have been resolved. They are therefore useful for understanding the structure of a knot. We aim to construct an algebra $\widetilde{\mathcal{TL}_n}$ where the simple n-diagrams are the elements. Using the dimensions of $\mathcal{TL}_n$ and $\widetilde{\mathcal{TL}_n}$ we then prove there is an algebra isomorphism $\phi : \mathcal{TL}_n \to \widetilde{\mathcal{TL}_n}$. This proves that the simple n-diagrams provide a diagrammatic interpretation for the elements of the Temperley-Lieb algebra $\mathcal{TL}_n$. The material continues to follow Chapter 5.7 of the book [KT08].

Definition 4.1.1. A *tangle* is an embedding of n arcs and m circles into $\mathbb{R}^2 \times [0, 1]$.

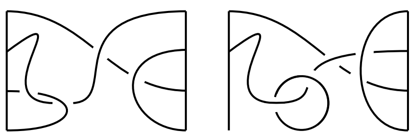


Figure 4.2: Tangles

Definition 4.1.2. A *simple n-diagram* is a tangle of n strands which has no crossings.

We consider simple n-diagrams up to isotopy.

Definition 4.1.3. Two simple n-diagrams D and D' are *isotopic* if they are related by a series of planar isotopy. We denote this $D \cong D'$.

Lemma 4.1.4. *[KT08, Lemma 5.33] The number of isotopy classes of simple n-diagrams is equal to the nth Catalan Number that is $\frac{1}{n+1}\binom{2n}{n}$.*

Proof. Let D be a simple n-diagram. By extending the strands of D from the endpoints and curving them down without crossing we obtain a union of n disjoint embedded arcs in $[0, +\infty) \times \mathbb{R}$. This is demonstrated in Figure 4.3. As there will be an extended endpoint for each strand there will total $2n$ endpoints along the straight line $\{0\} \times \mathbb{R}$.

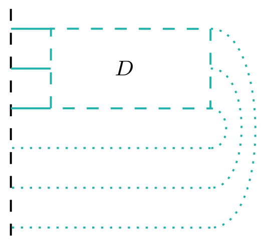


Figure 4.3: A simple 3-diagram extended to form arcs with 6 endpoints along $\{0\} \times \mathbb{R}$

There is a one-to-one correspondence between the simple n-diagrams and the systems of n disjoint arcs. We consider both up to isotopy where the n disjoint arcs can be continuously deformed without crossing. This implies it is enough to compute the number of systems of n disjoint arcs.

We label an endpoint of an arc by l or r if this point is the left or right endpoint of an ith strand. This is demonstrated in Figure 4.3 which we have rotated for convenience of reading. Reading the system of endpoints we obtain a word w of length $2n$ in the alphabet $\{l, r\}$. Every w is a *Dyck word* meaning it is a balanced string of l's and r's.

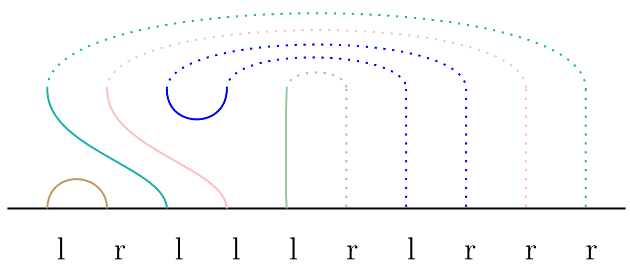


Figure 4.4: Example of a simple 5-diagram and the corresponding Dyck word

Each arc must have a balanced pair of endpoints. Therefore every such w of length $2n$ can be obtained uniquely from a system of n disjoint arcs.

For each Dyck word w of length $2n$ we associate a polygon path Γ_w in $\mathbb{R}^2$. The path has the vertices (x_k, y_k) for $k \in 0, 1, ..., 2n$ which are defined by $x_0 = y_0 = 0$ then inductively as follows:

if the kth letter in w is l: $x_k = x_{k-1} + 1$ and $y_k = y_{k-1}$

if the kth letter in w is r: $y_k = y_{k-1} + 1$ and $x_k = x_{k-1}$

Since there is a balanced number l and r each occurring n times the path Γ_w leads from (0,0) to (n, n). The path is admissible as it consists of orientated horizontal and vertical lines.

For example, consider the Dyck word $w = lrlllrlrrr$ in Figure 4.1. The associated polygonal path Γ_w will have the vertices $(0, 0), (1, 0), (1, 1), (2, 1), (3, 1), (4, 1), (4, 2), (5, 2), (5, 3), (5, 4), (5, 5)$.

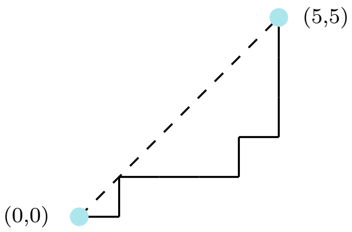


Figure 4.5: The path Γ_w associated to the simple n-diagram in Figure 4.1

In particular, Γ_w lies under the diagonal due the restrictions of w being a Dyck word. For example in Figure 4.1, it would not be possible for an r to appear as the third letter as it would not be possible for this to be balanced in the remainder of the word. Conversely, any admissible path from (0,0) to (n,n) lying under the diagonal is of the form Γ_w.

Therefore the number of simple n-diagrams up to isotopy is equal to the number of admissible paths from (0,0) to (n,n) lying under the diagonal. By Lemma 4.0.7 this number is equal to the nth Catalan number. ∎

By Lemma 4.1.4 we can calculate the number of isotopy classes of simple 3-diagrams as follows

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{3+1}\binom{6}{3} = \frac{20}{4} = 5 \tag{4.5}$$

Definition 4.1.5. Fix a non-zero $\tau \in \mathbb{C}$. Let $\widetilde{\mathcal{TL}}_n(\tau)$ be the complex vector space spanned by the isotopy classes of simple n-diagrams. Every simple n-diagram D represents a vector in $\widetilde{\mathcal{TL}}_n(\tau)$.

We equip $\widetilde{\mathcal{TL}}_n(\tau)$ with the structure of an associative algebra. As $\widetilde{\mathcal{TL}}_n(\tau)$ is defined to be a vector space it remains to equip a bilinear product as follows.

Definition 4.1.6. Define the composition of simple n-diagrams as

$$DD' = \tau^a D \circ D'$$

where $D \circ D'$ is the result of composing the strand at the end position i of D to the strand at the start position i of D' for $i \in \{1, ..., n-1\}$. The scalar a is the number of loops formed.

An example of this composition is given in Figure 4.8.

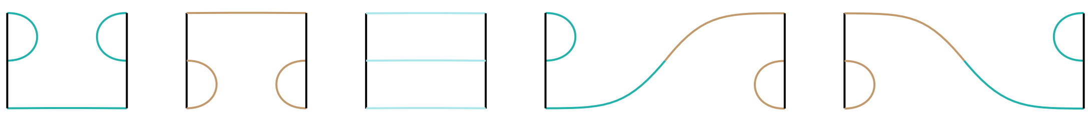


Figure 4.6: The complete set of simple 3-diagrams up to isotopy

By construction this is an associative product on $\widetilde{\mathcal{TL}}_n(\tau)$. Figure 4.6 presents the set of simple 3-diagrams up to isotopy. Notice the number of isotopy classes is five which agrees with Equation 4.5. The right two diagrams are both products of the left two diagrams where the order of composition has been reversed. This demonstrates that this product is not commutative.

The following theorem gives the diagrammatic interpretation of the Temperley-Lieb algebra.

Theorem 4.1.7. *[KT08, Theorem 5.34] For $i = 1, ..., n-1$, the assignment*

$$\phi(e_i) \to E_i$$

defines an algebra isomorphism $\phi : \mathcal{TL}_n(\tau) \to \widetilde{\mathcal{TL}}_n(\tau)$.

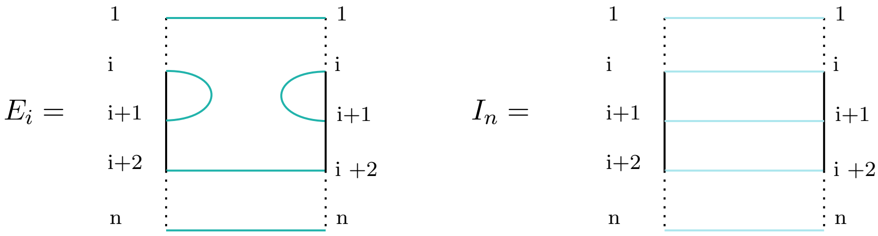


Figure 4.7: The simple n-diagram E_i for $i \in 1, ..., n-1$

Proof. The simple n-diagrams satisfy the relations in Definition 4.0.1 defining $\mathcal{TL}_n(\tau)$, as demonstrated in Figure 4.8. As we are considering up to isotopy the strands can be continuously deformed to identify them with their simple n-diagrams.

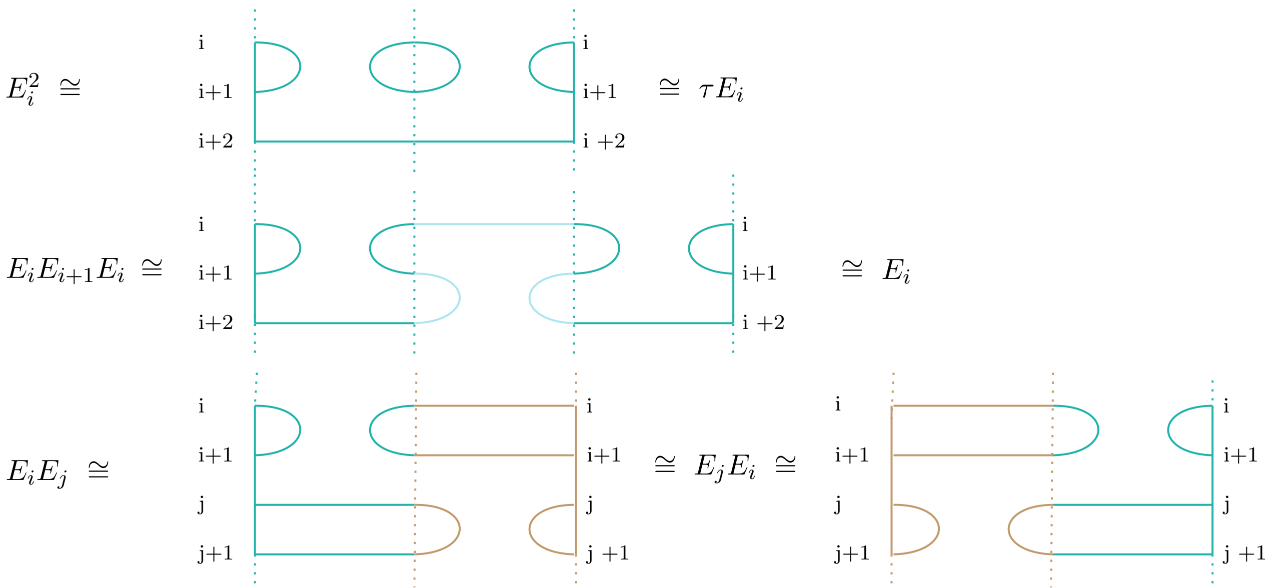


Figure 4.8: Relations (i), (ii), (iii) in Definition 4.0.1 satisfied by the generators of $\widetilde{\mathcal{TL}}_n(\tau)$

Therefore there is an algebra homomorphism $\phi : \mathcal{TL}_n(\tau) \to \widetilde{\mathcal{TL}}_n(\tau)$ such that $\phi(e_i) = E_i$ for $i \in \{1, ..., n-1\}$. We now verify that ϕ is an isomorphism.

We have that $\widetilde{\mathcal{TL}}_n(\tau)$ satisfies all the relations of $\mathcal{TL}_n(\tau)$ and by Lemmas 4.0.8 and 4.1.4 we have $\dim \mathcal{TL}_n(\tau) \leq \dim \widetilde{\mathcal{TL}}_n(\tau)$. Therefore ϕ is injective.

It remains to show the map is surjective. It suffices to show any $D \in \widetilde{\mathcal{TL}}_n$ not isotopic to I_n is a composition of diagrams $E_1, E_2, ..., E_{n-1}$. We shall prove this claim by induction on n.

If $n = 2$ then D must be isotopic to E_1 as it is the only non-identity element of $\widetilde{\mathcal{TL}}_2$.

Assume the claim is true $D \in \widetilde{\mathcal{TL}}_{(n-1)}$.

We will now prove the claim holds for $D \in \widetilde{\mathcal{TL}}_n$. Every start position issues a strand which implies the arcs must be disjoint. As simple n-diagrams contain no crossing strands the arcs must connect consecutive start points. By the assumption $D \ncong I_n$ there exists an arc connecting two consecutive start points. Denote $i(D) = i$ to be the minimal $i = \{1, ..., n-1\}$ such that there is an arc connecting positions i and $i+1$.

We can decompose D into a composition of diagrams $E_1, E_2, ..., E_{n-1}$ for the following cases. This is shown in Figure 4.9.

If $i(D) > 1$: Then $D \cong E_i D'$ which can be seen in the following deformation. Deform the non-crossing strand issued from $i-1$ to produce a local maxima and minima.

