Deligne Categories of Hyperoctahedral Groups

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Contents

0.1 Introduction

0.1.1 Interpolation categories

In this thesis we will discuss the representation theory of the Weyl groups of type B, more specifically we will discuss categories parameterized by a complex number t that interpolate the categories of representations $\mathrm{Rep}(B_n)$ for different $n \in \mathbb{N}$. Such interpolating categories were first defined by Deligne-Milne in [DM82] for the orthogonal and general linear groups and by Turaev for certain quantum groups [Tur90]. They were then systematically studied by Deligne in [Del07] for the Weyl groups of type A, the symmetric groups.

There where different generalisations afterwards, among them by Knop [Kno07] and Etingof [Eti14] and there are now many interpolation or Deligne categories. These Deligne categories have found applications in the theory of tensor categories, representations of supergroups [CH17][ES21] and invariant theory [Cou18].

Among the cases studied by Knop were representations of wreath products $G \wr S_n$. Knop constructed interpolating categories in this setting and determined the singular parameters (those t for which the categories are not semisimple). A different generalisation came from Mori in [Mor12]. Because the Weyl group B_n is isomorphic to the hyperoctahedral group $H_n = \mathbb{Z}_2 \wr S_n$, both theories yield constructions for how to interpolate the representation categories $\mathrm{Rep}(B_n) = \mathrm{Rep}(H_n)$. Recently Flake and Maasen [FM21] generalised the ideas of Deligne's $\underline{\mathrm{Rep}}(S_t)$ construction to interpolate the representation categories of easy quantum groups. Among their examples is yet another interpolation category $\underline{\mathrm{Rep}}(H_t)$ for $\mathrm{Rep}(H_n)$. Knop's construction was taken up recently by Likeng and Savage in [LS21], where they gave a description of interpolation categories for representation categories of wreath products $G \wr S_n$ using generators and relations.

0.1.2 Main results of this thesis

This thesis started with the observation that the interpolation categories $\underline{\mathrm{Rep}}(H_t)$ and $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ for B_n given in [FM21] and [LS21], interpolate the representation categories for the hyperoctahedral groups in different objects and for different parameters $t \in \mathbb{C}$. This means that the morphisms between the tensor powers of the generating objects in the categories $\underline{\mathrm{Rep}}_0(H_n)$ and $\mathrm{Par}(\mathbb{Z}_2, 2n)$, the pre-Karoubian envelope versions of the interpolation categories, mimic the behaviour of the morphism spaces between the tensor powers of the reflection representation u and of the permutation representation V respectively. So the interpolation functors $G : \underline{\mathrm{Rep}}(H_t) \to \mathrm{Rep}(H_n)$ and $H : \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar} \to \mathrm{Rep}(H_n)$ are defined on objects by $[k] \to u^{\otimes k}$ and $[\tilde{k}] \to V^{\otimes k}$ respectively. We compare these different constructions and formulate and prove for the first time clearly the universal properties of these categories. For this we derive in Theorem 2.6.21 a presentation via generators and relations for the reflection category $\underline{\mathrm{Rep}}(H_t)$.

The main result of this thesis is the following theorem (see Theorem 3.2.15 and Corollary 3.2.18).

Theorem 0.1.1. *There is a symmetric monoidal equivalence*

$$\Omega : \underline{Rep}(H_n) \simeq Par(\mathbb{Z}_2, 2n)^{Kar}$$

such that the diagram

$$\begin{array}{ccc} \underline{Rep}(H_n) & \xrightarrow{G} & Rep(H_n) \\ \downarrow \Omega & & \downarrow = \\ Par(\mathbb{Z}_2, 2n)^{kar} & \xrightarrow{H} & Rep(H_n) \end{array}$$

commutes for all $n \in \mathbb{N}$. *The functor* Ω *descends to an equivalence* $\widehat{\Omega}$ *between the semisimplifications of the interpolation categories. For the equivalences* $\widehat{G}$ *and* $\widehat{H}$*, the respective semisimplifications of the interpolation functors* G *and* H*, the functor* $\widehat{\Omega}$ *corresponds to the composition* $\widehat{H}^{-1} \circ \widehat{G}$.

We remark that it was already shown in [LS21] that there is an embedding of $\mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$ into the group Heisenberg category for the group $\mathbb{Z}/2\mathbb{Z}$ which intertwines the categorical actions of $\mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$ and $Heis(\mathbb{Z}/2\mathbb{Z})$ on modules of the hyperoctahedral group. Using the equivalence $\Omega : \underline{\mathrm{Rep}}(H_n) \simeq \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$ this implies now also that the categorical actions of $\underline{\mathrm{Rep}}(H_n)$ and the Heisenberg category are compatible.

As another application we obtain in Theorem 3.3.2 that the isomorphism classes of indecomposable objects of $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ are parametrized by the set of all bipartitions for $t \in \mathbb{C} \setminus \{0\}$. This was proven by Knop [Kno07] in the semisimple $t \neq 2n$-case (see also [LS21]).

0.1.3 Structure of the thesis

In Chapter 1, we introduce the categorical framework in which we will work during this thesis and discuss some relations between important representations of the hyperoctahedral groups. In Chapter 2 we will introduce the interpolation categories and some of their properties, as well as their universal properties which can be deduced from their presentations via generators and relations. We will also give a clear motivation for the definition of these categories, showing how the morphism spaces correspond to the morphisms between tensor powers of certain objects in the interpolated representation categories. Our approach in discussing $\underline{\mathrm{Rep}}(S_t)$, $\underline{\mathrm{Rep}}(H_t)$ and $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ will be one of unifying the language and the notation as much as possible. We hope that this lays bare the many similarities between the different interpolation categories and that this juxtaposition gives us new ideas and proofs. In Chapter 3 we will first discuss a naive way of defining a functor $\underline{\mathrm{Rep}}(H_n) \to \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$. This will be of limited use, because the functor will neither be full, nor compatible with the interpolation functors. We construct a functor Ω which does satisfy these properties and we will show that it is also faithful and essentially surjective, therefore an equivalence. The result is new and may seem counter-intuitive after a first look at the involved interpolation categories.

0.1.4 Outlook

It would be interesting to explore whether such equivalences occur in more general situations. One generalization would be to consider wreath product groups $G^n \wr S_n$ for other groups G. Maybe even more interesting would be to look at the complex reflection groups $G(r, p, d)$ where we have by their definition a reflection representation as well as a permutation representation, and study, whether their categorical incarnations generate the same symmetric monoidal category. In general, it seems unclear when the categorical analogues of two different faithful representations (or tensor generators) give rise to equivalent Deligne categories. We remark that even in the original article [Del07] Deligne does not use the reflection representation $\mathbb{C}^n$ of S_n and its tensor powers to build his category.

Another interesting direction is the connections to other settings of stable representation theory for the hyperoctahedral group. Wilson [Wil14] studied this from the perspective of stable sequences of Weyl group representations. Another setting are tensor representations of the infinite hyperoctahedral group, the inductive limit of the ascending tower of groups $H_1 \subset H_2 \subset ...$, similar to Sam-Snowden's category $\mathrm{Rep}(S_\infty)$ [SS13]. In the symmetric group case it is known that these different categories are closely related [BEH19], and one might expect that an analogous theorem holds for the hyperoctahedral case.

0.1.5 Acknowledgements

I would like to express my deep gratitude to my supervisor Dr. Thorsten Heidersdorf for his guidance and encouragement during the writing of this thesis. Not only was his feedback invaluable, I am also very grateful for the interesting insights and topics I got introduced to under his supervision. It has been an inspiring experience. I would also like to offer my special thanks to the second examiner Professor Dr. Catharina Stroppel for her time and effort.

Chapter 1

Background on Monoidal Categories and Representation Theory

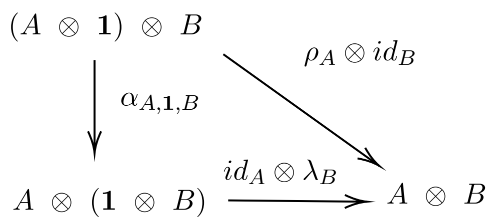
1.1 Category Theory

This section is mostly inspired on [BW99] and general knowledge of category theory. We introduce the category theoretic notions and statements which will be relevant for the later chapters of the thesis. Because this is not proven in the literature, we prove the well-known fact that the Karoubian envelope of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category is again a $\mathbb{C}$-linear spherical rigid symmetric monoidal category in Proposition 1.1.20, Proposition 1.1.24 and Proposition 1.1.28,

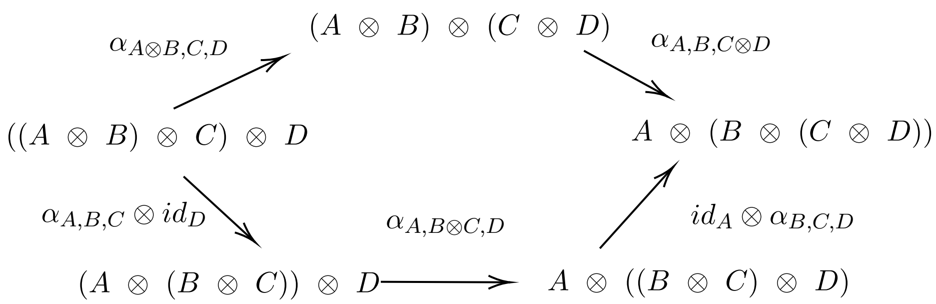
Definition 1.1.1. A monoidal category $\mathcal{C}$ is a category with

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- a unit object $\mathbf{1} \in C$
- an associator, which is a natural isomorphism $\alpha : ((- \otimes -) \otimes -) \to (- \otimes (- \otimes -))$
- a left unitor, which is a natural isomorphism $\lambda : (\mathbf{1} \otimes -) \to (-)$
- a right unitor, which is a natural isomorphism $\rho : (- \otimes \mathbf{1}) \to (-)$

such that he following diagrams commute for all objects $A, B, C, D \in \mathcal{C}$:



Triangle identity

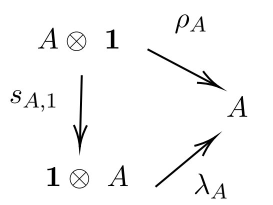


Pentagon identity

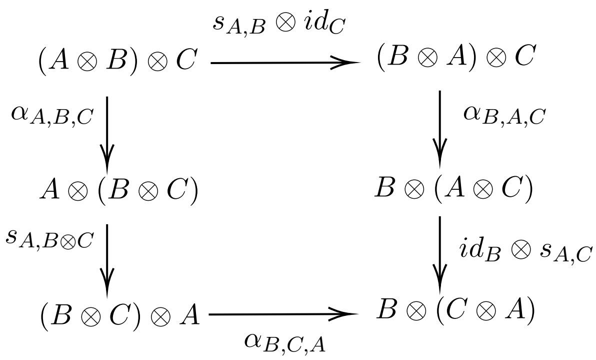
Remark 1.1.2. In this thesis we will assume for any monoidal category $\mathcal{C}$ that $End_{\mathcal{C}}(\mathbf{1}) \cong \mathbb{C}$ if $\mathcal{C}$ is $\mathbb{C}$-linear, i.e. the morphism spaces of $\mathcal{C}$ are $\mathbb{C}$-vector spaces and the composition is $\mathbb{C}$-bilinear. The sole reason for this, is that all $\mathbb{C}$-linear monoidal categories we will encounter in this thesis, will satsify property.

Example 1.1.3. Examples of monoidal categories are the category of vector spaces over some field $\mathbb{K}$. The bifunctor is the tensor product of vector spaces and the unit object is the one-dimensional vector space $\mathbb{K}$.

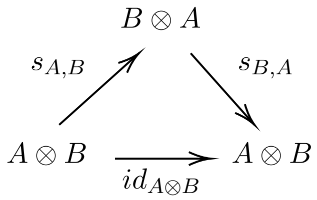
Definition 1.1.4. A symmetric monoidal category $\mathcal{C}$ is a monoidal category with a natural isomorphism $S_{A,B} : A \otimes B \to B \otimes A$ for all $A, B \in \mathcal{C}$ such that the following diagrams commute for all $A, B, C \in \mathcal{C}$:



Unit coherence



Associativity coherence



inverse law

Example 1.1.5. An example of a $\mathbb{C}$-linear symmetric monoidal category is the category $\mathrm{Rep}(G)$ of finite-dimensional representations over a group ring $\mathbb{C}G$ for some finite group G. The monoidal structure is given by the tensor product $\otimes_{\mathbb{C}}$ of the underlying $\mathbb{C}$-vector spaces and $\mathbf{1}_{\mathrm{Rep}(G)} = \mathbb{C}$ with the trivial G-action, because $\mathbb{C} \otimes_{\mathbb{C}} A = A = A \otimes_{\mathbb{C}} \mathbb{C}$ for all $A \in \mathrm{Rep}(G)$. Clearly there is an isomorphism $\mathrm{End}_{\mathrm{Rep}(G)}(\mathbb{C}) \cong \mathbb{C}$, see Remark 1.1.2. For all $A, B \in \mathrm{Rep}(G)$ there are isomorphisms

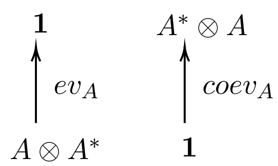
$$\begin{aligned} s_{A,B} : A \otimes_{\mathbb{C}} B &\to B \otimes_{\mathbb{C}} A \\ x \otimes y &\to y \otimes x \text{ for all } x \in A \text{ and } y \in B, \end{aligned}$$

which give the category a symmetric monoidal structure.

Let $(M, \circ, e)$ be a finite non-commutative monoid, for example the ring of 2×2 matrices over the finite field $\mathbb{F}_2$. Let $\mathcal{M}$ be a category with as objects elements in M and formal tensor products $m_1 \otimes \ldots \otimes m_k$ for $m_i \in M$. The morphisms of $\mathcal{M}$ are the identity morphisms and isomorphisms between tensor products if the corresponding products in M are the same. Then $(\mathcal{M}, \otimes, e)$ is a non-symmetric monoidal category.

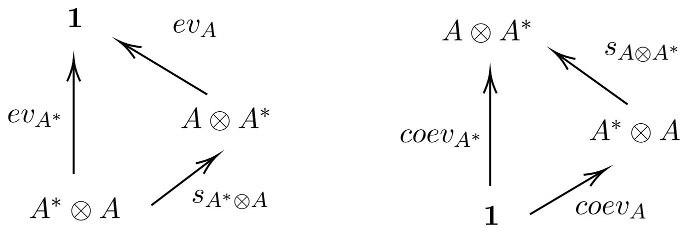
Definition 1.1.6. A rigid symmetric monoidal category $\mathcal{C}$ is a symmetric monoidal category with

- a contravariant monoidal endofunctor [1] $* : \mathcal{C} \to \mathcal{C}$. The object A^* is called the dual object of A.
- an isomorphism $\tau_A : A \cong ((A)^*)^*$ which is natural in all $A \in \mathcal{C}$, so a natural isomorphism $\tau : id_{\mathcal{C}} \to ((-)^*)^*$. If this axiom holds, we call the functor $*$ involutive.
- an isomorphism $\nu : \mathbf{1} \to \mathbf{1}^*$.
- isomorphisms $\gamma_{A,B} : (A \otimes B)^* \to B^* \otimes A^*$ which are natural in all $A, B \in \mathcal{C}$.
- morphisms called the evaluation and coevaluation

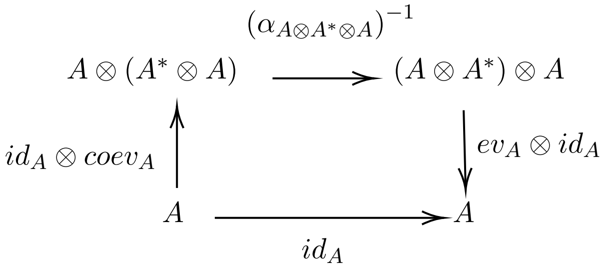


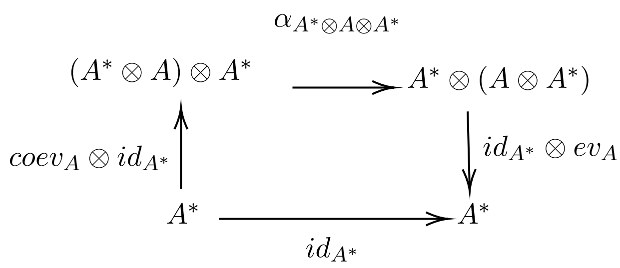
for all $A \in \mathcal{C}$. They satisfy the property that the following diagrams commute for all $A \in \mathcal{C}$:

[1]See Definition 1.1.9



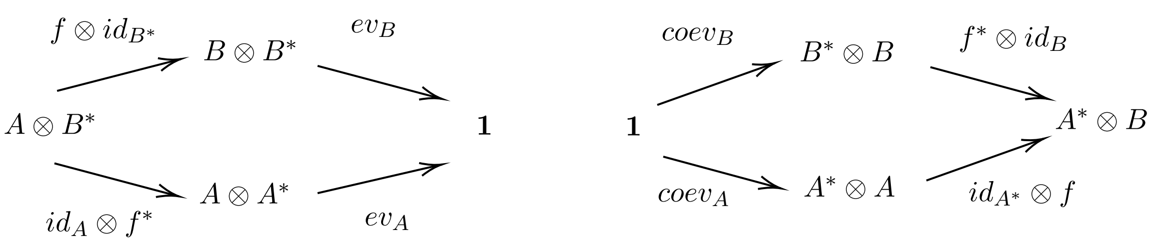
duality identities





Triangle identities

A $\mathbb{C}$-linear rigid symmetric monoidal category is a pivotal tensor category in the sense of [CO11] and we will call it a tensor category. We will also assume the following identities for rigid monoidal symmetric categories, because they will be needed to prove that the Karoubian envelope of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category $\mathcal{C}$ is also a $\mathbb{C}$-linear spherical rigid symmetric monoidal category. For all $f : A \to B$ the following diagrams commute:



Morphism identities

Example 1.1.7. The category of vector spaces over a field $\mathbb{K}$ is an example of a rigid symmetric monoidal category. Another example is the category $\mathrm{Rep}(G)$ of finite-dimensional representations over a group ring $\mathbb{C}G$ for some finite group G. Let $A \in \mathrm{Rep}(G)$ be a finite-dimensional representation over $\mathbb{C}G$ with basis $\{e_1, \dots, e_m\}$. Then the dual is defined by $A^* := Hom_{\mathbb{C}}(A, \mathbb{C})$ with the G-action defined by $g \cdot f(x) := f(g^{-1} \cdot x)$ for all $x \in A, g \in G$ and $f \in A^*$. It has a dual basis $\{e_1^*, \dots, e_m^*\}$. The evaluation is the $\mathbb{C}$-linear extension of

$$\begin{aligned} ev_A : A \otimes A^* &\to \mathbb{C} \\ x \otimes f &\mapsto f(x). \end{aligned}$$

The coevaluation is defined by the $\mathbb{C}$-linear extension of

$$\begin{aligned} coev_A : \mathbb{C} &\to A^* \otimes A \\ 1 &\mapsto \sum_{i=1}^{m} e_i^* \otimes e_i. \end{aligned}$$

Remark 1.1.8. The assumption of the morphism identities

$$ev_B \circ (f \otimes id_{B^*}) = ev_A \circ (id_A \otimes f^*) \text{ and } (f^* \otimes id_B) \circ coev_B = (id_{A^*} \otimes f) \circ coev_A$$

is not a far-fetched assumption since they hold for example in the category $\mathrm{Vect}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$ and the category $\mathrm{Rep}(G)$ of finite-dimensional representations over a group ring $\mathbb{C}G$ for some finite group G.