Now we may decompose this diagram into E_i and D'. The diagram D' has an arc issued from it's starting position $i-1$ therefore $i(D') = i(D) - 1$. If $i(D) - 1 > 1$, D' has a non-crossing strand at position 1. In other words $D' \in \widetilde{\mathcal{TL}}_{(n-1)}$ is embedded into $\widetilde{\mathcal{TL}}_n$. Therefore by the induction hypothesis D'' is a composition of $E_2, ..., E_{n-1}$. Hence D is a composition of $E_1, E_2, ..., E_{n-1}$ as required.

If $i(D') - 1 = 1$ we apply the following case to D'.

If $i(D) = 1$: Then $D \cong E_1 D''$ which can be seen in the following deformation. Stretch the arc descending from the minimum endpoint of D to the arc of D connecting positions 1 and 2.

Now we may decompose D into E_1 and D''. This implies D'' contains a strand connecting the minimum endpoints of D''. Therefore D'' is obtained by adding a non-crossing strand to a simple $(n-1)$-diagram. In other words $D'' \in \widetilde{\mathcal{TL}}_{(n-1)}$ is embedded into $\widetilde{\mathcal{TL}}_n$. The inductive assumption implies D'' is a product of elements of the form $E_2, ..., E_{n-1}$. Hence D is a composition of $E_1, E_2, ..., E_{n-1}$ as required.

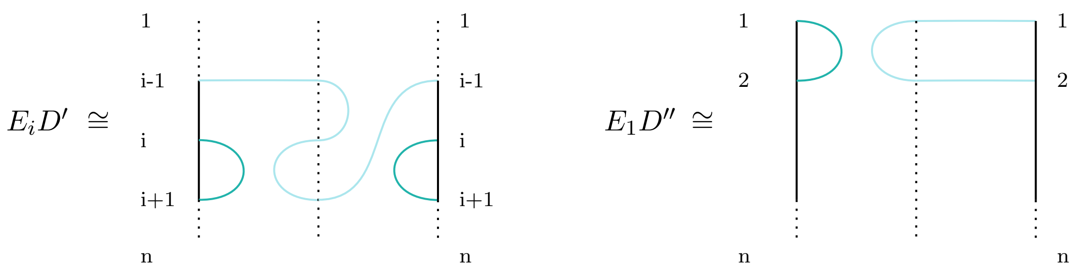


Figure 4.9: The decomposition of D into $E_i D'$ and $E_1 D''$ respectively

■

5 The Jones Revolution

5.1 Representation of the Artin Braid Group

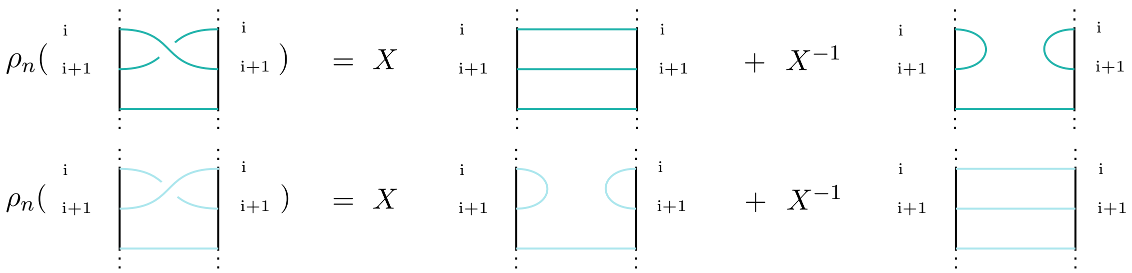
We are now able to compare the diagrammatic interpretation of the Temperley-Lieb algebra with the braid diagrams. The simple n-diagrams cannot contain crossings whereas the braid diagrams can. This implies in order to represent a braid diagram as a simple n-diagram we must resolve the crossings as in some way. The material in this section follows the paper [Fas05].

Definition 5.1.1. Define the homomorphism $\rho_n : \mathcal{B}_n \to \widetilde{\mathcal{TL}}_n(\tau)$ by

$$\rho_n(\sigma_i) = XI_n + X^{-1}E_i, \tag{5.1}$$

$$\rho_n(\sigma_i^{-1}) = XE_i + X^{-1}I_n \tag{5.2}$$

As a braid β can be written as a composition of elementary braids σ_i we can consider $\rho_n(\beta)$ as a sum of simple n-diagrams. We consider ρ_n as a function which resolves each crossing of the braid. There are two possible resolutions of the crossing; either the horizontal strands or the vertical strands are connected. This is given diagrammatically for $\rho_n(\sigma_i)$ and $\rho_n(\sigma_i^{-1})$ respectively by



Proposition 5.1.2. *[Fas05, Proposition 1] The map $\rho_n : \mathcal{B}_n \to \widetilde{\mathcal{TL}}_n(\tau)$ such that $\tau = -X^2 - X^{-2}$ is a representation of the Artin Braid Group.*

Proof. We must verify that ρ_n preserves the generator relations of $\mathcal{B}_n$ as given in Definition 3.1. We refer to the relations 4.1, 4.2 and 4.3 in Definition 4.0.1 which are given diagrammatically in Figure 4.8.

$$\begin{aligned}
\rho_n(\sigma_i)\rho_n(\sigma_i^{-1}) &= (XI_n + X^{-1}E_i)(XE_i + X^{-1}I_n) \\
&= X^2E_i + I_n + E_i^2 + X^{-2}E_i \\
&= I_n + (X^2 + X^{-2})E_i + \tau E_i \\
&= I_n + (X^2 + X^{-2})E_i + (-X^{-2} + -X^2)E_i \\
&= I_n \\
&= \rho_n(\sigma_i^{-1})\rho_n(\sigma_i)
\end{aligned}$$

For $|i - j| > 1$ by relation 4.3

$$\begin{aligned}
\rho_n(\sigma_i)\rho_n(\sigma_j) &= (XI_n + X^{-1}E_i)(XI_n + X^{-1}E_j) \\
&= X^2I_n + E_i + E_j + X^{-2}E_iE_j \\
&= X^2I_n + E_j + E_i + X^{-2}E_jE_i \\
&= (XI_n + X^{-1}E_j)(XI_n + X^{-1}E_i) \\
&= \rho_n(\sigma_j)\rho_n(\sigma_i)
\end{aligned}$$

For $|i-j|=1$ by applying relation 4.2 to the last term of each expansion

$$\begin{aligned}\rho_n(\sigma_i\sigma_{i+1}\sigma_i) &= (XI_n + X^{-1}E_i)(XI_n + X^{-1}E_{i+1})(XI_n + X^{-1}E_i)\\ &= (X^2I_n + E_{i+1} + E_i + X^{-2}E_iE_{i+1]})\\ &= X^3I_n + XE_i + XE_{i+1} + X^{-1}E_{i+1}E_i + XE_i + X^{-1}E_i^2\\ &\quad + X^{-1}E_iE_{i+1} + X^{-3}E_i\end{aligned} \tag{5.3}$$

$$\begin{aligned}\rho_n(\sigma_{i+1}\sigma_i\sigma_{i+1}) &= (XI_n + X^{-1}E_{i+1})(XI_n + X^{-1}E_i)(XI_n + X^{-1}E_{i+1})\\ &= X^3I_n + XE_{i+1} + XE_iX^{-1}E_iE_{i+1} + XE_{i+1} + X^{-1}E_{i+1}^2\\ &\quad + X^{-1}E_{i+1}E_i + X^{-3}E_{i+1}\end{aligned} \tag{5.4}$$

Therefore $\rho_n(\sigma_i\sigma_{i+1}\sigma_i) = \rho_n(\sigma_{i+1}\sigma_i\sigma_{i+1})$ can be seen by cancelling like terms in Equations 5.3 and 5.4 to give

$$X^{-1}E_{i+1}^2 + XE_{i+1} + X^{-3}E_{i+1} = X^{-1}E_i^2 + XE_i + X^{-3}E_i$$

then by applying relation 4.1 we have

$$(X^{-1} + \tau X + X^3)E_i = (X^{-1} + \tau X + X^3)E_{i+1}$$

these expressions equal as the common factor equals zero by

$$(X^{-1} + \tau X + X^3) = (X^{-1} + (-X^2 - X^{-2})X + X^3) = 0$$

■

5.2 Markov Trace

We are now able to represent a braid diagram via ρ_n as a sum of simple n-diagrams. We now define a Markov trace function on the simple n-diagrams. This will obtain a diagram isotopic to a number of loops. We assign to this diagram a Laurent polynomial in $\mathbb{C}[X, X^{-1}]$. We first generalise the notion of a trace function on an algebra. The material follows Chapter 2 of the paper [DA24].

Definition 5.2.1. A *trace* on an R-algebra $\mathcal{A}$ is a linear function $t : \mathcal{A} \to R$ such that $t(AB) = t(BA)$ for all $A, B \in \mathcal{A}$.

Definition 5.2.2. A *Markov Trace* on $\widetilde{\mathcal{TL}}_n(\tau)$ is a linear function $tr : \widetilde{\mathcal{TL}}_n(\tau) \to \mathbb{C}[X, X^{-1}]$ satisfying the following properties:

(*i*) $tr(I_{\widetilde{\mathcal{TL}}_n}) = 1$

(*ii*) $tr(AB) = tr(BA)$ for $A, B \in \widetilde{\mathcal{TL}}_n(\tau)$

(*iii*) $tr(AE_{n-1}) = \tau^{-1}tr(A)$ for $A \in \widetilde{\mathcal{TL}}_{n-1}(\tau)$

Lemma 5.2.3. *[DA24, Lemma 2.1] There is a unique linear function tr on $\bigcup_{n=1}^{\infty} \widetilde{\mathcal{TL}}_n(\tau)$ satisfying the axioms in Definition 5.2.2.*

Proof. By Lemma 4.0.4 for a word in the alphabet $\{I_n, E_1, ..., E_{n-1}\}$ the maximal generator appears exactly once. This follows from the algebra isomorphism defined in Theorem 4.1.7. Therefore a word $w \in \widetilde{\mathcal{TL}}_n(\tau)$ contains at most one E_{n-1}.

Now suppose $w \in \widetilde{\mathcal{TL}}_n(\tau) \backslash \widetilde{\mathcal{TL}}_{n-1}(\tau)$. All such elements in this set must contain E_{n-1} as it is the only non-common element as $E_{n-1} \notin \widetilde{\mathcal{TL}}_{n-1}(\tau)$. Combining this with the first statement the number of occurrences of E_{n-1} in w is exactly one. We can therefore write $w = w_1 E_{n-1} w_2$ where $w_1, w_2 \in \widetilde{\mathcal{TL}}_{n-1}(\tau)$. Then by the relations (ii) and (iii) in Definition 5.2.2 we have

$$tr(w) = tr(w_2 w_1 E_{n-1}) = \tau^{-1} tr(w_2 w_1)$$

Therefore for any word $w \in \widetilde{\mathcal{TL}}_n(\tau)$ we can reduce the trace computation to the trace of a word $w_2 w_1 \in \widetilde{\mathcal{TL}}_{n-1}(\tau)$. Iterating this process the trace of a word in $\widetilde{\mathcal{TL}}_n(\tau)$ is uniquely determined by the relations in Definition 5.2.2. By definition the trace is linear and the result follows. ■

Proposition 5.2.4. *The Markov trace tr is the function* $tr : \widetilde{\mathcal{TL}}_n(\tau) \to \mathbb{C}[X, X^{-1}]$ *given by*

$$tr(D) = \tau^{p-n} \tag{5.5}$$

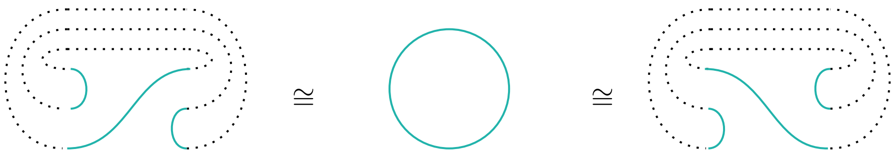
where p is the number of loops obtained connecting each starting point i with the endpoint i by non-crossing strands for all $i \in \{1, ..., n\}$ of the simple n-diagram D.

Proof. By uniqueness it is enough to show the function given by Equation 5.5 satisfies the axioms in Definition 5.2.2.

(i) A loop will form for each non crossing strand therefore $tr(I_n) = \tau^{n-n} = \tau^0 = 1$.

(ii) For $|i - j| > 1$ this follows immediately from the relations of $\widetilde{\mathcal{TL}}_n(\tau)$ in Figure 4.8. For $|i-j| = 1$, Figure 5.1 illustrates the equivalence of $tr(E_i E_{i+1}) = tr(E_{i+1} E_i)$. They both result in one loop being formed. Therefore we have shown (ii) holds for any combination of basis elements E_i. As a simple n-diagram may be written as a composition of basis elements the result follows.