Definition 1.1.9. We let a tensor functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between tensor categories be a $\mathbb{C}$-linear strong monoidal functor which respects the given tensor structures[2] of the involved tensor categories. This means that there exists a natural isomorphism $\zeta : F \circ (- \otimes_{\mathcal{C}} -) \to \mathcal{F}(-) \otimes_{\mathcal{D}} \mathcal{F}(-)$. The isomorphisms $\zeta_{A,B}$ are called coherence maps for all $A, B \in \mathcal{C}$. There also exists an isomorphism $\zeta : \mathcal{F}(\mathbf{1}_{\mathcal{C}}) \cong \mathbf{1}_{\mathcal{D}}$. We want the functor $\mathcal{F}$ to respect the properties of the tensor categories and refer for example to [Eti+15, Section 2.4, Definition 4.2.5] for more details. As an example we shortly state what it means for the functor to respect the symmetric structure and unit coherence property, all other structures and properties can be discussed similarly. Preserving the symmetric structure means that for all $A, B \in \mathcal{C}$ we have that

$$s^{\mathcal{D}}_{\mathcal{F}(A),\mathcal{F}(B)} \circ \zeta_{A,B} = \zeta_{B,A} \circ \mathcal{F}(s^{\mathcal{C}}_{B,A}).$$

This will imply together with the similar statements for the right and the left unitor

$$\begin{aligned} \mathcal{F} \circ \lambda_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}} \otimes -) &= \lambda_{\mathcal{D}}((\zeta \circ \mathcal{F})(\mathbf{1}_{\mathcal{C}}), \mathcal{F}(-)) = \lambda_{\mathcal{D}}(\mathbf{1}_{\mathcal{D}}, \mathcal{F}(-)) \text{ and} \\ \mathcal{F} \circ (- \otimes \rho_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}})) &= \rho_{\mathcal{D}}(\mathcal{F}(-), (\zeta \circ \mathcal{F})(\mathbf{1}_{\mathcal{C}})) = \rho_{\mathcal{D}}(\mathcal{F}(-), \mathbf{1}_{\mathcal{D}}), \end{aligned}$$

that unit coherence triangle of morphisms in $\mathcal{C}$ will be send to a triangle of morphisms in $\mathcal{D}$, which is isomorphic to a unit coherence triangle under the coherence maps.

Definition 1.1.10. We call the above tensor functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ strict if $\mathcal{F}(A \otimes B) = \mathcal{F}(A) \otimes \mathcal{F}(B)$

[2] If we have for example a functor between two tensor categories, which is an equivalence, then this equivalence induces a tensor structure of one category onto the other. This induced tensor structure is not necessarily the same as the original one. If we talk about a tensor functor, we assume that it respects the tensor structures that were considered before defining the tensor functor.

for all $A, B \in \mathcal{C}$, so if all coherence maps are equalities.

Definition 1.1.11. Let $\mathcal{C}$ be a $\mathbb{C}$-linear rigid symmetric monoidal category and $A \in \mathcal{C}$. Left and right traces are the $\mathbb{C}$-linear maps $Trace_l, Trace_r : \mathrm{End}_{\mathcal{C}}(A) \to \mathrm{End}_{\mathcal{C}}(\mathbf{1}) = \mathbb{C}$ which are defined for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ by the compositions:

$$\overbrace{\mathbf{1} \xrightarrow{coev_A} A^* \otimes A \xrightarrow{id_{A^*} \otimes f} A^* \otimes A \xrightarrow{ev_{A^*}} \mathbf{1}}^{Trace_r(f)}$$

$$\overbrace{\mathbf{1} \xrightarrow{coev_{A^*}} A^{**} \otimes A^* \xrightarrow{(\tau_A)^{-1} \otimes id_{A^*}} A \otimes A^* \xrightarrow{f \otimes id_{A^*}} A \otimes A^* \xrightarrow{ev_A} \mathbf{1}}^{Trace_l(f)}$$

Definition 1.1.12. A $\mathbb{C}$-linear spherical rigid symmetric monoidal category $\mathcal{C}$ is a $\mathbb{C}$-linear rigid symmetric monoidal category for which the left and the right traces coincide. In this case we denote the trace of morphism $f : A \to A$ in $\mathcal{C}$ by $tr(f)$.

Remark 1.1.13. Let $\mathcal{C}$ be a $\mathbb{C}$-linear spherical rigid symmetric monoidal category and $f : A \to B$ and $g : B \to A$ morphisms in $\mathcal{C}$. Then multiple applications of the morphism identities show us that $tr(g \circ f) = tr(f \circ g)$. For an object $A \in \mathcal{C}$ we define the categorical dimension of A by $\dim(A) := tr(id_A)$.

Example 1.1.14. An example of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category is the category $\mathrm{Rep}(G)$ of finite dimensional representations over a group algebra $\mathbb{C}\,G$ for a finite group G. We are interested in the category $\mathrm{Rep}(S_n)$ of finite dimensional representations over the group algebra $\mathbb{C}\,S_n$ for the n-th symmetric group S_n and the category $\mathrm{Rep}(H_n)$ of finite dimensional representations over the group algebra $\mathbb{C}\,H_n$ for the n-th hyperoctahedral group H_n.

Remark 1.1.15. Note that tensor functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between $\mathbb{C}$-linear rigid symmetrical monoidal categories preserve the traces. This holds because the properties of the tensor functor imply that $\mathcal{F}(Trace_l(f)) = Trace_l(\mathcal{F}(f))$ and $\mathcal{F}(Trace_r(f)) = Trace_r(\mathcal{F}(f))$.

Definition 1.1.16. An embedding $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a functor which is injective on objects and faithful. A full embedding is an embedding which is full.

Definition 1.1.17. Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. We define the additive envelope $\mathcal{C}^{add}$ to be the category with objects formal words $\overline{A} := A_1 \oplus \ldots \oplus A_n$ with $A_j \in \mathcal{C}$ for $1 \leqslant j \leqslant n$. The morphism sets are defined as

$$\mathrm{Hom}_{\mathcal{C}^{add}}(A_1 \oplus \ldots \oplus A_n, B_1 \oplus \ldots \oplus B_m) := \bigoplus_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n} \mathrm{Hom}_{\mathcal{C}}(A_j, B_i).$$

Therefore $\mathcal{C}^{add}$ is $\mathbb{C}$-linear. We write the elements of $\mathrm{Hom}_{\mathcal{C}}(\overline{A}, \overline{B})$ as $m \times n$-matrices $[f_{ij}]$, where $f_{ij} \in \mathrm{Hom}_{\mathcal{C}}(A_j, B_i)$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. This allows us to define the $\mathbb{C}$-linear composition map

$$\circ : \mathrm{Hom}_{\mathcal{C}^{add}}(\overline{B}, \overline{C}) \otimes \mathrm{Hom}_{\mathcal{C}^{add}}(\overline{A}, \overline{B}) \to \mathrm{Hom}_{\mathcal{C}^{add}}(\overline{A}, \overline{C})$$

for some object $\overline{C} := C_1 \oplus \ldots \oplus C_p$ and $[g_{ij}] \in \mathrm{Hom}_{\mathcal{C}^{add}}(\overline{B}, \overline{C})$ as

$$[g_{ij}] \circ [f_{ij}] := \Big[\sum_{k=1}^{m} g_{ik} \circ f_{kj} \Big].$$

The additive envelope $\mathcal{C}^{add}$ comes together with an obvious $\mathbb{C}$-linear functor $\iota_{add} : \mathcal{C} \to \mathcal{C}^{add}$ which sends

$$A \mapsto A$$
$$f \mapsto f$$

for all objects $A \in \mathcal{C}$ and all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Thus ι_{add} is a $\mathbb{C}$-linear full embedding.

Definition 1.1.18. A $\mathbb{C}$-linear category $\mathcal{C}$ is called additive if $\iota_{add} : \mathcal{C} \simeq \mathcal{C}^{add}$ is an equivalence, i.e. it admits all finite biproducts. A functor between additive categories is called additive if it preserves all finite biproducts.

Remark 1.1.19. Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. Then the additive envelope $\mathcal{C}^{add}$ satisfies the following universal property. Let $\mathcal{D}$ be an additive category and $\alpha : \mathcal{C} \to \mathcal{D}$ a $\mathbb{C}$-linear functor. Then there exists a $\mathbb{C}$-linear additive functor $\alpha' : \mathcal{C}^{add} \to \mathcal{D}$ such that $\alpha = \alpha' \circ \iota_{add}$. The functor α is unique up to natural isomorphism.

Proposition 1.1.20. *The additive envelope $\mathcal{C}^{add}$ of a $\mathbb{C}$-linear spherical rigid monoidal category $\mathcal{C}$ can again be given the structure of a $\mathbb{C}$-linear spherical rigid monoidal category.*

Proof. The $\mathbb{C}$-linearity is given by the definition of the additive envelope of $\mathbb{C}$-linear category. We define the necessary morphisms in terms of corresponding morphisms in $\mathcal{C}$. It will be clear by these definitions that the necessary equalities and identities follow from the fact that the corresponding ones hold in $\mathcal{C}$. We define for all $\overline{A} := A_1 \oplus \ldots \oplus A_n, \overline{B} := B_1 \oplus \ldots \oplus B_m$ and $\overline{C} := C_1 \oplus \ldots \oplus C_p$ in $\mathcal{C}^{add}$

- $\overline{A} \otimes \overline{B} := \oplus_{i,j=1}(A_i \otimes B_j)$.
- $\mathbf{1}_{\mathcal{C}} := \mathbf{1}$.
- $\alpha_{\overline{A},\overline{B},\overline{C}} := \oplus_{i,j,k} \alpha_{(A_i,B_j,C_k)} := (\overline{A} \otimes \overline{B}) \otimes \overline{C} \to \overline{A} \otimes (\overline{B} \otimes \overline{C})$.
- $\lambda_{\overline{A}} := \oplus_{i=1}^{n} \lambda_{A_i} : \mathbf{1} \otimes \overline{A} \to \overline{A}$.
- $\rho_{\overline{A}} := \oplus_{i=1}^{n} \rho_{A_i} : \overline{A} \otimes \mathbf{1} \to \overline{A}$.
- $s_{\overline{A},\overline{B}} := \oplus_{i,j} s_{A_i,B_j} : \overline{A} \otimes \overline{B} \to \overline{B} \circ \overline{A}$.
- $(\overline{A})^* := \oplus_{i=1}^{n} A_i^*$ and $(f = [f_{ij}] : \overline{A} \to \overline{B})^* := [f_{ij}^*]^T$, where T is taking the transposed of the matrix. So $(f^*)_{ji} = (f_{ij})^* : B_i^* \to A_j^*$.

- $\tau_{\overline{A}} := \oplus_{i=1}^{n} \tau_{A_i} : \overline{A} \cong ((\overline{A})^*)^*$.
- The evaluation is defined as the map $ev_{\overline{A}} :=: \overline{A} \otimes \overline{A}^* \to \mathbf{1}$ so that

$$
\begin{aligned}
ev_{\overline{A}} \circ p_{i,i} &:= ev_{A_i} \text{ for all } i \in \{1, \dots, n\}, \\
ev_{\overline{A}} \circ p_{i,j} &:= 0 \text{ if } i \neq j.
\end{aligned}
$$

  for the inclusions $p_{i,j} : A_i \otimes A_j^* \to \overline{A} \otimes \overline{A}^*$.
- The coevaluation is defined as the map $coev_{\overline{A}} : \mathbf{1} \to \overline{A}^* \otimes \overline{A}$ so that

$$
\begin{aligned}
q_{i,i} \circ coev_{\overline{A}} &:= coev_{A_i} \text{ for all } i \in \{1, \dots, n\}, \\
q_{i,j} \circ coev_{\overline{A}} &:= 0 \text{ if } i \neq j
\end{aligned}
$$

  for the projections $q_{i,j} : \overline{A}^* \otimes \overline{A} \to A_i^* \otimes A_j$.
- $\nu_{\mathcal{C}^{add}} := \nu$.
- $\gamma_{\overline{A},\overline{B}} := \oplus_{i,j}(\gamma_{A_i,B_j}) : (\overline{A} \otimes \overline{B})^* \to \overline{B}^* \otimes \overline{A}^*$, which is again an isomorphism.
- Let $f = [f_{ij}] : \overline{A} \to \overline{A}$. The right trace is defined by

$$Trace_r(f) := ev_{\overline{A}^*} \circ (id_{\overline{A}^*} \otimes f) \circ coev_{\overline{A}}$$

  .
- Let $f = [f_{ij}] : \overline{A} \to \overline{A}$. The left trace is defined by

$$Trace_l(f) := ev_{\overline{A}} \circ (f \otimes id_{\overline{A}^*}) \circ (\tau_{\overline{A}}^{-1} \otimes id_{\overline{A}^*}) \circ coev_{\overline{A}^*}.$$

Monoidal category: The triangle and pentagon identities follow directly from the definitions and the fact that $\mathcal{C}$ is monoidal.

Symmetric Monoidal category: Unit coherence, Associative coherence and the inverse law follow directly from the definitions and the fact that $\mathcal{C}$ is symmetric monoidal.

Rigid Symmetric Monoidal category: The dual and triangle identities follow directly from the definitions and the corresponding identities for $\mathcal{C}$. We see for example for the first duality identity that

$$
\begin{aligned}
ev_{\overline{A}} \circ s_{\overline{A}^* \otimes \overline{A}} &= \oplus_{i=1}^{n} (ev_{A_i} \circ s_{A_i^* \otimes A_i}) \\
&= \oplus_{i=1}^{n} ev_{A_i^*} \\
&= ev_{\overline{A}^*}
\end{aligned}
$$

where we used the first duality identity in $\mathcal{C}$ for the second equality. We see that the morphism identities also hold for $f : \overline{A} \to \overline{B}$ because

$$
\begin{aligned}
ev_{\overline{B}} \circ (f \otimes id_{\overline{B}^*}) &= (\oplus_{i=1}^{m} ev_{B_i}) \circ ([f_{rs}] \otimes (\oplus_{j=1}^{m} id_{B_j^*})) \\
&= \oplus_{i=1}^{m} (ev_{B_i} \circ ([f_{rs}] \otimes (\oplus_{j=1}^{m} id_{B_j^*})) \\
&= \oplus_{i=1}^{m} \oplus_{j=1}^{m} (ev_{B_i} \circ ([f_{rs}] \otimes id_{B_j^*}))
\end{aligned}
$$

$$
\begin{aligned}
&= \oplus_{i=1}^m \oplus_{j=1}^m (ev_{B_i} \circ (\sum_{k=1}^n f_{jk} \otimes id_{B_j^*})) \\
&= \oplus_{i=1}^m \sum_{k=1}^n (ev_{B_i} \circ (f_{ik} \otimes id_{B_i^*})) \\
&= \oplus_{k=1}^n \sum_{i=1}^m (ev_{A_k} \circ (id_{A_k} \otimes f_{ki}^*)) \\
&= \oplus_{k=1}^n \oplus_{j=1}^n (ev_{A_k} \circ (id_{A_j} \otimes \sum_{i=1}^m f_{ji}^*)) \\
&= (\oplus_{k=1}^n ev_{A_k}) \circ ((\oplus_{j=1}^n id_{A_j}) \otimes [f_{rs}^*])) \\
&= ev_{\overline{A}} \circ (id_{\overline{A}} \otimes f^*)
\end{aligned}
$$

where we applied the first morphism identity in $\mathcal{C}$ for the sixth equality. The other morphism identity is proven similarly. Clearly $\mathrm{End}(\mathbf{1}) = \mathbb{C}$.

Spherical Rigid Symmetric Monoidal category: The fact that the left and right trace coincide follows directly from the definitions and the fact that $\mathcal{C}$ is spherical.

□

Definition 1.1.21. Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. The $\mathbb{C}$-linear category $\mathcal{C}^\natural$, called the idempotent completion of $\mathcal{C}$, has as objects pairs (A, e), where $A \in \mathrm{ob}(\mathcal{C})$ and $e = e^2 \in \mathrm{Hom}_\mathcal{C}(A, A)$ is an idempotent. When the context is clear, we will sometimes write A for (A, id_A). The morphism sets are obtained by pre- and postcomposing the morphisms of the corresponding morphism sets in $\mathcal{C}$ with the idempotents of the pairs, thus

$$\mathrm{Hom}_{\mathcal{C}^\natural}((A, e), (B, f)) := f\, \mathrm{Hom}_\mathcal{C}(A, B)e.$$

The composition in $\mathcal{C}^\natural$ coincides with the composition of morphisms in $\mathcal{C}$. The identity morphism of $(A, e) \in \mathcal{C}^\natural$ is $id_{(A,e)} := e$. The idempotent completion $\mathcal{C}^\natural$ comes together with a $\mathbb{C}$-linear functor $\iota_\natural : \mathcal{C} \to \mathcal{C}^\natural$ given by

$$
\begin{aligned}
A &\mapsto (A, id_A) \\
f &\mapsto f
\end{aligned}
$$

for all $A \in \mathcal{C}$ and $f \in \mathrm{Hom}_\mathcal{C}(B, C)$. The functor $\iota_\natural$ is a $\mathbb{C}$-linear full embedding.

Definition 1.1.22. A $\mathbb{C}$-linear category $\mathcal{C}$ is idempotent complete if $\iota_\natural : \mathcal{C} \simeq \mathcal{C}^\natural$, is an equivalence i.e. all idempotents split. An idempotent $e^2 = e : (A, id_A) \to (A, id_A)$ splits because e is the identity of (A, e) and because it factors over the morphisms $e : (A, id_A) \to (A, e)$ and $e : (A, e) \to (A, id_A)$.

Remark 1.1.23. Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. Then the idempotent completion $\mathcal{C}^\natural$ satisfies the following universal property. Let $\mathcal{D}$ be an idempotent complete category and $\beta : \mathcal{C} \to \mathcal{D}$ a $\mathbb{C}$-linear functor. Then there exists a $\mathbb{C}$-linear functor $\beta' : \mathcal{C}^\natural \to \mathcal{D}$ such that $\beta = \beta' \circ \iota_\natural$. The functor β is unique up to natural isomorphism.

Proposition 1.1.24. *The idempotent completion $\mathcal{C}^\natural$ of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category $\mathcal{C}$ can again be given the structure of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category.*

Proof. The $\mathbb{C}$-linearity is given by the definition of the idempotent completion of a $\mathbb{C}$-linear category. We define the necessary morphisms in terms of corresponding morphisms in $\mathcal{C}$. It will be clear by these definitions that the necessary equalities and identities follow from the fact that the corresponding ones hold in $\mathcal{C}$. Let $(A,e),(B,f),(C,g) \in \mathcal{C}^\natural$. We define:

- $(A,e) \otimes (B,f) := (A \otimes B, e \otimes f)$
- by abuse of notation $\mathbf{1} := (\mathbf{1}, 1_\mathbb{C})$
- $\alpha_{(A,e),(B,f),(C,g)} := (e \otimes (f \otimes g)) \circ \alpha_{A,B,C} \circ ((e \otimes f) \otimes g)$ which is a map form $((A,e) \otimes (B,f)) \otimes (C,g)$ to $(A,e) \otimes ((B,f) \otimes (C,g))$. Note that by naturality
$$\alpha_{A,B,C} \circ ((e \otimes f) \otimes g) = (e \otimes (f \otimes g)) \circ \alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C).$$
- $\lambda_{(A,e)} := e \circ \lambda_A(1_\mathbb{C} \otimes e) : (\mathbf{1}, 1_\mathbb{C}) \otimes (A,e) \to (A,e)$
- $\rho_{(A,e)} := e \circ \rho_A(e \otimes 1_C) : (A,e) \otimes (\mathbf{1}, 1_\mathbb{C}) \to (A,e)$
- $s_{(A,e),(B,f)} := (f \otimes e) \circ s_{A,B} \circ (e \otimes f) : (A,e) \otimes (B,f) \to (B,f) \otimes (A,e)$. Note that by naturality $s_{A,B} \circ (e \otimes f) = (f \otimes e) \circ s_{A,B} : A \otimes B \to B \otimes A$
- $(A,e)^* := (A^*, e^*)$. Note that the dual of an idempotent is again an idempotent.
- $ev_{(A,e)} := 1_\mathbb{C} \circ ev_A \circ (e, \otimes e^*) : (A,e) \otimes (A^*, e^*) \to (\mathbf{1}, 1_\mathbb{C})$.
- $coev_{(A,e)} := (e^* \otimes e) \circ coev_A \circ 1_\mathbb{C}. : (\mathbf{1}, 1_\mathbb{C}) \to (A^*, e^*) \otimes (A,e)$.
- $\tau_{(A,e)} := (e^*)^* \circ \tau_A \circ e : (A,e) \to ((A,e)^*)^*$. Note that by naturality
$$\tau_A \circ e = (e^*)^* \circ \tau_A : A \to (A^*)^*.$$
So $\tau_{(A,e)}$ is still an isomorphism with inverse $e \circ \tau_A^{-1} \circ (e^*)^*$ because
$$\begin{aligned}\tau_{(A,e)} \circ \tau_{(A,e)}^{-1} &= (e^*)^* \circ \tau_A \circ e \circ e \circ \tau_A^{-1} \circ (e^*)^* = (e^*)^* \circ \tau_A \circ \tau_A^{-1} \circ (e^*)^* \\ &= (e^*)^* = id_{((A,e)^*)^*} \text{ and} \\ \tau_{(A,e)}^{-1} \circ \tau_{(A,e)} &= e \circ \tau_A^{-1} \circ (e^*)^* \circ (e^*)^* \circ \tau_A \circ e = e \circ \tau_A^{-1} \tau_A \circ e = e = id_{(A,e)}.\end{aligned}$$
- $\nu_{\mathcal{C}^\natural} := 1_\mathbb{C} \circ \nu \circ 1_\mathbb{C}$
- $\gamma_{(A,e),(B,f)} := (f^* \otimes e^*) \circ \gamma_{A,B} \circ (e \circ f)^* : ((A,e) \otimes (B,f))^* \to (B,f)^* \otimes (A,e)^*$. Note that $\gamma_{A,B} \circ (e \circ f)^* = (f^* \otimes e^*) \circ \gamma_{A,B} : (A \otimes B)^* \to B^* \otimes A^*$ by naturality. Similarly as for $\tau_{(A,e)}$, this implies that $\gamma_{(A,e),(B,f)}$ is still an isomorphism.
- Let $f = efe : (A,e) \to (A,e)$. The right trace is defined by
$$Trace_r^\natural(f) := ev_{(A,e)^*} \circ (id_{A^*} \otimes f) \circ coev_{(A,e)}.$$
- Let $f = efe : (A,e) \to (A,e)$. The left trace is defined by
$$Trace_l^\natural(f) := ev_{A,e} \circ (f \otimes id_{(A,e)^*}) \circ (\tau_{(A,e)}^{-1} \otimes id_{(A,e)^*}) \circ coev_{(A,e)^*}.$$