$tr(E_i E_{i+1})$:

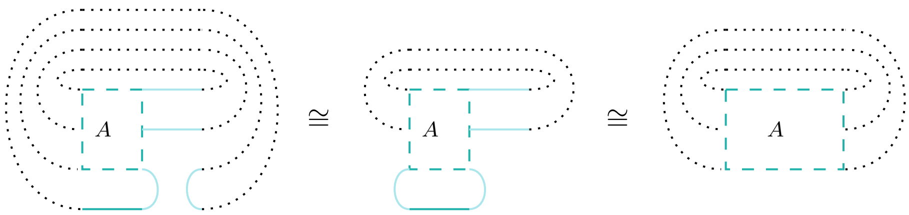


: $tr(E_{i+1} E_i)$

Figure 5.1: The trace on elements $E_i E_{i+1}$ and $E_{i+1} E_i \in \widetilde{\mathcal{TL}}_n(\tau)$

(iii) Let $A \in \widetilde{\mathcal{TL}}_{n-1}(\tau)$. We can embed $A \in \widetilde{\mathcal{TL}}_n(\tau)$ by introducing a non crossing nth strand. The trace of A composed with $E_{n-1} \in \widetilde{\mathcal{TL}}_n(\tau)$ is demonstrated in Figure 5.2. Suppose $tr(AE_{n-1}) = \tau^{p-n}$. Then $tr(A) = \tau^{p-(n-1)} = \tau\tau^{p-n}$ implies $\tau^{-1} tr(A) = \tau^{p-n}$ as required.

$tr(AE_{n-1})$:



: $\tau^{-1} tr(A)$

Figure 5.2: The trace of $AE_{n-1} \in \widetilde{\mathcal{TL}}_n(\tau)$ is equivalent to $\tau^{-1} tr(A)$ for $A \in \widetilde{\mathcal{TL}}_{n-1}(\tau)$

■

5.3 Writhe

We are now able to move from braids to Laurent polynomials by the composition

$$tr \circ \rho_n : \mathcal{B}_n \to \widetilde{\mathcal{TL}}_n(\tau) \to \mathbb{C}[X, X^{-1}]$$

Recall for a polynomial to be a knot invariant by Markov's Theorem 3.3.1 it must be invariant under a series of Markov modifications. We will discover that the construction $tr \circ \rho$ is not invariant under the Markov move $MIII$. To adjust this we introduce the writhe factor. We define this on an arbitrary braid which by Alexander's Theorem 3.2.3 becomes a property of a knot. The material follows Chapter 3 of the book [JM19].

Definition 5.3.1. For a braid $\beta \cong \sigma_{i_1}^{\alpha_1}\sigma_{i_2}^{\alpha_2}\cdots\sigma_{i_m}^{\alpha_m}$ the *writhe* of the braid is defined as a map $w : \mathcal{B}_n \to \mathbb{Z}$ such that

$$w(\beta) = \sum_{i=1}^{m} \alpha_i$$

For example, consider the braid corresponding to the trefoil knot in Figure 3.6. The braid is a composition of three basis elements denoted $\sigma_1^3 \in \mathcal{B}_2$. The writhe is calculated to be $w(\sigma_1^3) = 3$.

Definition 5.3.2. Let K be a knot diagram given with an orientation. The *writhe* $w(K)$ is the sum of the signs of all crossings of K.

Remark 5.3.3. Computing the writhe directly from the knot results in a sum of positive and negative crossings. This requires a knot diagram with an orientation. By convention the sign of the crossing is determined by the right-hand rule. To find the sign of the crossing follow the string of the knot continuously in one direction marking this with arrows. Then follow the string in this direction again and observe at each over crossing whether the piece of string crossing above travels to the right (positive) or left (negative) relative to the string being followed.

The writhe of a knot diagram is *not* a knot invariant as it is not invariant under Reidemeister move RI. This is shown in Figure 5.3 where $K \cong K'$ differ only in the local areas as shown below.

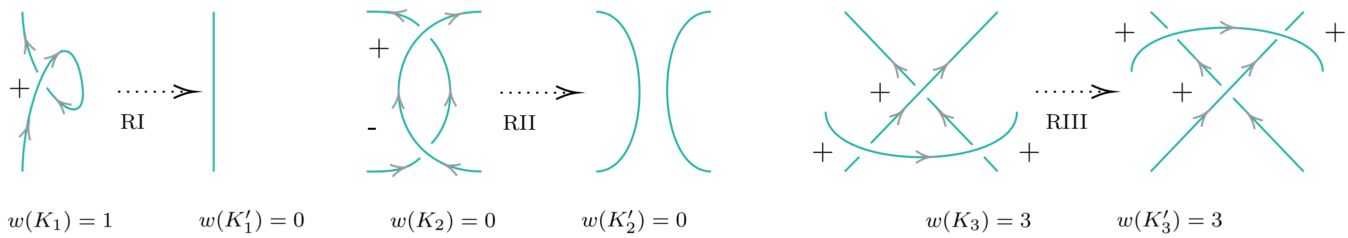


Figure 5.3: The *writhe* of the Reidemeister moves given in Definition 2.2.2

Remark 5.3.4. Recall the moves from Markov's Theorem 3.3.1. As shown in Figure 3.7 $MIII$ modifies the knot in the same way as RI in Figure 5.3. The loop is simply unravelled to remove the crossing. Therefore $MIII$ alters the writhe of the braid by ± 1 depending on whether the nth strand is crossed over or under. This is given by

$$w(\beta\sigma_n^{\pm 1}) = w(\beta) \pm 1$$

Recall Theorem 3.0.4. The writhe under MI will not be effected by the above RII and $RIII$ and planar isotopy does not effect crossings. The writhe under MII will not be effected as

$$w(\alpha\beta\alpha^{-1}) = w(\alpha) + w(\beta) - w(\alpha) = w(\beta)$$

5.4 Jones Polynomial

We will discover in the proof of Theorem 5.4.2 that the Markov move $MIII$ changes $tr \circ \rho$ by a factor of $(-X^{\pm 3})$ depending on whether the crossing has been made over or under. Therefore by Remark 5.3.4 introducing $(-X^{\pm 3})^{w(\beta)}$ nullifies this change. By Remark 5.3.4 this factor will not be effected under equivalence, MI, or conjugation, MII.

It is now possible to define the Jones polynomial as a knot invariant in terms of the closure of a braid. We are able to define a function

$$V_K : \{Knots\} \to \mathcal{B}_n \to \widetilde{\mathcal{TL}}_n(\tau) \to \mathbb{C}[X, X^{-1}]$$

which is well defined as by Alexander's Theorem 3.2.3 every knot can be considered as the closure of a braid. The material in this section follows the paper [Fas05].

Definition 5.4.1. Let $\beta \in \mathcal{B}_n$ be a braid and K a knot diagram. The *Jones Polynomial* $V_K(X)$ of a knot $K \cong \bar{\beta}$ is the function $V_K : \{Knots\} \to \mathbb{C}[X, X^{-1}]$ defined by

$$V_K(X) = (-X^3)^{-w(\beta)} tr(\rho_n(\beta))\tau^{n-1} \tag{5.6}$$

where $\tau = (-X^2 - X^{-2})$.

Theorem 5.4.2. *[Fas05, Proposition 2] The Jones Polynomial $V_K(X)$ is a knot invariant.*

Proof. By Theorem 2.1.5 we can consider every knot as a knot diagram K. By Alexander's Theorem 3.2.3 for every knot K there exists a braid $\beta \in \mathcal{B}_n$ such that $K \cong \overline{\beta}$. Let $K \cong \overline{\beta}$ and $K' \cong \overline{\beta'}$. By Markov's Theorem 3.3.1 if it is possible to obtain β' from β by a series of Markov moves then their closures are ambient isotopic knots $\overline{\beta'} \cong \overline{\beta}$. Therefore it suffices to show that $V_K(X)$ is invariant under a series of Markov moves given in Markov's Theorem 3.3.1.

Case MI : Equivalence in a given braid group follows immediately from ρ_n being a representation therefore is invariant under equivalent braids.

Case MII : Let $K \cong \overline{\beta}$ and $K' \cong \overline{\alpha\beta\alpha^{-1}}$ for braids $\alpha, \alpha^{-1} \in \mathcal{B}_n$. For conjugation in a given braid group the following calculation shows $\rho_n(\beta) = \rho_n(\sigma_i\beta\sigma_i^{-1})$. As any braid can be written as a composition of elementary braids this implies $\rho_n(\beta) = \rho_n(\alpha\beta\alpha^{-1})$. We apply relation 4.1 of Definition 4.0.1 to the last term in the following expansion.

$$\begin{aligned}
\rho_n(\sigma_i\beta\sigma_i^{-1}) &= \rho_n(\sigma_i)\rho_n(\beta)\rho_n(\sigma_i^{-1}) \\
&= (XI_n + X^{-1}E_i)\rho_n(\beta)(XE_i + X^{-1}I_n) \\
&= (XI_n\rho_n(\beta) + X^{-1}E_i\rho_n(\beta))(XE_i + X^{-1}I_n) \\
&= I_n\rho_n(\beta) + X^2E_iI_n\rho_n(\beta) + X^{-2}E_iI_n\rho_n(\beta) + E_i^2\rho_n(\beta) \\
&= \rho_n(\beta) + (X^2 + X^{-2})E_i\rho_n(\beta) + (-X^2 - X^{-2})E_i\rho_n(\beta) \\
&= \rho_n(\beta) \\
\implies V_K(X) &= (-X^3)^{-w(\beta)}tr(\rho_n(\beta))\tau^{n-1} \\
&= (-X^3)^{-w(\beta)}tr(\rho_n(\alpha\beta\alpha^{-1}))\tau^{n-1} = V_{K'}(X)
\end{aligned}$$

Case MIII : Let $K \cong \overline{\beta}$ and $K' \cong \overline{\beta\sigma_n}$. By Remark 5.3.4 we have $w(\beta\sigma_n) = w(\beta) + 1$ and $w(\beta\sigma_n^{-1}) = w(\beta) - 1$. The case that the nth strand passes over the $(n+1)$th strand is calculated first. In the case of the nth strand passing under the $(n+1)$th strand we use the second representation in Definition 5.1.1. This demonstrates why the factor $(-X^3)^{-w(\beta)}$ is necessary for the Jones Polynomial to be defined as a knot invariant.

$$\begin{aligned}
(-X^3)^{-w(\beta\sigma_n)}tr(\rho_{n+1}(\beta\sigma_n))\tau^{n+1-1} &= (-X^3)^{-(w(\beta)+1)}tr(\rho_n(\beta)\rho_{n+1}(\sigma_n))\tau^n \\
&= (-X^3)^{-(w(\beta)+1)}tr(\rho_n(\beta)(XI_n + X^{-1}E_n))\tau^n \\
&= (-X^3)^{-(w(\beta)+1)}tr(\rho_n(\beta))(X\tau^0 + X^{-1}\tau^{n-(n+1)})\tau^n \\
&= (-X^3)^{-w(\beta)-1}tr(\rho_n(\beta))(X(-X^2 - X^{-2}) + X^{-1})\tau^{n-1} \\
&= (-X^3)^{-w(\beta)}(-X^{-3})tr(\rho_n(\beta))(-X^3)\tau^{n-1} \\
&= (-X^3)^{w(\beta)}tr(\rho_n(\beta))\tau^{n-1} \\
(-X^3)^{-w(\beta\sigma_n^{-1})}tr(\rho_{n+1}(\beta\sigma_n^{-1}))\tau^n &= (-X^3)^{-(w(\beta)-1)}tr(\rho_n(\beta)\rho_{n+1}(\sigma_n^{-1}))\tau^n \\
&= (-X^3)^{-(w(\beta)-1)}tr(\rho_n(\beta)(XE_n + X^{-1}I_n))\tau^n \\
&= (-X^3)^{-(w(\beta)-1)}tr(\rho_n(\beta))(X\tau^{n-(n+1)} + X^{-1}\tau^0)\tau^n \\
&= (-X^3)^{-(w(\beta)-1)}tr(\rho_n(\beta))(X\tau^{-1} + X^{-1})\tau^n \\
&= (-X^3)^{-w(\beta)+1}tr(\rho_n(\beta))(X + X^{-1}(-X^2 - X^{-2}))\tau^{n-1} \\
&= (-X^3)^{-w(\beta)}(-X^3)tr(\rho_n(\beta))(-X^{-3})\tau^{n-1} \\
&= (-X^3)^{w(\beta)}tr(\rho_n(\beta))\tau^{n-1} \\
\implies V_K(X) &= (-X^3)^{w(\beta)}tr(\rho_n(\beta))\tau^{n-1} \\
&= (-X^3)^{w(\beta\sigma_n^{\pm 1})}tr(\rho_{n+1}(\beta\sigma_n^{\pm 1}))\tau^n = V_{K'}(X)
\end{aligned}$$

■

This completes the construction of the Jones polynomial $V_K(X)$. We will now give two examples to demonstrate calculating the Jones polynomial of a knot.

Example 5.4.3. Consider the trefoil knot diagram given in Figure 2.1. By Alexander's Theorem we have shown in Example 3.2.4 that the trefoil knot is equivalent to the closure of a braid $\overline{\sigma_1^3}$. We therefore consider the braid σ_1^3. As there are three crossings we expect there to be 2^3 terms in the expansion of $\rho_2(\sigma_1^3)$. We have $\sigma_1^3 \in \mathcal{B}_2$ as the braid has two strands.

$$\begin{aligned}
\rho_2(\sigma_1^3) &= (XI_2 + X^{-1}E_1)^3 \\
&= X^3I_2^3 + XI_2^2E_1 + E_1XI_2^2 + X^{-1}E_1^2I_2 + XI_2^2E_1 + I_2X^{-1}E_1^2 + X^{-1}E_1^2I_2 + X^{-3}E_1^3 \\
&= X^3I_2 + 3XI_2E_1 + 3I_2X^{-1}E^2 + X^{-3}E_1^3
\end{aligned}$$

Now we have a sum of simple n-diagrams which are elements of $\widetilde{\mathcal{TL}}_2(\tau)$.

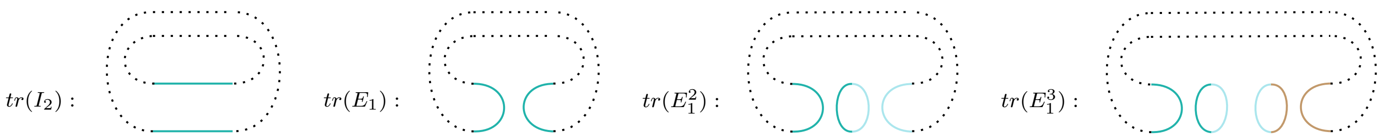


Figure 5.4: The trace of the required elements of $\widetilde{\mathcal{TL}}_2(\tau)$

The process of taking the trace of each simple n-diagram is shown in Figure 5.4.
We count the loops in each diagram to calculate the trace as follows.

$$\begin{aligned} tr(I_2) &= \tau^{2-2} = \tau^0 \\ tr(E_1) &= \tau^{1-2} = \tau^{-1} \\ tr(E_1^2) &= \tau^{2-2} = \tau^0 \\ tr(E_1^3) &= \tau^{3-2} = \tau \end{aligned}$$

Now we can give the trace of the representation of the braid as

$$tr(\rho_2(\sigma_1^3)) = (X^3 + 3X\tau^{-1} + 3X^{-1} + X^{-3}\tau) \tag{5.7}$$

It remains to multiply by the factor $(-X^3)^{-w(\sigma_1^3)}\tau^{2-1}$ to calculate the Jones polynomial. Multiplying by τ^{2-1} will cancel any negative powers of τ in Equation 5.7. We use by Definition 5.4.1 $\tau = (-X^2 - X^{-2})$ which allows us to expanded and simplify the polynomial.

$$\begin{aligned} (-X^3)^{-w(\sigma_1^3)}tr(\rho_2(\sigma_1^3))\tau &= (-X^3)^{-3}(X^3 + 3X\tau^{-1} + 3X^{-1} + X^{-3}\tau)\tau \\ &= (-X^3)^{-3}(X^3\tau + 3X + 3X^{-1}\tau + X^{-3}\tau^2) \\ &= (-X^3)^{-3}(X^3(-X^2 - X^{-2}) + 3X + 3X^{-1}(-X^2 - X^{-2}) \\ &\quad + X^{-3}(-X^2 - X^{-2})^2) \\ &= (-X^3)^{-3}(X^{-7} - X^{-3} - X^5) \\ &= (-X^{-16} + X^{-12} + X^{-4}) \end{aligned}$$

The factor of τ^{n-1} is to set the convention that the Jones Polynomial of the unknot is equal to one. This is demonstrated in the following example.

Example 5.4.4. Consider the knot K given in Figure 5.5. By Alexander's Theorem 3.2.3 we can form the closure of the braid $\overline{\beta} \cong K$.

The braid β can be seen in the highlighted section of the knot shown in Figure 5.5. The closure strands surround this highlighted section. We present this more clearly on the right.

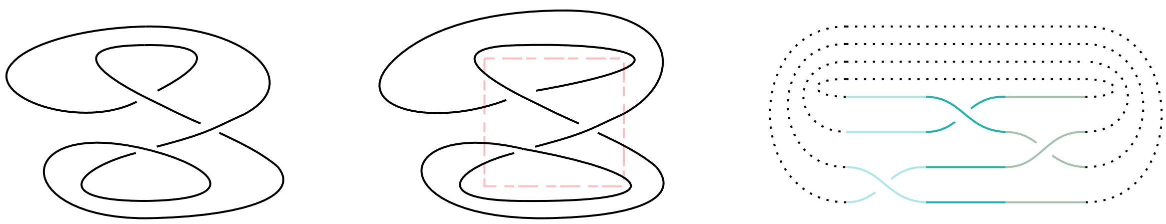


Figure 5.5: The knot K equivalent to the closure of a braid $\overline{\beta}$ respectively

The braid β is the exact braid illustrated in Figure 3.2. Therefore β is written as the composition of elementary braids given by

$$\beta \cong \sigma_3\sigma_1\sigma_2^{-1}$$

We have that $\beta \in \mathcal{B}_4$. The writhe of β is calculated $w(\beta) = 1 + 1 - 1 = 1$.

We calculate the Jones Polynomial in stages as follows.

$$\begin{aligned}
\rho_4(\sigma_3\sigma_1\sigma_2^{-1}) &= (X + X^{-1}E_3)(X + X^{-1}E_1)(X^{-1} + XE_2) \\
&= X + X^3E_2 + X^{-1}E_1 + XE_1E_2 + X^{-1}E_3 + XE_3E_2 \\
&\quad + X^{-3}E_3E_1 + X^{-1}E_3E_1E_2 \\
tr(\rho_4(\sigma_3\sigma_1\sigma_2^{-1})) &= X + X^3\tau^{-1} + X^{-1}\tau^{-1} + X\tau^{-2} + X^{-1}\tau^{-1} + X\tau^{-2} \\
&\quad + X^{-3}\tau^{-2} + X^{-1}\tau^{-3} \\
tr(\rho_4(\sigma_3\sigma_1\sigma_2^{-1}))\tau^3 &= X\tau^3 + X^3\tau^2 + X^{-1}\tau^2 + X\tau + X^{-1}\tau^2 + X\tau + X^{-3}\tau + X^{-1} \\
&= X\tau^3 + X^3\tau^2 + 2X^{-1}\tau^2 + 2X\tau X^{-3}\tau + X^{-1} \\
&= X(-X^6 - 3X^2 - 3X^{-2} - X^{-6}) + X^3(X^4 + 2 + X^{-4}) \\
&\quad + 2X^{-1}(X^4 + 2 + X^{-4}) + 2X(-X^2 - X^{-2}) + X^{-3}(-X^2 - X^{-2}) + X^{-1} \\
&= -X^7 - 3X^3 - 3X^{-1} - X^{-5} + X^7 + 2X^3 + X^{-1} + 2X^3 + 4X^{-1} + 2X^{-5} \\
&\quad - 2X^3 - 2X^{-1} - X^{-1} - X^{-5} + X^{-1} \\
&= -X^3 \\
\implies V_K(X) &= (-X^3)^{-w(\sigma_3\sigma_1\sigma_2^{-1})} tr(\rho_4(\sigma_3\sigma_1\sigma_2^{-1}))\tau^{4-1} \\
&= (-X^3)^{-1} tr(\rho_4(\sigma_3\sigma_1\sigma_2^{-1}))\tau^3 \\
&= (-X^3)^{-1}(-X^3) \\
&= 1
\end{aligned}$$

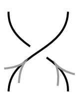
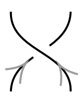
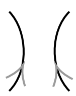
This leads us to suspect that this knot is inequivalent to the unknot. Indeed we can apply the Reidemeister move RI to the three crossings of K in Figure 5.5. By Reidemeister's Theorem 2.2.3 this will prove that K is in fact equivalent to the unknot.

6 Applications of the Jones Polynomial

6.1 Skein Relation

A skein relation is a term used in knot theory for a linear relation between three knot diagrams which differ in a small region. Skein relations often simplify knot invariants. The following skein relation is sufficient for computing the Jones polynomial. The skein relation gradually decreases the number of crossings of the knot and allows us to compute the Jones polynomial recursively.

Definition 6.1.1. The knots $K_+,\ K_-,\ K_I$ are the same except for a small neighbourhood of a point where they are given by

$K_+ =$  $K_- =$  $K_I =$ 

Theorem 6.1.2. *[Lic97, Proposition 3.7] Let K be a knot and $K_\circ$ be the unknot. The Jones polynomial $V_K(X)$ of a knot K is given by the relations*

$$V_{K_\circ}(X) = 1 \tag{6.1}$$

$$(-X^4)V_{K_+} + (X^{-4})V_{K_-} = (X^2 - X^{-2})V_{K_I} \tag{6.2}$$

Proof. For the first equation let $\beta \in \mathcal{B}_1$, then

$$V_{K_\circ} = (-X^3)^{-w(\beta)} tr(\rho_n(\beta))\tau^{n-1} = 1$$

By Theorem 5.4.2, $V_{K_\circ}(X)$ is invariant under equivalent knots and the claim follows.

The second relation is given as follows. By Alexander's Theorem 3.2.3 we can consider the additional crossing as a crossing of the braid.

$$\begin{aligned}
&(-X^4)V_{K_+} + (X^{-4})V_{K_-} \\
&= -X^4(-X^3)^{-w(K_+)} tr(\rho_n(\beta\sigma_i))\tau^{n-1} + X^{-4}(-X^3)^{-w(K_-)} tr(\rho_n(\beta\sigma_i^{-1}))\tau^{n-1} \\
&= -X^4(-X^3)^{-w(K_+)}(X + X^{-1}\tau^{-1}) tr(\rho_n(\beta))\tau^{n-1} + X^{-4}(-X^3)^{-w(K_-)}(X^{-1} + X\tau^{-1}) tr(\rho_n(\beta))\tau^{n-1} \\
&= -X^4(-X^3)^{-w(K_+)}(X\tau + X^{-1}) tr(\rho_n(\beta))\tau^{n-2} + X^{-4}(-X^3)^{-w(K_-)}(X^{-1}\tau + X) tr(\rho_n(\beta))\tau^{n-2} \\
&= -X^4(-X^3)^{-w(K_+)}(-X^3) tr(\rho_n(\beta))\tau^{n-2} + X^{-4}(-X^3)^{-w(K_-)}(-X^{-3}) tr(\rho_n(\beta))\tau^{n-2} \\
&= -X^4(-X^3)^{-(w(\beta)+1)}(-X^3) tr(\rho_n(\beta))\tau^{n-2} + X^{-4}(-X^3)^{-(w(\beta)-1)}(-X^{-3}) tr(\rho_n(\beta))\tau^{n-2} \\
&= -X^4(-X^{-3})(-X^3)^{-w(\beta)}(-X^3) tr(\rho_n(\beta))\tau^{n-2} + X^{-4}(-X^3)(-X^3)^{-w(\beta)}(-X^{-3}) tr(\rho_n(\beta))\tau^{n-2} \\
&= -X^4(-X^3)^{-w(\beta)} tr(\rho_n(\beta))\tau^{n-2} + X^{-4}(-X^3)^{-w(\beta)} tr(\rho_n(\beta))\tau^{n-2} \\
&= (-X^4 + X^{-4})(-X^3)^{-w(\beta)} tr(\rho_n(\beta))\tau^{n-2} \\
&= (X^2 - X^{-2})(-X^2 - X^{-2})(-X^3)^{-w(\beta)} tr(\rho_n(\beta))\tau^{n-2} \\
&= (X^2 - X^{-2})\tau(-X^3)^{-w(\beta)} tr(\rho_n(\beta))\tau^{-1}\tau^{n-1} \\
&= (X^2 - X^{-2})(-X^3)^{-w(\beta)} tr(\rho_n(\beta))\tau^{n-1} \\
&= (X^2 - X^{-2})V_{K_I}
\end{aligned}$$

■

The knot diagram K_+ given in Figure 6.1 is the trefoil knot given in Figure 2.1. The knots K_- and K_I differ only in the highlighted region. They differ in this region by the diagrams given in Definition 6.1.1. The crossings correspond to the orientation convention given in Remark 5.3.3.

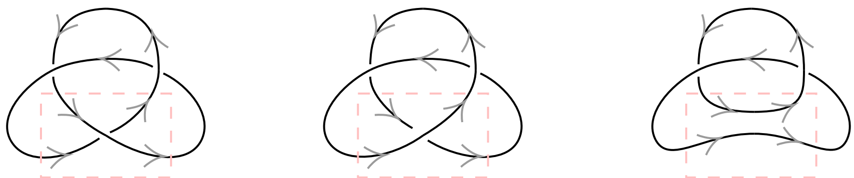
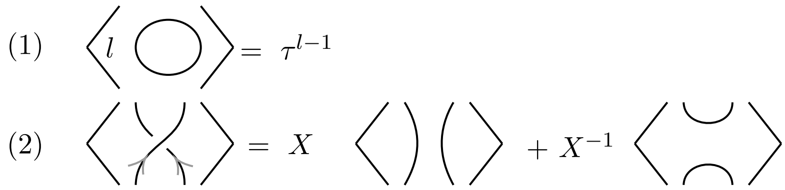


Figure 6.1: The trefoil knot diagram K_+ and the knot diagrams K_- and K_I respectively

6.2 Kauffman Bracket

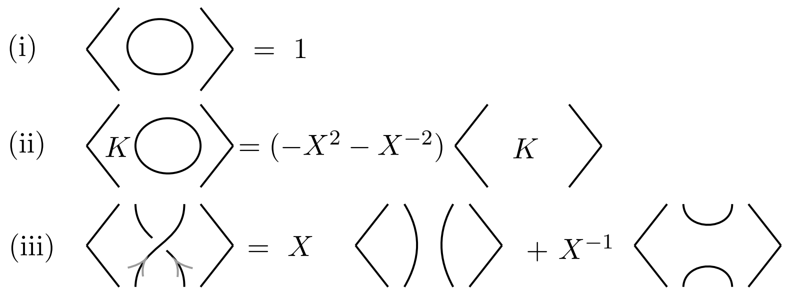
In this section we present the Kauffman bracket $\langle K \rangle$. We will use the Kauffman bracket to construct a function that is a knot invariant. In this case the function must be invariant under a series of Reidemeister moves as we are applying the function to the knot directly. We conclude by proving that the function is in fact a variation of the Jones polynomial. This is an even simpler form of the Jones polynomial than the skein relation given in Theorem 6.1.2. This is due to the number of crossings of the knot diagram being strictly reduced when the Kauffman bracket is applied. The material in this section is primarily from Chapter 5 of the book [PBI+23].