Monoidal category: The first triangle identities follows from the naturality of α and the identities for $\mathcal{C}$:

$$\begin{aligned}
&(id_{(A,e)} \otimes \lambda_{(B,f)}) \circ \alpha_{(A,e),(\mathbf{1},1_{\mathbb{C}}),(B,f)} \\
&\qquad = (e \otimes f) \circ (id_A \otimes \lambda_B) \circ (e \otimes (1_{\mathbb{C}} \otimes f)) \circ \alpha_{A,\mathbf{1},B} \circ ((e \otimes 1_{\mathbb{C}}) \otimes f) \\
&\qquad = (e \otimes f) \circ (id_A \otimes \lambda_B) \circ \alpha_{A,\mathbf{1},B} \circ ((e \otimes 1_{\mathbb{C}}) \otimes f) \\
&\qquad = (e \otimes f) \circ (\rho_A \otimes id_B) \circ ((e \otimes 1_{\mathbb{C}}) \otimes f) \\
&\qquad = \rho_{(A,e)} \otimes id_{(B,e)}.
\end{aligned}$$

The other triangle identity and the pentagon identity are proven in a similar way.

Symmetric Monoidal category: The unit coherence axiom is proven by the naturality of s and the corresponding identity for $\mathcal{C}$:

$$\begin{aligned}
\lambda_{(A,e)} \circ s_{(A,e),(\mathbf{1},1_{\mathbb{C}})} &= e \circ \lambda_A \circ (1_{\mathbb{C}}, e) \circ s_{A,\mathbf{1}} \circ (e, 1_{\mathbb{C}}) \\
&= e \circ \lambda_A \circ s_{A,\mathbf{1}} \circ (e, 1_{\mathbb{C}}) \\
&= e \circ \rho_A \circ (e, 1_{\mathbb{C}}) \\
&= \rho_{(A,e)}.
\end{aligned}$$

The associativity coherence and the inverse law are proven in a similar fashion. Clearly $\mathrm{End}((\mathbf{1}, 1_C)) = \mathbb{C}$.

Rigid Symmetric Monoidal category: The first duality axiom holds because of the naturality of s and the corresponding identity for $\mathcal{C}$

$$\begin{aligned}
ev_{(A,e)} \circ s_{(A,e)^*,(A,e)} &= 1_{\mathbb{C}} \circ ev_A \circ (e \otimes e^*) \circ s_{A^*,A} \circ (e^* \otimes e) \\
&= 1_{\mathbb{C}} \circ ev_A \circ s_{A^*,A} \circ (e^* \otimes e) \\
&= 1_{\mathbb{C}} \circ ev_{A^*} \circ (e^* \otimes e) \\
&= ev_{(A,e)^*}.
\end{aligned}$$

The other duality identity is proven similarly.
The morphism identites in $\mathcal{C}$ implies that for some idempotent $e^2 = e : A \to A$ the following equalities hold:

$$\begin{aligned}
ev_A \circ (e \otimes e^*) = ev_A \circ (id_A \otimes (e^*)^2) = ev_A \circ (id_A \otimes e^*) \text{ and} \\
(e^* \otimes e) coev_A = (id_{A^*} \otimes e^2) \circ coev_A = (id_{A^*} \otimes e) \circ coev_A.
\end{aligned}$$

The first of the triangle identities follows from the morphism identities in $\mathcal{C}$, the naturality of α and the corresponding identity in $\mathcal{C}$:

$$\begin{aligned}
&(ev_{(A,e)} \otimes id_{(A,e)}) \circ (\alpha_{(A,e),(A,e)^*,(A,e)})^{-1} \circ (id_{(A,e)} \otimes coev_{(A,e)}) \\
&\qquad = e \circ (ev_A \otimes id_A) \circ ((e \otimes e^*) \otimes e) \circ (\alpha_{A,A^*,A})^{-1} \circ (e \otimes (e^* \otimes e)) \circ (id_A \otimes coev_a) \circ e \\
&\qquad = e \circ (ev_A \otimes id_A) \circ ((e \otimes e^*) \otimes e) \circ (\alpha_{A,A^*,A})^{-1} \circ (id_A \otimes coev_a) \circ e \\
&\qquad = e \circ (ev_A \otimes id_A) \circ ((id_A \otimes e^*) \otimes e) \circ (\alpha_{A,A^*,A})^{-1} \circ (id_A \otimes coev_a) \circ e \\
&\qquad = e \circ (ev_A \otimes id_A) \circ (\alpha_{A,A^*,A})^{-1} \circ (id_A \otimes (e^* \otimes e)) \circ (id_A \otimes coev_a) \circ e
\end{aligned}$$

$$
\begin{aligned}
&= e \circ (ev_A \otimes id_A) \circ (\alpha_{A,A^*,A})^{-1} \circ (id_A \otimes (id_{A^*} \otimes e)) \circ (id_A \otimes coev_a) \circ e \\
&= e \circ (ev_A \otimes id_A) \circ ((id_A \otimes id_{A^*}) \otimes e) \circ (\alpha_{A,A^*,A})^{-1} \circ (id_A \otimes coev_a) \circ e \\
&= e \circ (ev_A \otimes e) \circ (\alpha_{A,A^*,A})^{-1} \circ (id_A \otimes coev_a) \circ e \\
&= e \circ (ev_A \otimes id_A) \circ (\alpha_{A,A^*,A})^{-1} \circ (id_A \otimes coev_a) \circ e \\
&= e \circ id_A \circ e \\
&= id_{(A,e)}.
\end{aligned}
$$

The other triangle identity can be proven in a similar way. The first morphism identity follows for $g = f \circ g \circ e : (A, e) \to (B, f)$ from the morphism identities in $\mathcal{C}$:

$$
\begin{aligned}
ev_{(B,f)} \circ (g \otimes id_{(B,f)^*}) &= ev_B \circ (f \otimes f^*) \circ (g \otimes id_{B^*}) \circ (e \otimes f^*) \\
&= ev_B \circ ((f \circ g \circ e) \otimes id_{B^*}) \circ (e, f^*) \\
&= ev_A \circ (id_A \otimes (f \circ g \circ e)^*) \circ (e, f^*) \\
&= ev_A \circ (e, e^*) \circ (id_A \otimes g^*) \circ (e, f^*) \\
&= ev_{(A,e)} \circ (id_{(A,e)} \otimes g^*).
\end{aligned}
$$

The second one is proven similarly.

Spherical Rigid Symmetric Monoidal category: Let $f = e \circ f \circ e : (A, e) \to (A, e)$ be an endomorphism, then by the naturality of τ, the morphism identities and the fact that $\mathcal{C}$ is spherical we see that

$$
\begin{aligned}
Trace_l^\natural f &= ev_{A,e} \circ (f \otimes id_{(A,e)^*}) \circ (\tau_{(A,e)}^{-1} \otimes id_{(A,e)^*}) \circ coev_{(A,e)^*} \\
&= ev_A \circ (e \otimes e^*) \circ (f \otimes id_{A^*}) \circ (e \otimes e^*) \otimes ((\tau_A)^{-1} \otimes id_{A^*}) \circ ((e^*)^* \otimes e^*) \circ coev_{A^*} \\
&= ev_A \circ (e \otimes e^*) \circ (f \otimes id_{A^*}) \circ (e \otimes e^*) \otimes ((\tau_A)^{-1} \otimes id_{A^*}) \circ coev_{A^*} \\
&= ev_A \circ (e \otimes e^*) \circ ((f \circ e) \otimes id_{A^*}) \otimes ((\tau_A)^{-1} \otimes id_{A^*}) \circ coev_{A^*} \\
&= ev_A \circ (e \otimes id_{A^*}) \circ ((f \circ e) \otimes id_{A^*}) \otimes ((\tau_A)^{-1} \otimes id_{A^*}) \circ coev_{A^*} \\
&= ev_A \circ ((e \circ f \circ e) \otimes id_{A^*}) \otimes ((\tau_A)^{-1} \otimes id_{A^*}) \circ coev_{A^*} \\
&= Trace_l(efe) \\
&= Trace_r(efe) \\
&= ev_{A^*} \circ (id_{A^*} \otimes (e \circ f \circ e)) \circ coev_A \\
&= ev_{A^*} \circ (id_{A^*} \otimes e) \circ (id_{A^*} \otimes (f \circ e)) \circ coev_A \\
&= ev_{A^*} \circ (e^* \otimes e) \circ (id_{A^*} \otimes (f \circ e)) \circ coev_A \\
&= ev_{A^*} \circ (e^* \otimes e) \circ (id_{A^*} \otimes f) \circ (e^* \otimes e) \circ coev_A \\
&= ev_{(A,e)^*} \circ (id_{(A,e)^*} \otimes f) \circ coev_{(A,e)} \\
&= Trace_r^\natural(f)
\end{aligned}
$$

□

Definition 1.1.25. We define the Karoubian envelope of a $\mathbb{C}$-linear category $\mathcal{C}$ by $\mathcal{C}^{Kar} := (\mathcal{C}^{add})^\natural$ and $\iota_{Kar} := \iota_\natural \circ \iota_{add}$, which is a $\mathbb{C}$-linear full embedding.