Consider a knot diagram K. Each crossing of K is resolved by a choice from the set { ≍ ,) (}. This results in a knot diagram with no crossings that is a *state* of the knot K. As a choice of two is made at each crossing a knot diagram with n crossings will have a summation of 2^n states. A number of closed loops will form in each state which we denote l. This is given as the following skein relation along with the convention that the unknot gives a bracket of one. Each closed loop in a diagram is removed as a power of the variable τ.



The aim would be to find the necessary coefficients X and Y and variable τ in order for the bracket to become a knot invariant. We will derive the conditions given in Definition 6.2.1. However we will see that the Kauffman bracket $\langle K \rangle$ cannot be a knot invariant without introducing a factor which nullifies the change under the Reidemeister move *RI*. This follows from Remark 5.3.4.

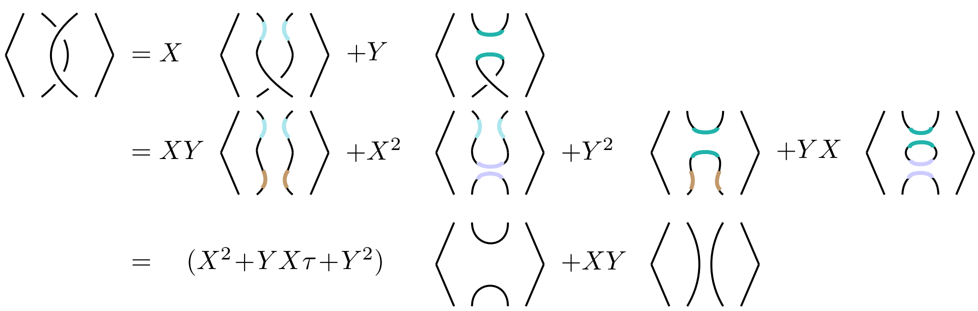
Definition 6.2.1. For a knot diagram K the *Kauffman bracket* $\langle K \rangle \in \mathbb{C}[X, X^{-1}]$ of K is defined recursively by the follow properties:



The sum of all states is referred to as the *state sum* formulation of the Kauffman Bracket $\langle K \rangle$. An orientation is not required for the result (iii) in Definition 6.2.1 as each state is a number of closed loops.

Recall the Reidemeister moves given in Definition 2.2.2. In order to construct the Kauffman bracket as a knot invariant we must show that the Kauffman bracket of the knot diagram is preserved when each Reidemeister move is applied locally. Notice that $\langle\ )\ (\ \rangle$ in the skein relation resembles the result of Reidemeister move RII. We use this equality to calculate the coefficients required for $\langle K \rangle$ to be invariant under RII.

Case RII : Notice for the negative crossing of RII in Figure 5.3 we must exchange the coefficients X and Y as the orientation is reversed. The Kauffman bracket is given by

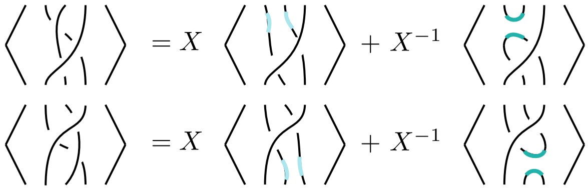


For RII we require equality between the left bracket and second term therefore

$$Y = X^{-1}$$

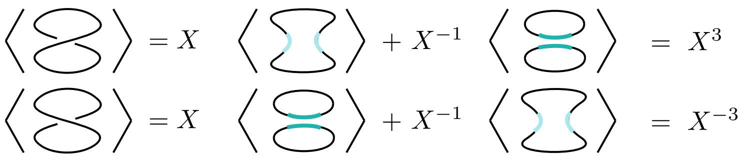
$$X^2 + YX\tau + Y^2 = 0 \implies \tau = -X^2 - X^{-2}$$

Case RIII : We show the skein relation with the above coefficients is invariant under $RIII$ by



The first terms equal by planar isotopy. The second terms equal by applying *Case RII*.

Case RI : The Kauffman bracket is *not* invariant under RI as shown in the following diagram.



By Reidemeister's Theorem 2.2.3 the left knot diagrams are both equivalent to the unknot. Therefore a knot invariant on these knots must be equivalent to the unknot that is by assumption one. However the Reidemeister move RI causes an additional factor of $X^{\pm 3}$ depending on whether it is a positive or negative crossing. To nullify this change we introduce the factor $(-X^3)^{-w(K)}$. Consider the right hand rule in Remark 5.3.3. As RI has a factor $(-X^3)$ for a positive crossing $(-X^3)^{-1}$ will cancel this. The negative crossing is cancelled similarly. We also know by Remark 5.3.4 that the writhe is not effected under moves RII, $RIII$ or planar isotopy. Therefore the Kauffman bracket multiplied by this factor becomes a knot invariant.

Corollary 6.2.2. *For a knot K then $(-X^3)^{-w(K)}\langle K\rangle$ is a knot invariant.*

Theorem 6.2.3. *[DA24, Lemma 2.2] The Jones Polynomial of a knot K is given by*

$$V_K(X) = (-X^3)^{-w(K)}\langle K\rangle \tag{6.3}$$

Proof. Let K be a knot and β a braid such $\overline{\beta} \cong K$. It is immediate that $w(\overline{\beta}) = w(\beta)$ as the closure is formed by non-crossing strands. We have that $(-X^3)^{-w(K)}$ and $(-X^3)^{-w(\beta)}$ are introduced to nullify the effect of RI and $MIII$ respectively. By Remark 5.3.4 the moves RI and $MIII$ have the same effect on the knot therefore any such moves will be nullified in $V_K(X)$ by these factors and can therefore be considered equal. Therefore in comparison to Definition 5.6 it is required to show

$$\langle K\rangle = \tau^{n-1} tr(\rho_n(\beta))$$

There is a one to one correspondence between the simple n-diagrams as a result of $\rho_n(\beta)$ and the states in the expansion of $\langle K\rangle$. The difference at this point is we consider the crossings on the knot instead of on the braid. In each case there is 2^n terms in the expansion as at each crossing we have a decision of the same two resolutions.

By linearity of the trace function it suffices to show each term in the Kauffman bracket expansion is equal to the trace of the simple n-diagram multiplied by the factor τ^{n-1}. To compute the term of the Kauffman bracket we resolve each of the crossings then apply the relations in Definition 6.2.1 until we we obtain a diagram of closed loops.

We claim that the Kauffman bracket of l closed loops is $(-X^2 - X^{-2})^{l-1}$. We apply Definition 6.2.1 inductively as follows.

We show this holds for 1 loop: $\langle \circ \rangle = 1$

Assume this holds for l loops: $\langle l\circ \rangle = (-X^2 - X^{-2})^{l-1}$

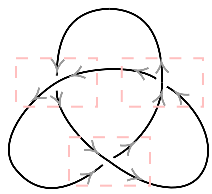
This implies: $\langle (l+1)\circ \rangle = (-X^2 - X^{-2})\langle l\circ \rangle = (-X^2 - X^{-2})(-X^2 - X^{-2})^{l-1} = (-X^2 - X^{-2})^{(l+1)-1}$

In comparison, taking the trace of the simple n-diagram will produce a number of loops which will be equivalent to the number of loops in the corresponding Kauffman bracket term. Multiplying this value by the factor τ^{n-1} for $\tau = (-X^2 - X^{-2})$ implies

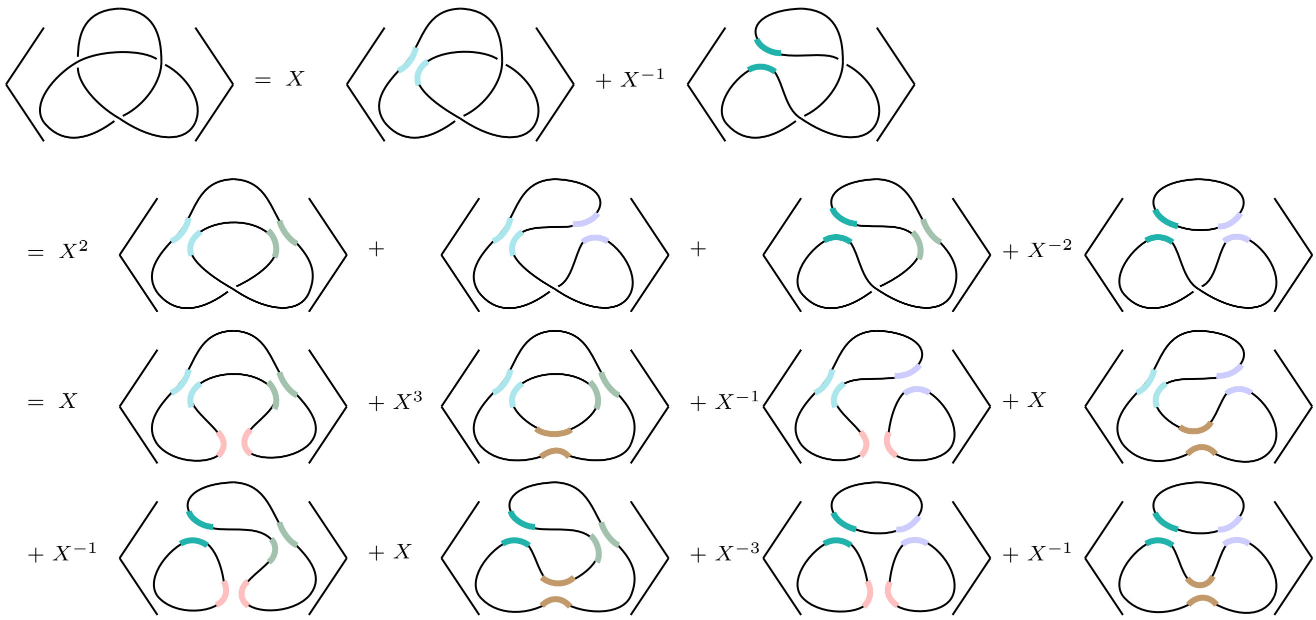
$$\tau^{n-1}\tau^{l-n} = \tau^{l-1} = (-X^2 - X^{-2})^{l-1}$$

Therefore each term of $\langle K \rangle$ equals the corresponding term in $\tau^{n-1} tr(\rho_n(\beta))$ as required. ■

Example 6.2.4. Let K be the trefoil knot in Figure 2.1. We orientate the knot diagram to give



and apply (iii) in Definition 6.2.1 to each of the highlighted areas. Therefore the Kauffman bracket $\langle K \rangle$ is given by



There are 3 crossings which give 2^3 states in the bracket expansion. By applying the relations (i) and (ii) in Definition 6.2.1 this results in the following expansion.

$$\begin{aligned}\langle K\rangle &= X^3(-X^2 - X^{-2}) + X + X + X^{-1}(-X^2 - X^{-2}) + X + X^{-1}(-X^2 - X^{-2})\\ &\quad + X^{-1}(-X^2 - X^{-2}) + X^{-3}(-X^2 - X^{-2})^2\\ &= X^3(-X^2 - X^{-2}) + 3X + 3X^{-1}(-X^2 - X^{-2}) + X^{-3}(-X^2 - X^{-2})^2\\ &= (X^{-7} - X^{-3} - X^5)\end{aligned} \tag{6.4}$$

Therefore

$$(-X^3)^{-3}\langle K\rangle = (-X^{-16} + X^{-12} + X^{-4}) \tag{6.5}$$

Notice the resulting Laurent polynomial in Equation 6.5 is the same as the Jones polynomial of the trefoil knot in Example 5.4.3 as expected.

6.3 The Jones Unknotting Conjecture

The classical problem of knot theory is to decide whether two knots are equivalent. The Jones polynomial allows us to detect distinct knots. However the Jones polynomial does not detect equivalent knots. Therefore the Jones polynomial is *not* a complete knot invariant. There exists infinitely many distinct knots with equal Jones Polynomials. This can be shown by *Conway mutation* which is detailed in Chapter 7 of the book [Cro04].

An example is the Kinoshita-Terasaka knot and Conway knot illustrated in Figure 6.2. The length of the calculation of the Jones polynomial increases exponentially as the number of crossings of the knot increases. There would be 2^{11} states in the Jones polynomial of the Kinoshita-Terasaka and Conway knots. Therefore we do not include the calculation of the Jones polynomial of these knots.

The Jones polynomial for the knots K and K' in Figure 6.2 are equal, that is $V_K(X) = V_{K'}(X)$. However the knot K is *not* equivalent to the knot K' in Figure 6.2. We omit the proofs which can be found in the paper [Lic88].