Definition 1.1.26. A $\mathbb{C}$-linear category $\mathcal{C}$ is Karoubi if $\iota_{Kar} : \mathcal{C} \simeq \mathcal{C}^{Kar} = (\mathcal{C}^{add})^\natural$ is an equivalence, i.e. all idempotents split and all finite biproducts exist.

Remark 1.1.27. Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. Then the Karoubian envelope $\mathcal{C}^{Kar}$ satisfies the following universal property. Let $\mathcal{D}$ be a Karoubi category and $\gamma : \mathcal{C} \to \mathcal{D}$ a $\mathbb{C}$-linear functor. Then there exists a $\mathbb{C}$-linear additive functor $\gamma' : \mathcal{C}^{Kar} \to \mathcal{D}$ such that $\gamma = \gamma' \circ \iota_{Kar}$. The functor γ is unique up to natural isomorphism.

Proposition 1.1.28. *The Karoubian envelope $\mathcal{C}^{Kar}$ of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category $\mathcal{C}$ can again be given the structure of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category.*

Proof. This follows immediately from Proposition 1.1.20 and Proposition 1.1.24. □

Proposition 1.1.29. *Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. Let $\mathcal{D}$ be a Karoubi category and $\gamma : \mathcal{C} \to \mathcal{D}$ a $\mathbb{C}$-linear functor. Let $\gamma' : \mathcal{C}^{Kar} \to \mathcal{D}$ be the $\mathbb{C}$-linear additive functor given by the universal property of the Karoubian envelope such that $\gamma = \gamma' \circ \iota_{Kar}$. Assume that γ is faithful. Then the induced functor γ' is also faithful. Assume that γ is fully faithful. Then the induced functor γ' is also fully faithful.*

Proof. We first assume that γ is faithful and show the faithfulness of γ'. Because $\mathcal{D}$ is Karoubi, we can replace it by $\mathcal{D}^{Kar}$. We do this to be able to get a more manageable description of γ'. The functor γ' is given up to isomorphism, but we work with the concrete choice which sends an object $(A, e) \in \mathcal{C}^{Kar}$ to $\gamma'((A, e)) := (\gamma(A), \gamma(e)) \in \mathcal{D}^{Kar}$. Let $g, h : (A, e) \to (B, f)$ be morphisms in $\mathcal{C}^{Kar}$ with $\gamma'(g) = \gamma'(h) : (\gamma(A), \gamma(e)) \to (\gamma(B), \gamma(f))$. By the fact that the morphisms $g, h : A \to B$ also lie in $\mathcal{C}$, γ' sends the commutative square

$$\begin{array}{ccc} (A,e) & \xrightarrow{g,h} & (B,f) \\ {\scriptstyle e}\uparrow & & {\scriptstyle f}\uparrow \\ (A, id_A) & \xrightarrow{g,h} & (B, id_B). \end{array}$$

in $\mathcal{C}^{Kar}$ to the commutative square

$$\begin{array}{ccc} (\gamma(A),\gamma(e)) & \xrightarrow{\gamma'(g),\gamma'(h)} & (\gamma(B),\gamma(f)) \\ {\scriptstyle \gamma(e)}\uparrow & & {\scriptstyle \gamma(f)}\uparrow \\ (\gamma(A), id_{\gamma(A)}) & \xrightarrow{\gamma(g),\gamma(h)} & (\gamma(B), id_{\gamma(B)}) \end{array}$$

in $\mathcal{D}$. This shows that

$$\begin{aligned} \gamma(g) &= \gamma(f \circ g) = \gamma(f) \circ \gamma(g) = \gamma'(g) \circ \gamma(e) \\ &= \gamma'(h) \circ \gamma(e) = \gamma(f) \circ \gamma(h) = \gamma(f \circ g) = \gamma(h). \end{aligned}$$

Because γ is faithful by assumption, we get $g = h$. This shows that γ' is faithful.

Now assume that γ is fully faithful. Let $g : (\gamma(A), \gamma(e)) \to (\gamma(B), \gamma(f))$ be a morphism. Then g is also a morphism $(\gamma(A), id_{\gamma(A)}) \to (\gamma(B), id_{\gamma(A)})$. Because γ is full there exists a $g' : A \to B$ such that $\gamma(g') = g$. Because

$$\gamma(f \circ g' \circ e) = \gamma(f) \circ \gamma(g') \circ \gamma(e) = \gamma(g')$$

and γ is faithful, we see that $g' = f \circ g' \circ e$. So g' is a morphism $(A, e) \to (B, f)$ with $\gamma'(g') = g$. This shows that γ' is full.

□

Definition 1.1.30. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is called essentially injective if for all objects $A, B \in \mathcal{C}$ with $\mathcal{F}(A) \cong \mathcal{F}(B)$, also $A \cong B$ holds.

Lemma 1.1.31. *If $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a fully faithful functor, then it is essentially injective.*

Proof. Let $A, B \in \mathcal{C}$ with $\mathcal{F}(A) \cong \mathcal{F}(B)$. Then there exist $\alpha : \mathcal{F}(A) \to \mathcal{F}(B)$ and $\beta : \mathcal{F}(B) \to \mathcal{F}(A)$ such that $\beta \circ \alpha = id_{\mathcal{F}(A)}$ and $\alpha \circ \beta = id_{\mathcal{F}(B)}$. Because $\mathcal{F}$ is full, there exist $\alpha' : A \to B$ and $\beta' : B \to A$ such that $\mathcal{F}(\alpha') = \alpha$ and $\mathcal{F}(\beta') = \beta$. Because $\mathcal{F}$ is faithful, $\mathcal{F}(\beta' \circ \alpha') = \beta \circ \alpha = id_{\mathcal{F}(A)} = \mathcal{F}(id_A)$ implies that $\beta' \circ \alpha' = id_A$. Similarly we obtain $\alpha' \circ \beta' = id_B$, so $A \cong B$, which shows that $\mathcal{F}$ is essentially injective. □

Definition 1.1.32. A $*$-*operation*[3] on a $\mathbb{C}$-linear monoidal category $\mathcal{C}$ is a contravariant involutive antilinear monoidal endofunctor $* : \mathcal{C} \to \mathcal{C}$ which is the identity on objects. Involutive means that $* \circ * = Id_{\mathcal{C}}$.

Definition 1.1.33. A $*$-*category*[4] is a $\mathbb{C}$-linear monoidal category with an $*$-operation.

Definition 1.1.34. A functor between two $*$-categories is called a $*$-*functor* if it preserves the $*$-operation.

Definition 1.1.35. A $*$-operation is called *positive* if for any morphism f we have that $f^* \circ f = 0$ implies $f = 0$.

1.2 Representation Theory

The representation categories $\mathrm{Rep}(S_n)$ and $\mathrm{Rep}(H_n)$ over the complex numbers are $\mathbb{C}$-linear semisimple categories. This follows from Maschkes Theorem. In this section we want to take a closer look at some representations of the symmetric and hyperoctahedral groups.

1.2.1 The symmetric and hyperoctahedral groups

Definition 1.2.1. Let $n \in \mathbb{N}$. The n-th permutation group S_n is the group of bijective functions of some set with n elements into itself.

Remark 1.2.2. As is seen in [AA00] the symmetric groups can be described using the following generators and relations:

$$\begin{aligned} S_n = \langle x_1, x_2, ..., x_{n-1} | x_i^2 &= 1 \text{ for all } i \in \{1, \dots, n-1\}, \\ (x_i x_{i+1})^3 &= 1 \text{ for all } i \in \{1, \dots, n-2\}, \\ (x_i x_j)^2 &\text{ for } 1 \leqslant i < j-1 \leqslant n-2 \rangle. \end{aligned}$$

[3]This and the following definitions are taken from [FM21, Chapter 3.2]

[4]See Remark 2.5.23 for an example.

Definition 1.2.3. Let G and H be groups. Let A be a finite set equiped with an H-action and with cardinality $|A| = n$. The wreath product $G \wr_A H$ is the semi direct product with underlying set $G^n \times H$ and product given by

$$(g_1, \dots, g_n, h) \cdot (g'_1, \dots, g'_n, h') := (g_1 g'_{h^{-1}(1)}, \dots, g_n g'_{h^{-1}(n)}, hh')$$

for all $g_1, \dots, g_n, g'_1, \dots, g'_n \in G$ and $h, h' \in H$.

Remark 1.2.4. The $n-th$ symmetric group S_n is isomorphic to the wreath product $\mathbf{1} \wr S_n$, where $\mathbf{1}$ is the group with one element. As a matrix group, it consists of the permutation matrices of size $n \times n$. Note that S_n is also an example of a complex reflection group, namely $G(1, 1, n)$, see [Tay12].

Definition 1.2.5. The n-th hyperoctahedral group H_n is the group of bijective functions π on $\{-n, \dots, -1, 1, \dots, n\}$ where $\pi(i) = -\pi(-i)$ for all $i \in \{-n, \dots, -1, 1, \dots, n\}$.