Figure 6.2: Kinoshita-Terasaka Knot K and Conway Knot K' respectively

A more specific problem of knot theory is whether a knot is equivalent to the unknot. We would like to know whether the Jones polynomial detects the unknot. For this problem there is currently no known counter example. That is there is no known non-trivial knot K such that the Jones Polynomial of K is the same as the Jones Polynomial of the unknot.

Conjecture 6.3.1. *[PBI+23, Conjecture 5.2.9] The Jones unknotting Conjecture states*

$$V_K(X) = 1 \iff K \textit{ is the unknot}$$

It has been shown that the conjecture holds for knots of up to 24 crossings; see the paper [RET20]. The conjecture in general remains an open problem in knot theory.

6.4 Chiral Knots

The aim of this subsection is to understand whether the Jones polynomial distinguishes a knot K from it's mirror image K^*. We now understand that if the Jones polynomials $V_K(X)$ and $V_{K^*}(X)$ are distinct then the knots are distinct. We return to the familiar trefoil knot as first presented in Figure 1.1. We will compare the Jones polynomials of the trefoil knot and it's mirror image. We will observe a certain relation which will allow us to understand how the Jones polynomial can in general distinguish whether a knot is equivalent to it's mirror image. The material follows from Chapter 11 of the book [Mur93].

Definition 6.4.1. A knot K is *amphichiral* if it is ambient isotopic to it's mirror image K^*. Otherwise the knot is *chiral*.

In order to study chirality it is necessary to understand how to construct the mirror image of a knot K. The mirror image of a knot is accessed more easily when considering the braid β such $K \cong \overline{\beta}$. The mirror image of the braid β is β^{-1} by definition. By Alexander's Theorem 3.2.3 this implies $K^* \cong \overline{\beta^{-1}}$.

Example 6.4.2. Consider the trefoil knot as the closure of a braid $K \cong \beta$ from Figure 3.2.4. The mirror image β^{-1} is achieved by replacing each of the elementary braid elements with the inverse. In this case each strand in position 1 will now pass under the strand in position 2. This is illustrated in Figure 6.4.2. Taking the closure of this braid produces the mirror image of the trefoil knot K^*.

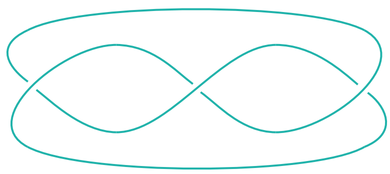
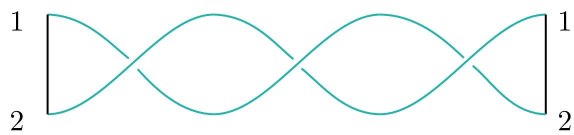
$K^* = \overline{\beta^{-1}} =$  , $\beta^{-1} =$ 

Figure 6.3: The mirror image of the trefoil knot K^* with the corresponding braid β^{-1}

Proposition 6.4.3. *The trefoil knot is chiral.*

Proof. We aim to prove that the trefoil knot K is *not* equivalent to it's mirror image K^*. We have calculated the Jones Polynomial of the trefoil knot in Example 5.4.3. We will calculate the Jones Polynomial of the mirror image of the trefoil knot. By Theorem 5.4.2 if the resulting Laurent polynomials are distinct then the knots are distinct.

For the first line in Equation 6.6 the negative power requires the representation of the negative crossing given in Definition 5.1.1.

$$\begin{aligned}\rho_2(\sigma_1^{-3}) &= (XE_1 + X^{-1}I_2)^3 \\ &= X^3E_1^3 + XE_1^2I_2 + XE_1^2I_2 + X^{-1}E_1I_2^2 + XE_1^2I_2 + X^{-1}E_1I_2^2 + X^{-1}E_1I_2^2 + X^{-3}I_2^3 \\ &= X^3E_1^3 + 3XE_1^2I_2 + 3X^{-1}E_1I_2^2 + X^{-3}I_2^3\end{aligned} \tag{6.6}$$

By Figure 5.4 we have $tr(I_2) = \tau^0$, $tr(E_1) = \tau^{-1}$, $tr(E_1^2) = \tau^0$, $tr(E_1^3) = \tau^1$. Therefore

$$\begin{aligned}(-X^3)^{-w(\sigma_1^{-3})}tr(\rho_2(\sigma_1^{-3}))\tau &= (-X^3)^{-(-3)}(X^3\tau + 3X + 3X^{-1}\tau^{-1} + X^{-3})\tau \\ &= (-X^3)^3(X^3(-X^2 - X^{-2})^2 + 3X(-X^2 - X^{-2}) + 3X^{-1} + X^{-3}(-X^2 - X^{-2})) \\ &= (-X^3)^3(X^7 - X^3 - X^{-5}) \\ &= (-X^{16} + X^{12} + X^4)\end{aligned}$$

This implies

$$V_K(X) = (-X^{-16} + X^{-12} + X^{-4}) \neq (-X^{16} + X^{12} + X^4) = V_{K^*}(X) \tag{6.7}$$

Therefore the trefoil is not equivalent to it's mirror image. This implies the trefoil knot is chiral. ■

Notice in Equation 6.7 that $V_K(X) = V_{K^*}(X^{-1})$. The formal variable in the Jones polynomial of K has been replaced by the inverse of such in Jones polynomial of K^*. This holds for all knots by the following theorem.

Theorem 6.4.4. *[Mur93, Theorem 11.2.5] Suppose that the knot K^* is the mirror image of the knot K then*

$$V_{K^*}(X^{-1}) = V_K(X)$$

Furthermore if K is amphichiral then

$$V_K(X) = V_K(X^{-1})$$

Proof. Let K be a knot and $\beta \in \mathcal{B}_n$ a braid such that $K \cong \overline{\beta}$. Recall from Equation 3.2 a braid can be written as a composition of elementary braids

$$\beta \cong \sigma_{i_1}^{\alpha_1}\sigma_{i_2}^{\alpha_2}\cdots\sigma_{i_m}^{\alpha_m}$$

To construct the mirror image of the braid at each crossing in β we replace σ_i with it's inverse σ_i^{-1}

$$\beta^{-1} \cong \sigma_{i_1}^{-\alpha_1}\sigma_{i_2}^{-\alpha_2}\cdots\sigma_{i_m}^{-\alpha_m}$$

The mirror image of the knot K denoted K^* is ambient isotopic to $\overline{\beta^{-1}}$.

We compute the Jones Polynomial using Definition 5.4.1. Recall the representations in Definition 5.1.1. We aim to calculate $\rho_n(\beta^{-1})$. The representations $(XI_n + X^{-1}E_i)$ and $(X^{-1}I_n + XE_i)$ in $\rho_n(\beta^{-1})$ will interchange if an over crossing has been replaced with an under crossing and vice versa. It is possible α_i may be negative in the first instance however this does not effect the proof as it would become positive which will obtain the alternative representation as required.

$$\begin{aligned}\rho_n(\beta) &= \rho_n(\sigma_{i_1}^{\alpha_1}\sigma_{i_2}^{\alpha_2}\cdots\sigma_{i_m}^{\alpha_m}) \\ &= (XI_n + X^{-1}E_{i_1})^{\alpha_1}(XI_n + X^{-1}E_{i_2})^{\alpha_2}\cdots(XI_n + X^{-1}E_{i_m})^{\alpha_m}\end{aligned}$$

$$
\begin{aligned}
\rho_n(\beta^{-1}) &= \rho_n(\sigma_{i_1}^{-\alpha_1}\sigma_{i_2}^{-\alpha_2}\cdots\sigma_{i_m}^{-\alpha_m}) \\
&= (X^{-1}I_n + XE_{i_1})^{\alpha_1}(X^{-1}I_n + XE_{i_2})^{\alpha_2}\cdots(X^{-1}I_n + XE_{i_m})^{\alpha_m} \\
&= ((X^{-1})I_n + (X^{-1})^{-1}E_{i_1})^{\alpha_1}((X^{-1})I_n + (X^{-1})^{-1}E_{i_2})^{\alpha_2}\cdots((X^{-1})I_n + (X^{-1})^{-1}E_{i_m})^{\alpha_m}
\end{aligned}
$$

Therefore each formal variable X has been replaced with X^{-1} in the representation of $\rho_n(\beta)$ to achieve the representation of $\rho_n(\beta^{-1})$. Hence $V_{K^*}(X^{-1}) = V_K(X)$ as required.

If $V_K(X^{-1}) = V_K(X)$ by the previous calculation this implies $\rho_n(\beta) = \rho_n(\beta^{-1})$. As ρ_n is defined to be a representation on $\mathcal{B}_n$ this implies $\beta \cong \beta^{-1}$. Therefore $K \cong \overline{\beta} \cong \overline{\beta^{-1}} \cong K^*$ as required.

■

Therefore the Jones Polynomial allows us to study amphichirality. It follows from Theorem 6.4.4 that

$$V_K(X) \neq V_K(X^{-1}) \implies K \textit{ is chiral}$$

However the Jones Polynomial cannot detect a mirror image. That is $V_{K^*}(X^{-1}) = V_K(X)$ does *not* imply that K^* is the mirror image of K. There is currently no known knot invariant which is able to fully detect chirality.

7 Beyond the Jones Polynomial

The Jones polynomial led to the discovery of a new knot invariant, the HOMFLY polynomial $P_K(z,m)$. The construction of the HOMFLY polynomial is similar to the Jones polynomial. We will construct a representation φ_m on the braid group into the Hecke algebra $\mathcal{H}_n(z,m)$ and compose this with the Markov trace

$$Tr \circ \varphi_m : \mathcal{B}_n \to \mathcal{H}_n(z,m) \to \mathbb{C}[z^{\pm 1}, m^{\pm 1}]$$

to obtain a Laurent polynomial in two variables. It remains to find the necessary conditions such that the function is preserved under a series of Markov moves.

The relations of $\mathcal{H}_n(z,m)$ are similar to those of $\mathcal{TL}_n$. The Temperley-Lieb algebra is a quotient of the Hecke algebra. We will see that the Markov trace Tr on $\mathcal{H}_n$ in Definition 7.1.10 satisfies the same relations as the Markov trace tr on $\mathcal{TL}_n$ in Definition 5.2.2. Therefore the resulting HOMFLY polynomial is closely related to the Jones polynomial. We will prove that the HOMFLY polynomial contains the Jones polynomial as a special case for $z = (X^2 - X^{-2})$ and $m = (-X^{-4})$.

The Hecke-algebras are closely related to the symmetric group $\mathcal{S}_n$ and braid group $\mathcal{B}_n$ given in Section 3.1. For $\mathcal{H}_n(z = 0, m = 1)$ we achieve the group algebra of $\mathcal{S}_n$. The Hecke algebras share certain properties of $\mathcal{S}_n$. We use these properties to gain an understanding of the elements of $\mathcal{H}_n(z,m)$. In particular this will allow us to define the Markov trace Tr recursively. The material follows Chapter 17 of the book [BZH13].

7.1 The Hecke Algebra

We return to the Symmetric Group $\mathcal{S}_n$ which admits a presentation

$$\mathcal{S}_n = \langle \delta, ..., \delta_{n-1} \mid \delta_1^2 = I,\ \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1},\ \delta_i\delta_j = \delta_j\delta_i \text{ for } |i-j| > 1\rangle$$

Group composition is defined as the usual permutation composition in this case from left to right which aligns with the composition of braids. Every permutation $\pi \in \mathcal{S}_n$ can be written as a composition of transpositions. We define the unique representation $b_\pi(\delta_i)$ of $\mathcal{S}_n$ as follows. For the maximum transposition $(n-1, n)$ where $\pi^{-1}(n) = j$ we write

$$b_\pi(\delta_i) = (j, j+1)(j+1, j+2)...(n-1, n) \cdot b_{\pi'}(\delta_i) \tag{7.1}$$

for $\pi' \in \mathcal{S}_{n-1}$. We can embed $\mathcal{S}_{n-1}$ as a subgroup of $\mathcal{S}_n$ such that the nth generator remains fixed. This is shown in Figure 7.1.

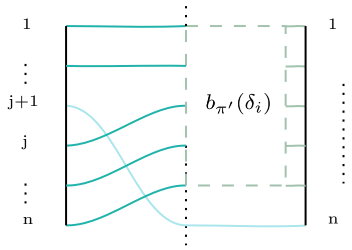


Figure 7.1: The diagram of $b_\pi(\delta_i)$ for some $\pi \in \mathcal{S}_n$

Define $\mathcal{W}_n(\delta_i) = \{b_\pi(\delta_i) \mid \pi \in \mathcal{S}_n\}$. It follows from $\mathcal{S}_n$ having order $n!$ that $\mathcal{W}_n$ has $n!$ elements which we call *words*.

Example 7.1.1. Consider the cycle $(1, 5, 2, 3)$. Then $\pi^{-1}(5) = 1$. Therefore by Equation 7.1 $b_\pi(\delta_i) = (1, 2)(2, 3)(3, 4)(4, 5) \cdot (3, 4)(1, 2)(2, 3)$. This is shown in the following diagram.