Remark 1.2.6. If we consider $\mathbb{Z}_2$ as the multiplicative group $\{-1, 1\} \subset \mathbb{C}$, then we can describe the group $\mathbb{Z}_2^n$ by the following generators and relations:

$$\begin{aligned} \mathbb{Z}_2^n = \langle y_1, y_2, \dots, y_n | y_i^2 = 1 \text{ for all } i \in \{1, \dots, n\}, \\ (y_i y_j)^2 = 1 \text{ for } 1 \leqslant i < j \leqslant n \rangle. \end{aligned}$$

Note that the generators equal $y_i = (1, \dots, 1, \underset{i-th}{-1}, 1, \dots, 1)$ for all $i \in \{1, \dots, n\}$. We will sometimes write $1 := (1, \dots, 1) \in \mathbb{Z}_2^n$ for the neutral element of the group. As a wreath product H_n equals $\mathbb{Z}_2 \wr S_n = \mathbb{Z}_2^n \rtimes S_n$. The elements are of the form $a = (a_1, \dots, a_n, \sigma)$, with $a_i \in \mathbb{Z}_2$ and $\sigma \in S_n$. The product is given by

$$(a_1, \dots, a_n, \sigma)(b_1, \dots, b_n, \rho) = (a_1 b_{\sigma^{-1}(1)}, \dots, a_n b_{\sigma^{-1}(n)}, \sigma\rho).$$

Then H_n can be described using the following generators and relations:

$$\begin{aligned} \mathbb{Z}_2 \wr S_n = \langle (y_1, 1), (1, x_1), \dots, (1, x_{n-1}) | ((y_1, 1)(1, x_1))^4 = 1, \\ ((1, x_i)(1, x_{i+1}))^3 = 1 \text{ for all } i \in \{1, \dots, n-2\}, \\ (y_1, 1)^2 = 1, (1, x_i)^2 = 1 \text{ for all } i \in \{1, \dots, n-1\}, \\ ((y_1, 1)(1, x_i))^2 = 1 \text{ for all } i \in \{2, \dots, n-1\}, \\ ((1, x_i)(1, x_j))^2 = 1 \text{ for } 1 \leqslant i < j-1 \leqslant n-2 \rangle. \end{aligned}$$

This shows that H_n is isomorphic to the Coxeter group of type B

$$\begin{aligned} B_n = \langle r_1, \dots, r_n | (r_1 r_2)^4 = 1, \\ (r_i r_{i+1})^3 = 1 \text{ for all } i \in \{2, \dots, n\}, \\ (r_i)^2 = 1 \text{ for all } i \in \{1, \dots, n\}, \\ (r_i r_j)^2 = 1 \text{ for } 1 \leqslant i < j-1 \leqslant n-1 \rangle. \end{aligned}$$

Remark 1.2.7. There is an injective group homomorphism $\lambda : S_n \to H_n$ given by $\lambda(x_i) = (1, x_i)$. As a matrix group, H_n consists of the permutation matrices of size $n \times n$

with possible entries $\{-1,1\}$. Note that H_n is also an example of a complex reflection group, namely $G(2,1,n)$, see [Tay12].

1.2.2 Reflection and permutation representations

Definition 1.2.8. Let $\sigma \in S_n$ and $(c_1,\dots,c_n) \in \mathbb{C}^n$. The permutation representation $u' = \mathbb{C}^n$ is given by

$$\sigma(c_1,\dots,c_n) = (c_{\sigma^{-1}(1)},\dots,c_{\sigma^{-1}(n)}).$$

If $e_i = (0,\dots,1,\dots,0)$ is the canonical basis element with 1 on the i-th place, then $\sigma(e_i) = e_{\sigma(i)}$.

Lemma 1.2.9. *The permutation representation of S_n is a faithful representation and it is self-dual, i.e. $u' \cong (u')^*$ as representations.*

Proof. The fact that u' is faithful follows directly from its definition. Let $\{e^i | i \in \{1,\dots,n\}\}$ be the canonical basis of $(\mathbb{C}^n)^*$. The action of S_n on $(u')^*$ is given by $\sigma \cdot (e^i) := e^{\sigma(i)}$. We define the following isomorphism of vector spaces

$$\begin{aligned} \phi : u' &\to (u')^* \\ e_i &\mapsto e^i \text{ for all } i \in \{1,\dots,n\}. \end{aligned}$$

For all $\sigma \in S_n$ and $a = (a_1,\dots,a_n), c = (c_1,\dots,c_n) \in u$ we have that

$$\begin{aligned} \phi(\sigma \cdot a)(c) &= \phi(\sigma \cdot (a_1,\dots,a_n))(c_1,\dots,c_n) \\ &= \phi((a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(n)}))(c_1,\dots,c_n) \\ &= \sum_{i=1}^{n} a_{\sigma^{-1}(i)} c_i \\ &= \sum_{i=1}^{n} a_i c_{\sigma(i)} \\ &= \phi((a_1,\dots,a_n))(\sigma^{-1} \cdot (c_1,\dots,c_n)) \\ &= \sigma \cdot \phi(a)(c). \end{aligned}$$

This shows that ϕ is an isomorphism of representations. ☐

Definition 1.2.10. Let $a = (a_1,\dots,a_n,\sigma) \in H_n$ and $(c_1,\dots,c_n) \in \mathbb{C}^n$. Then the reflection representation $u = \mathbb{C}^n$ is defined by

$$a \cdot (c_1,\dots,c_n) = (a_1 c_{\sigma^{-1}(1)},\dots,a_n c_{\sigma^{-1}(n)})$$

where we consider $\mathbb{Z}_2 = \{-1,1\} \subset \mathbb{C}$. If e_i is a canonical basiselement of $\mathbb{C}^n$, then $a \cdot e_i = a_{\sigma(i)} e_{\sigma(i)}$.

Remark 1.2.11. Note that this corresponds to the multiplication of the corresponding permutation matrices with entries $\{-1,1\}$ with vectors of $\mathbb{C}^n$.

Lemma 1.2.12. *The reflection representation of H_n is a faithful representation and it is self-dual, i.e. $u \cong u^*$ as representations.*

Proof. The fact that u is faithful follows directly from its definition. Let $\{e^i | i \in \{1, \dots, n\}\}$ be the canonical basis of $(\mathbb{C}^n)^*$. The action of H_n on u^* is given by $a \cdot (e^i) := a_{\sigma(i)} e^{\sigma(i)}$. We define the following isomorphism of vector spaces

$$\begin{aligned}\psi : u &\to u^* \\ e_i &\mapsto e^i \text{ for all } i \in \{1, \dots, n\}.\end{aligned}$$

For all $a = (a_1, \dots, a_n, \sigma) \in H_n$ we have that

$$\begin{aligned}a^{-1} &= (a_{\sigma(1)}^{-1}, \dots, a_{\sigma(n)}^{-1}, \sigma^{-1}) \\ &= (a_{\sigma(1)}, \dots, a_{\sigma(n)}, \sigma^{-1})\end{aligned}$$

and for all $b = (b_1, \dots, b_n), c = (c_1, \dots, c_n) \in u$ we have

$$\begin{aligned}\psi(a \cdot b)(c) &= \psi((a_1, \dots, a_n, \sigma) \cdot (b_1, \dots, b_n))(c_1, \dots, c_n) \\ &= \psi((a_1 b_{\sigma^{-1}(1)}, \dots, a_n b_{\sigma^{-1}(n)}))(c_1, \dots, c_n) \\ &= \sum_{i=1}^{n} a_i b_{\sigma^{-1}(i)} c_i \\ &= \sum_{i=1}^{n} b_i a_{\sigma(i)} c_{\sigma(i)} \\ &= \psi((b_1, \dots, b_n))((a_{\sigma(1)} c_{\sigma(1)}, \dots, a_{\sigma(n)} c_{\sigma(n)})) \\ &= \psi((b_1, \dots, b_n))((a_{\sigma(1)}, \dots, a_{\sigma(n)}, \sigma^{-1}) \cdot (c_1, \dots, c_n)) \\ &= \psi(b)(a^{-1} \cdot c) \\ &= a \cdot \psi(b)(c).\end{aligned}$$

This shows that ψ is an isomorphism of representations. □

Definition 1.2.13. Let $a = (a_1, \dots, a_n, \sigma) \in H_n$. The permutation representation $V = \mathbb{C}^{2n} = (\mathbb{C}\,\mathbb{Z}_2)^n = \oplus_{i=1}^n (\mathbb{C}\, e_1^i \oplus \mathbb{C}\, e_{-1}^i)$ of H_n is defined by the $\mathbb{C}$-linear extension of the action

$$a \cdot e_j^i = e_{a_{\sigma(i)} \cdot j}^{\sigma(i)}$$

for $j \in \{-1, 1\}$ and $i \in \{1, \dots, n\}$. Let $c_j^i \in \mathbb{C}$ for $j \in \mathbb{Z}_2$ and $i \in \{1, \dots, n\}$. Then we get for $c = (c_1^1 e_1^1 + c_{-1}^1 e_{-1}^1, \dots, c_1^n e_1^n + c_{-1}^n e_{-1}^n \in V)$ that

$$a \cdot c = (c_{a_1 \cdot 1}^{\sigma^{-1}(1)} e_1^1 + c_{a_1 \cdot (-1)}^{\sigma^{-1}(1)} e_{-1}^1, \dots, c_{a_n \cdot 1}^{\sigma^{-1}(n)} e_1^n + c_{a_n \cdot (-1)}^{\sigma^{-1}(n)} e_{-1}^n).$$

Lemma 1.2.14. *The permutation representation of H_n is a faithful representation and it is self-dual, i.e. $V \cong V^*$ as representations.*

Proof. The fact that V is faithful follows directly from its definition.
Let $\{e_i^j | i \in \{1, \dots, n\}, j \in \{-1, +1\}\}$ be the canonical basis of V^*. The action of H_n on

V^* is given by $a \cdot (e_i^j) := e_{\sigma(i)}^{a_{\sigma(i)} \cdot j}$. We define the following isomorphism of vector spaces

$$\begin{aligned} \rho : V &\to V^* \\ e_j^i &\mapsto e_i^j \text{ for all } i \in \{1, \dots, n\}, j \in \{-1, 1\}. \end{aligned}$$

For all $a = (a_1, \dots, a_n, \sigma) \in H_n$, $b = (b_1^1 e_1^1 + b_{-1}^1 e_{-1}^1, \dots, b_1^n e_1^n + b_{-1}^n e_{-1}^n) \in V$ and $c = (c_1^1 e_1^1 + c_{-1}^1 e_{-1}^1, \dots, c_1^n e_1^n + c_{-1}^n e_{-1}^n) \in V$ we have that

$$\begin{aligned} \rho(a \cdot b)(c) &= \rho((a_1, \dots, a_n, \sigma) \cdot (b_1^1 e_1^1 + b_{-1}^1 e_{-1}^1, \dots, b_1^n e_1^n + b_{-1}^n e_{-1}^n))(c) \\ &= \rho(b_{a_1 \cdot 1}^{\sigma^{-1}(1)} e_1^1 + b_{a_1 \cdot (-1)}^{\sigma^{-1}(1)} e_{-1}^1, \dots, b_{a_n \cdot 1}^{\sigma^{-1}(n)} e_1^n + b_{a_n \cdot (-1)}^{\sigma^{-1}(n)}) e_{-1}^n)(c) \\ &= \sum_{\substack{i=1 \\ j \in \mathbb{Z}_2}}^{n} b_{a_i \cdot j}^{\sigma^{-1}(i)} c_j^i \\ &= \sum_{\substack{i=1 \\ j \in \mathbb{Z}_2}}^{n} b_{a_{\sigma(i)} \cdot j}^{i} c_j^{\sigma(i)} \\ &= \sum_{\substack{i=1 \\ j \in \mathbb{Z}_2}}^{n} b_j^i c_{a_{\sigma(i)} \cdot j}^{\sigma(i)} \\ &= \rho(b)(c_{a_{\sigma(1)} \cdot 1}^{\sigma(1)} e_1^1 + c_{a_{\sigma(1)} \cdot (-1)}^{\sigma(1)} e_{-1}^1, \dots, c_{a_{\sigma(n)} \cdot 1}^{\sigma(n)} e_1^n + c_{a_{\sigma(n)} \cdot (-1)}^{\sigma(n)} e_{-1}^n) \\ &= \rho(b)((a_{\sigma(1)}, \dots, a_{\sigma(n)}, \sigma^{-1}) \cdot (c_1^1 e_1^1 + c_{-1}^1 e_{-1}^1, \dots, c_1^n e_1^n + c_{-1}^n e_{-1}^n)) \\ &= \rho(b)(a^{-1} \cdot c) \\ &= a \cdot \rho(b)(c). \end{aligned}$$