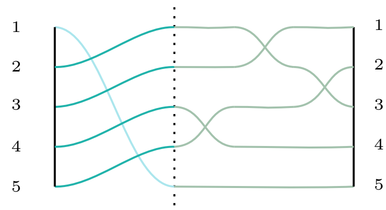


Figure 7.2: The cycle $(1523) \in \mathcal{S}_n$ written in the form $b_\pi(\delta_i)$

Definition 7.1.2. Define $\widetilde{\mathcal{S}_n}$ as the semigroup given by the presentation

$$\widetilde{\mathcal{S}_n} = \langle \tilde{\delta}, ..., \tilde{\delta}_{n-1} \mid \tilde{\delta}_i \tilde{\delta}_{i+1} \tilde{\delta}_i = \tilde{\delta}_{i+1} \tilde{\delta}_i \tilde{\delta}_{i+1},\ \tilde{\delta}_i \tilde{\delta}_j = \tilde{\delta}_j \tilde{\delta}_i\ for\ |i-j| \geq 2 \rangle$$

$\widetilde{\mathcal{S}_n}$ is a semigroup of $\mathcal{S}_n$ which means it has the same group composition as $\mathcal{S}_n$ but does not have a multiplicative inverse. Therefore the elements of $\widetilde{\mathcal{S}_n}$ correspond to the braids $\beta \cong \sigma_{i_1}^{\alpha_1} \sigma_{i_2}^{\alpha_2} \cdots \sigma_{i_m}^{\alpha_m}$ where the powers $\{\alpha_1, ..., \alpha_m\}$ are strictly positive.

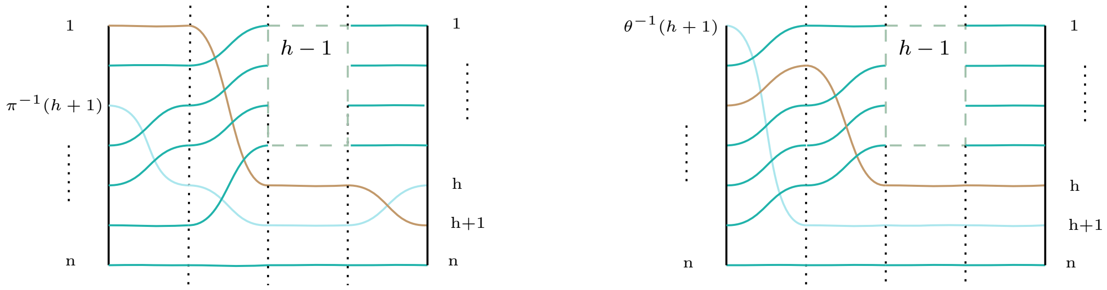
There is a canonical homomorphism $\phi : \widetilde{\mathcal{S}_n} \to \mathcal{S}_n$ given by

$$\phi(\tilde{\delta}_i) = \delta_i \tag{7.2}$$

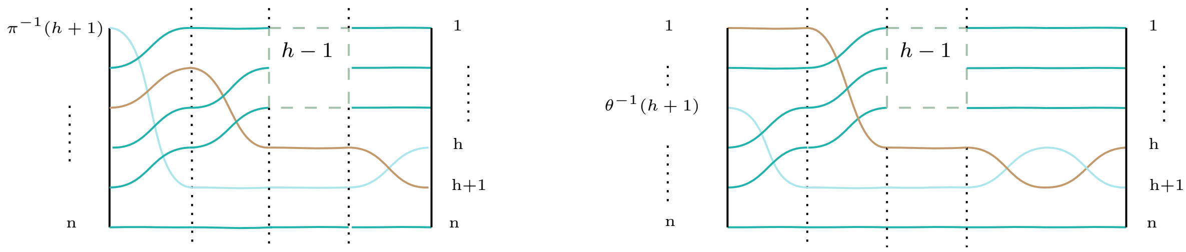
We can define the representation $\widetilde{b_\pi} = b_\pi(\tilde{\delta}_i)$ on $\widetilde{\mathcal{S}_n}$. This forms the set $\widetilde{\mathcal{W}_n} = \{\widetilde{b_\pi} \mid \pi \in \mathcal{S}_n\}$. We now construct the product $\widetilde{b_\pi} \cdot \tilde{\delta}_h$. This continues to follow the construction in Chapter 17 of the book [BZH13].

Definition 7.1.3. We consider the following two cases. *Case I* holds if the strands crossing at δ_h do not cross with each other in b_π. *Case II* holds if they do.

$$Case\ I\ \ : \widetilde{b_\pi} \cdot \tilde{\delta}_h = \widetilde{b_\theta} \qquad \text{for } \theta = \pi\delta_h$$



$$Case\ 2 : \widetilde{b_\pi} \cdot \tilde{\delta}_h = \widetilde{b_\theta}\tilde{\delta}_h^{\ 2} \qquad \text{for } \theta\delta_h = \pi$$



We now use the $n!$ words of $\mathcal{W}_n$ to construct the Hecke algebra. We first define M_n to be the free module over a unitary commutative ring R on the words of $\mathcal{W}_n$. We now denote each generator $\tilde{\delta}_i$ by g_i. Let M_n have the basis $\mathcal{W}_n(g_i) = \{b_\pi(g_i) |\ \pi \in \mathcal{S}_n\}$. Each of the generators g_j for $j \in \{1, ..., n-1\}$ are contained in this basis as each can be written in the form $b_{\delta_j}(g_i) = g_j$. This follows from $b_{\delta_j}(\delta_i) = \delta_j$ and the canonical homomorphism $\phi(\tilde{\delta}_j) = \delta_j$. To achieve the structure of an associative R-algebra it remains to define an associative product $\cdot$ on M_n. By distributivity it is enough to define the product on the basis elements. This is obtained by iterating the following definition for the basis elements $b_\pi(g_i) \in \mathcal{W}_n(g_i)$.

Definition 7.1.4. Let $g_h^2 = zg_h + 1$ for some fixed element $z \in R$. Then Definition 7.1.3 takes the form

$$Case\ I\ \ : b_\pi(g_i) \cdot g_h = b_\theta(g_i) \qquad \text{for } \theta = \pi\delta_h$$

$$Case\ II : b_\pi(g_i) \cdot g_h = zb_\pi(g_i) + b_\theta(g_i) \qquad \text{for } \theta\delta_h = \pi$$

The *Case II* follows from $b_\pi(g_i) \cdot g_h = zb_\theta(g_i) \cdot g_h + b_\theta(g_i)$ where $b_\theta(g_i) \cdot g_h = b_{\theta\delta_h}(g_i) = b_\pi(g_i)$ by construction on the basis $\mathcal{W}_n(g_i)$ and the condition $\theta\delta_h = \pi$.

Lemma 7.1.5. *[BZH13, Lemma 17.6] The product in Definition 7.1.4 is associative on $\mathcal{W}_n(g_i)$.*

We have now constructed an R-algebra of rank $n!$ which is known as the Hecke algebra. We will now denote it by $\mathcal{H}_n(z, m)$.

Theorem 7.1.6. *[BZH13, Theorem 17.7] Let R be a commutative unitary ring of Laurent polynomials in two variables $R = \mathbb{C}[z^{\pm 1}, m^{\pm 1}]$ such that z and m are formal variables. The unitary associative R-algebra with generators $g_1, ..., g_{n-1}$ defined by the relations:*

$$g_i^2 = zg_i + 1 \tag{7.3}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{7.4}$$

$$g_i g_j = g_j g_i \qquad \text{for } |i-j| > 1 \tag{7.5}$$

is isomorphic to the Hecke algebra $\mathcal{H}_n(z, m)$.

Remark 7.1.7. The inverse g_i^{-1} of g_i follows from Relation 7.3 where $g_i^2 - zg_i = g_i(g_i - z) = 1$ implies $g_i^{-1} = g_i - z$.

We now use the Hecke algebra $\mathcal{H}_n(z, m)$ to construct a representation of the braid group $\mathcal{B}_n$.

Definition 7.1.8. Define the homomorphism $\varphi_m : \mathcal{B}_n \to \mathcal{H}_n(z, m)$ by

$$\varphi_m(\sigma_j) = mg_j \tag{7.6}$$

$$\varphi_m(\sigma_j^{-1}) = m^{-1} g_j^{-1} \tag{7.7}$$

Proposition 7.1.9. *The map $\varphi_m : \mathcal{B}_n \to \mathcal{H}_n(z, m)$ is a representation of the Artin Braid Group.*

Proof. We must verify that φ_m preserves the generator relations of $\mathcal{B}_n$ as given in Definition 3.1. By Remark 7.1.7 we have the following.

$$\varphi_m(\sigma_i)\varphi_m(\sigma_i^{-1}) = mg_i m^{-1} g_i^{-1} = g_i(g_i - z) = 1 = (g_i - z)g_i = m^{-1} g_i^{-1} m g_i = \varphi_m(\sigma_i^{-1})\varphi_m(\sigma_i)$$

The below equations follow immediately from the Relations 7.4 and 7.5 in Definition 7.1.6.

$$\varphi_m(\sigma_i \sigma_{i+1} \sigma_i) = m^3 g_i g_{i+1} g_i = m^3 g_{i+1} g_i g_{i+1} = \varphi_m(\sigma_{i+1} \sigma_i \sigma_{i+1})$$

$$\varphi_m(\sigma_i \sigma_j) = \varphi_m(\sigma_i)\varphi_m(\sigma_j) = m^2 g_i g_j = m^2 g_j g_i = \varphi_m(\sigma_j)\varphi_m(\sigma_i) = \varphi_m(\sigma_j \sigma_i) \text{ for } |i-j| > 1$$

■

For the construction of the trace we temporarily introduce a third variable C to the ring R.

Definition 7.1.10. A *Markov trace* on $\mathcal{H}_n(z, m)$ is a linear function $Tr : \mathcal{H}_n(z, m) \to \mathbb{C}[z^{\pm 1}, m^{\pm 1}, C]$ satisfying the following properties:

(*i*) $Tr(I_{\mathcal{H}_n}) = 1$

(*ii*) $Tr(ab) = Tr(ba)$ for $a, b \in \mathcal{H}_n(z, m)$

(*iii*) $Tr(xg_{n-1}) = C \cdot Tr(x)$ for $x \in \mathcal{H}_{n-1}(z, m)$

Remark 7.1.11. By Remark 7.1.7, $Tr(xg_n^{-1}) = Tr(xg_n) - z \cdot Tr(x) = (C - z) \cdot Tr(x)$ for $x \in \mathcal{H}_n(z, m)$.

Lemma 7.1.12. *[BZH13, Lemma 17.10] There is a unique function Tr on $\bigcup_{n=1}^{\infty} H_n(z, m)$ satisfying the axioms in Definition 7.1.10.*

Remark 7.1.13. The proof shows that a trace on $\mathcal{H}_{n-1}$ can be uniquely extended to a trace on $\mathcal{H}_n$. Let $w \in \mathcal{H}_n / \mathcal{H}_{n-1}$. Then w must contain g_{n-1} as it is the only non-common element as $g_{n-1} \notin \mathcal{H}_{n-1}$. By Equation 7.1 we have that δ_{n-1} appears in $w \in \mathcal{W}_n$ at most once. Therefore combining this with the first statement we have g_{n-1} occurs in w exactly once. Therefore w can be written $x_1 g_{n-1} x_2$ for $x_1 x_2 \in \mathcal{H}_{n-1}$. By relations (*ii*) and (*iii*) in Definition 7.1.10 we have the extension of Tr given by

$$Tr(x_1 g_{n-1} x_2) = C \cdot Tr(x_1 x_2) \text{ for } x_1 g_{n-1} x_2 \in \mathcal{H}_n \text{ and } x_1, x_2 \in \mathcal{H}_{n-1} \tag{7.8}$$

7.2 The HOMFLY Polynomial

We now have the function $Tr \circ \varphi_m : \mathcal{B}_n \to \mathcal{H}_n(z,m) \to \mathbb{C}[z^{\pm 1}, m^{\pm 1}, C]$. We are aiming to construct a knot invariant. Consider the function defined by

$$c_n \cdot Tr(\varphi_m(\beta)) \tag{7.9}$$

for some $c_n \in \mathbb{C}[z^{\pm 1}, m^{\pm 1}, C]$. We aim to define c_n such that this function is invariant under the Markov moves. Therefore by Markov's Theorem 3.3.1 this will be a knot invariant.

Case MI : Equivalence within a given braid group follows immediately from φ_m being a representation and so invariant under equivalent braids.

Case MII : This follows from (ii) in Definition 7.1.10.

$$c_n \cdot Tr(\varphi(\alpha^{-1}\beta\alpha)) = c_n \cdot m^{-1} \cdot m \cdot Tr(\sigma_{i_1}^{-1} \dots \sigma_{i_n}^{-1} \varphi_m(\beta)\sigma_{i_1} \dots \sigma_{i_n}) = c_n \cdot Tr(\varphi_m(\beta))$$

Case MIII : This Markov move involves embedding the braid $\beta \in \mathcal{B}_n$ into $\mathcal{B}_{n+1}$. We assume

$$c_n \cdot Tr(\varphi_m(\beta)) = c_{n+1} \cdot Tr(\varphi_m(\beta\sigma_n^{\pm 1})) \tag{7.10}$$

For the positive crossing we apply (iii) in Definition 7.1.10 where $g_n = g_{n+1-1}$ we have

$$c_{n+1} \cdot Tr(\varphi_m(\beta\sigma_n)) = c_{n+1} \cdot m \cdot Tr(\varphi_m(\beta)g_n) = c_{n+1} \cdot m \cdot C \cdot Tr(\varphi_m(\beta))$$

Combining with Equation 7.10 this implies

$$c_n = c_{n+1} \cdot m \cdot C \tag{7.11}$$

For the negative crossing by applying Remark 7.1.11 we have

$$c_{n+1} \cdot Tr(\varphi_m(\beta\sigma_n^{-1})) = c_{n+1} \cdot m^{-1} \cdot Tr(\varphi_m(\beta)c_n^{-1}) = c_{n+1} \cdot m^{-1} \cdot (C - z) \cdot Tr(\varphi_m(\beta))$$

Combining with Equation 7.10 this implies

$$c_n = c_{n+1} \cdot m^{-1} \cdot (C - z) \tag{7.12}$$

To find the necessary value of c_n we equate Equations 7.11 and 7.12 to give

$$\begin{aligned} c_{n+1} \cdot m \cdot C &= c_{n+1} \cdot m^{-1} \cdot (C - z) \\ \implies C &= \frac{zm^{-1}}{(m^{-1} - m)} \end{aligned} \tag{7.13}$$

It remains to solve Equation 7.11 using the above value of C.

$$c_{n+1} = c_n \cdot \frac{1}{m \cdot C} = c_n \cdot \frac{m^{-1}(m^{-1} - m)}{zm^{-1}} = c_n \cdot \frac{m^{-1} - m}{z} \tag{7.14}$$

We use Equation 7.14 to define c_n inductively for n as follows.

$$c_n = c_{n-1} \cdot \frac{m^{-1} - m}{z} = (\frac{m^{-1} - m}{z})^{n-1} \tag{7.15}$$

This is the value of c_n required for $c_n \cdot Tr(\varphi_m(\beta))$ to be invariant under $MIII$.

We now have the necessary conditions such that the function given in Equation 7.9 is invariant under the Markov moves.

Definition 7.2.1. Let $\beta \in \mathcal{B}_n$ be a braid and K a knot diagram. The *HOMFLY polynomial* $P_K(z,m)$ of the knot $K \cong \overline{\beta}$ is the function $P_K : \{Knots\} \to \mathbb{C}[z^{\pm 1}, m^{\pm 1}]$ defined by

$$P_K(z,m) = \Big(\frac{m^{-1}-m}{z}\Big)^{n-1} \cdot Tr(\varphi_m(\beta)) \tag{7.16}$$

Theorem 7.2.2. *[BZH13, Theorem 17.13] The HOMFLY polynomial $P_K(z,m)$ is a knot invariant.*

Proof. The proof follows from the above construction. ∎

Example 7.2.3. We compute the HOMFLY polynomial of the trefoil knot. By Alexander's Theorem 3.2.3 and Figure 3.6 we have the trefoil knot $K \cong \overline{\beta}$ for $\beta \cong \sigma_1^3 \in \mathcal{B}_2$. In order to simplify the trace we use Equation 7.3 to give

$$(g_1^2)g_1 = (zg_1+1)g_1 = zg_1^2 + g_1 = z(zg_1+1) + g_1 = (z^2+1)g_1 + z$$

We use (iii) in Definition 7.1.10 as $g_{2-1} = g_1$ this implies $tr(g_1) = C$. Recall $C = zm^{-1}/(m^{-1}-m)$. By linearity of the Markov trace we have the following calculation.

$$\begin{aligned}
P_K(z,m) &= \frac{(m^{-1}-m)^{2-1}}{z^{2-1}} \cdot tr(\varphi(\sigma_1^3)) \\
&= z^{-1}(m^{-1}-m) \cdot m^3 \cdot Tr(g_1^3) \\
&= z^{-1}(m^{-1}-m) \cdot m^3 \cdot Tr((z^2+1)g_1 + z) \\
&= z^{-1}(m^{-1}-m) \cdot m^3 \cdot Tr((z^2+1)g_1 + z) \\
&= z^{-1}(m^{-1}-m) \cdot m^3 \cdot ((z^2+1)Tr(g_1) + z) \\
&= z^{-1}(m^{-1}-m) \cdot m^3 \cdot ((z^2+1)C + z) \\
&= z^{-1} \cdot m^3 \cdot (zm^{-1}(z^2+1) + z(m^{-1}-m)) \\
&= z^{-1} \cdot m^3 \cdot (z^3m^{-1} + zm^{-1} + zm^{-1} - zm) \\
&= z^2m^2 + 2m^2 - m^4
\end{aligned}$$

We introduced skein relations in Section 6.1. The skein relation given in Proposition 7.2.4 suffices to calculate the HOMFLY polynomial of a knot recursively.

Proposition 7.2.4. *[BZH13, Proposition 17.17] Let K_+, K_-, K_I be the sections of knot diagrams given in Theorem 6.1.2. There is a skein relation given by*

$$m^{-1}P_{K_+} - mP_{K_-} = zP_{K_I}$$

Proof. To compute P_K we are considering the knot as the closure of a braid $K \cong \beta$. We can write β such that K_+ and K_- correspond to over and under crossings of the braid and K_I is the area of no crossing. We use the fact that Remark 7.1.7 implies $g_i - g_i^{-1} = g_i - g_i + z$. We apply properties (ii) and (iii) from Definition 7.1.10 to give the following equality.

$$\begin{aligned}
m^{-1}P_{K_+} - mP_{K_-} &= m^{-1}c_n \cdot Tr(\varphi_m(\beta'\sigma_i)) - mc_n \cdot Tr(\varphi_m(\beta'\sigma_i^{-1})) \\
&= m^{-1}c_n \cdot m \cdot Tr(g_i\varphi(\beta')) - mc_n \cdot m^{-1} \cdot Tr(g_i^{-1}\varphi(\beta')) \\
&= c_n \cdot Tr(g_i - g_i^{-1}) \cdot Tr(\varphi(\beta')) \\
&= c_n z \cdot Tr(\varphi(\beta')) \\
&= zP_{K_I}
\end{aligned} \tag{7.17}$$

■

Theorem 7.2.5. *[BZH13, Theorem 17.21] The Jones polynomial can be obtained in terms of the HOMFLY polynomial by the change of variables*

$$V_K(X) = P_K(z = X^2 - X^{-2}, m = -X^{-4}) \tag{7.18}$$

Proof. We claim $P_K(z, m)$ in Definition 7.2.1 specialises to $V_K(X)$ given in Definition 5.4.1 under the change of variables in Equation 7.18.

Recall C and c_n from Equations 7.13 and 7.15. The change of variables implies the following two equations where $\tau = (-X^2 - X^{-2})$.

$$c_n = \Big(\frac{m^{-1} - m}{z}\Big)^{n-1} = \Big(\frac{-X^4 + X^{-4}}{X^2 - X^{-2}}\Big)^{n-1} = \Big(\frac{(X^2 - X^{-2})(-X^2 - X^{-2})}{(X^2 - X^{-2})}\Big)^{n-1} = \tau^{n-1} \tag{7.19}$$

$$C = \frac{zm^{-1}}{m^{-1} - m} = \frac{(X^2 - X^{-2})(-X^4)}{-X^4 + X^{-4}} = \frac{(X^2 - X^{-2})(-X^4)}{(X^2 - X^{-2})(-X^2 - X^{-2})} = (-X^4)\tau^{-1} \tag{7.20}$$

By Alexander's Theorem 3.2.3 we consider the braid $\beta \in \mathcal{B}_n$ such that $K \cong \overline{\beta}$. We aim to prove the claim holds for all such braids. We proceed inductively on n.

If $n = 1$ then for all $\beta \in \mathcal{B}_1$ we have $\tau^0 \cdot Tr(I_{\mathcal{H}_1}) = 1 = (-X^3)^0 tr(I_1)\tau^0$ and the claim holds.

Assume for all $\beta \in \mathcal{B}_n$ the claim holds.

We prove that the claim holds for all $\beta \in \mathcal{B}_{n+1}$. That is we claim

$$\tau^{(n+1)-1} Tr(\varphi_m(\beta)) = (-X^3)^{-w(\beta)+1} tr(\rho_{n+1}(\beta))\tau^{(n+1)-1} \tag{7.21}$$

We can write β as a composition of elementary braids. If β does not contain σ_n then the $(n+1)th$ strand does not cross anywhere in the braid. In this case $\beta \in \mathcal{B}_n$ is embedded into $\mathcal{B}_{n+1}$ by a non-crossing $(n+1)th$ strand and the claim follows from the induction hypothesis.

Suppose β contains a crossing of the $(n+1)$th strand. We can write $\beta \cong \alpha\sigma_n^{\pm 1}\alpha'$ for braids $\alpha, \alpha' \in \mathcal{B}_{n+1}$.

Case 1: If neither α or α' contain a crossing with the $(n+1)$th strand then $\alpha, \alpha' \in \mathcal{B}_n$ are embedded into $\mathcal{B}_{n+1}$. By Theorem 5.4.2 and Theorem 7.2.2 we have invariance under Markov move $MIII$ therefore the following holds for σ_n and σ_n^{-1} similary.

On the left hand side of Equation 7.21 we use the Equations 7.19 and 7.20. We apply Remark 7.1.13 for $x_1, x_2 \in \mathcal{H}_n$ to give

$$\begin{aligned}
\tau^{(n+1)-1} \cdot Tr(\varphi_m(\alpha\sigma_n\alpha')) &= \tau^{n-1}\tau(-X^{-4}) \cdot Tr(x_1 g_n x_2) \\
&= \tau^{n-1}\tau(-X^{-4}) \cdot C \cdot Tr(x_1 x_2) \\
&= \tau^{n-1}\tau(-X^{-4}) \cdot C \cdot Tr(\varphi_m(\alpha\alpha')) \\
&= \tau^{n-1}\tau(-X^{-4})(-X^4)\tau^{-1} \cdot Tr(\varphi_m(\alpha\alpha')) \\
&= \tau^{n-1} \cdot Tr(\varphi_m(\alpha\alpha'))
\end{aligned}$$

On the right hand side of Equation 7.21 we have

$$(-X^3)^{-w(\beta)+1}tr(\rho_{n+1}(\alpha\sigma_n\alpha'))\tau^{(n+1)-1} = (-X^3)^{-w(\beta)}(-X^{-3})(X+X^{-1}\tau^{-1})tr(\rho_n(\alpha\alpha'))\tau^n$$
$$= (-X^3)^{-w(\beta)}tr(\rho_n(\alpha\alpha'))\tau^{n-1}$$

as $\alpha, \alpha' \in \mathcal{B}_n$ equality holds by the induction hypothesis.

Case 2: If either α or α' do contain a crossing with the $(n+1)$th strand then we can write $\alpha = \alpha''\sigma_n^{\pm 1}\alpha'''$ and repeat this process. By properties (ii) and (iii) in Definition 7.1.10 we can then apply Case 1 and the claim follows.

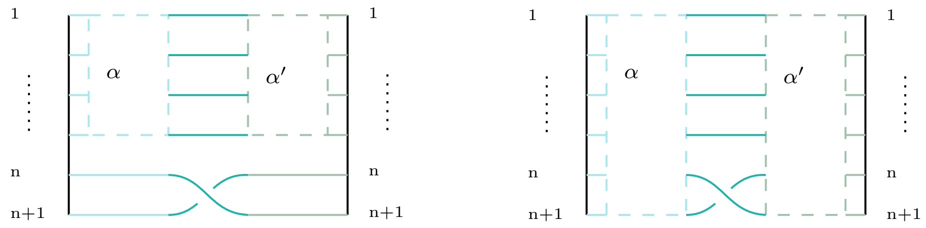


Figure 7.3: Case 1 and Case 2 respectively

Alternatively the claim follows immediately from the skein relation of the Jones polynomial given in Theorem 6.1.2 and the HOMFLY polynomial given in Proposition 7.2.4.

$$-X^4P_{K_+} + X^{-4}P_{K_-} - (X^2 - X^{-2})PK_I = (-X^4)V_{K_+} + (X^{-4})V_{K_-} - (X^2 - X^{-2})V_{K_I}$$

■

We have now defined a knot invariant using representations of the braid group on either the Temperley-Lieb algebra or the Hecke algebra. There are conceptual explanations as to why such algebras are chosen. This involves the study of quantum groups. The theory of quantum groups is closely related to the theory of representations of semisimple Lie algebras and the topology of knots. Vaughan Jones' discovery of the Jones polynomial whilst studying von Neumann algebras was a significant event in the early history of this theory.

Inspired by Vaughan Jones' work, Reshetikhin and Turaev showed how to use quantum groups to construct finite-dimensional representations of the braid group; see the paper [NR90]. Any finite-dimensional representation V of a quantum group gives rise to a representation of the braid group $\mathcal{B}_n$ on $V^{\otimes n}$. The Temperley-Lieb algebra and Hecke algebra appear as endomorphism algebras of such tensor powers.

For the Temperley-Lieb algebra this is given by

$$End_{U_q(\mathfrak{g})}(V_q^{\otimes n}) \cong \mathcal{TL}_n(q)$$

where V_q is a representation of $U_q(\mathfrak{sl}(2))$, the quantum enveloping algebra of $\mathfrak{sl}(2)$. The representation V_q is the quantum analogue of the standard representation $V = \mathbb{C}^2$ of the Lie algebra $\mathfrak{sl}(2)$.

Therefore the Jones polynomial and the HOMPFLY polynomial are special cases of quantum invariants. An introduction to quantum invariants can be found in the book [Kas95].

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