This shows that ρ is an isomorphism of representations. □

Remark 1.2.15. Note that $u \cong \tilde{u} := \oplus_{i=1}^n \mathbb{C}(e_1^i - e_{-1}^i) \subset V$ is a subrepresentation. Another interesting n-dimensional subrepresentation of V is $v = \oplus_{i=1}^n \mathbb{C}(e_1^i + e_{-1}^i)$, this is the complement of $\tilde{u}$ in V. By considering S_n as a subgroup of H_n via $\lambda : S_n \to H_n$, we see that $\mathrm{Res}_{S_n}^{H_n}(u) = u'$. The induced representation $\mathrm{Ind}_{S_n}^{H_n}(u') = \mathbb{C}H_n \otimes_{\mathbb{C}S_n} u'$ has V as a subrepresentation, as can be seen by the $\mathbb{C}$-linear map

$$e_j^i \mapsto (1, \dots, \underset{i-th}{j}, \dots, 1, id_{S_n}) \otimes e_i.$$

Chapter 2

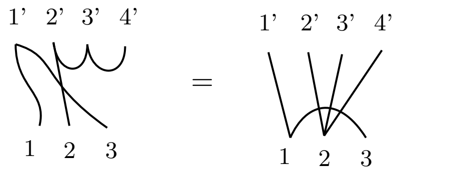
Interpolation Categories

In this chapter we are defining three interpolation categories for all $t \in \mathbb{C}$: the categories $\underline{\mathrm{Rep}}(S_t)$ of Deligne [Del07], $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ of Nyobe Likeng-Savage [LS21] and $\underline{\mathrm{Rep}}(H_t)$ of Flake-Maassen [FM21]. The latter two can be seen as different interpolation categories for the hyperoctahedral groups, respectively modelled by the tensor products of the permutation representations and the tensor products of the reflection representations.

2.1 Partition Theory

In this section we introduce the notions of partition theory that we will use later. The general partition theory is based on [CW12], but we will introduce the notation in such a way that it can be made compatible with notions from other sources. We define so-called permutation partitions in Definition 2.1.2. The discussion about the even partitions is based on [FM21]. In Proposition 2.1.7 we show that P_{even} is closed under the partition operations and in Proposition 2.1.8 that P_{even} is generated by a finite set of partitions, They are assumed in the mentioned source, but not proven. The definitions for the $\mathbb{Z}_2$-coloured partitions and the idea for the proof of Proposition 2.1.11 are taken from [LS21], though the notation has been adjusted in both cases. Proposition 2.1.10 was assumed in the source, but not proven. We define normal forms for coloured and non-coloured partitions. This was inspired on the idea of a standard decomposition given in the proof of [LS21, Theorem 4.4] and the normal form in [Koc03, Remark 1.4.16].

Definition 2.1.1. Let $k, l \in \mathbb{N}$. We denote the set of all partitions of the set $\{1, \ldots, k, 1', \ldots, l'\}$ by $P(k, l)$ and the set of all partitions by $P := \sqcup_{k,l \in \mathbb{N}} P(k, l)$. Note that we will add or remove accents in the sets $\{1, \ldots, k, 1', \ldots, l'\}$ depending on the context we are working in. We call (k, l) the size of the partitions in $P(k, l)$. The elements of some partition $p \in P(k, l)$ are called components, parts or blocks. For partitions $p, q \in P(k, l)$ we call q coarser as p if every part in p is a subset of some part in q. We can associate a partition diagram to each partition $p \in P(k, l)$ by placing k vertices in a horizontal row and l vertices in a horizontal row, above the first one. We label the vertices from left to right by the elements of the set $\{1, \ldots, k, 1', \ldots, l'\}$. We draw a line between to vertices in the diagram if and only if the corresponding labels lie in the same component. We call two such partition diagrams equivalent if the set partitions are the same.



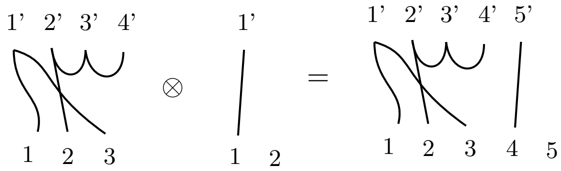
Example: $\{\{1,3,1'\},\{2,2',3',4'\}\} \in P(3,4)$

There are some important operations on the sets of partitions.

Horizontal concatenation: Let $p \in P(k,l)$ and $q \in P(m,n)$, where we let the set of partitions contain partitions of $\{1,\dots,k,1'',\dots,l''\}$ and $\{1',\dots,m',1''',\dots,n'''\}$ respectively. Then we can describe $P(k+m,l+n)$ as the set of partitions of

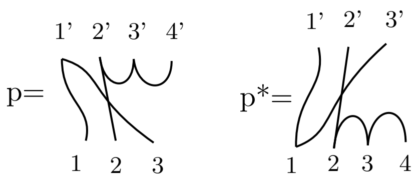
$$\{1,\dots,k,1',\dots,m',1'',\dots,l'',1''',\dots,n'''\}.$$

We define $p \otimes q \in P(k+m,l+n)$ by: $a \in p \otimes q$ if and only if $a \in p$ or $a \in q$.



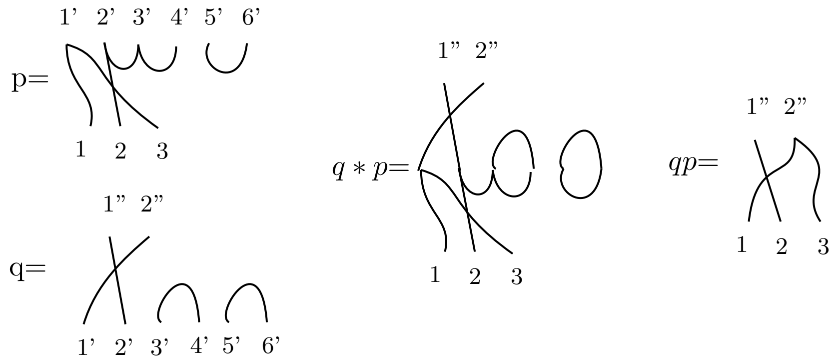
Example: Horizontal Concatenation

Involution: Let $p \in P(k,l)$ and let us describe $P(l,k)$ as the set of partitions of $\{1',\dots,l',1,\dots,k\}$. We define $p^* \in P(l,k)$ by: $a \in p$ if and only if $a \in p^*$.



Example: Involution

Vertical concatenation: Let $p \in P(k,l)$ and $q \in P(l,m)$ where we let the set of partitions contain partitions of $\{1,\dots,k,1',\dots,l'\}$ and $\{1',\dots,l',1'',\dots,m''\}$ respectively. Then we can describe $P(k,m)$ as the set of partitions of $\{1,\dots,k,1'',\dots,m''\}$ and define qp by: $\{i,j''\} \subseteq A$ for some $A \in qp$ if and only if there exists $h' \in \{1'\dots,l'\}$ such that there exist $B \in p$ and $C \in q$ with $\{i,h'\} \subseteq B$ and $\{h',j''\} \subseteq C$.
The stacking $q \star p$ is the diagram which is attained of putting q on top of p.

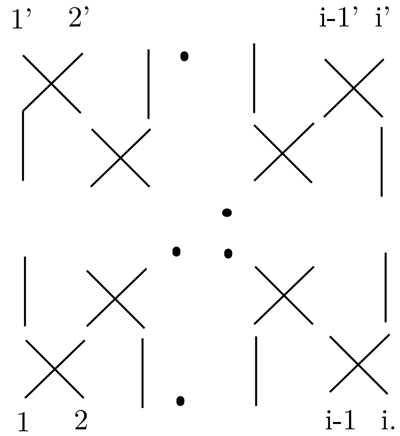


Example: Vertical Concatenation with $l(p,q) = 1$.

A loop in qp is a subset $L \subseteq \{1' \dots, l'\}$ satisfying two conditions: firstly it consists of elements in $h' \in \{1' \dots, l'\}$ for which there doesn't exist an $i \in \{1 \dots, k\}$ and a $B \in p$ such that $\{i, h'\} \subseteq B$ or an element $j'' \in \{1'' \dots, m''\}$ and a $C \in q$ such that $\{h, j''\} \subseteq C$. Secondly assume that L is a non-empty loop containing some element h'. Then $g' \in L$ if and only if there exist some B $\in p$ such that $\{h', g'\} \subseteq B$ or $C \in q$ such that $\{h', g'\} \subseteq C$. We denote the number of loops in qp by $l(q,p)$.

Before we continue discussing the even partitions we discuss the permutation partitions and the non-crossing forms of a partition.

Definition 2.1.2. For every $n \in \mathbb{N}$, there is an injective monoid homomorphism $\phi : S_n \hookrightarrow P(n,n)$, which is defined by sending the cycles $(1,i)$ to

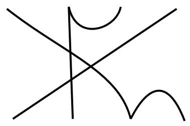


for all $2 \leqslant i \leqslant n$. The image $\phi((1,i))$ is the partition

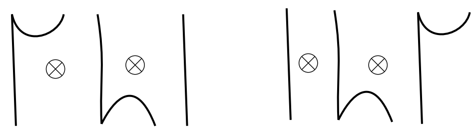
$$\{\{1, i'\}, \{2, 2'\}, \dots \{i, 1'\}, \dots, \{n, n'\}\} \in P(n,n).$$

Because the cycles $\{(1,i) | 2 \leqslant i \leqslant n\}$ generate S_n, this defines the map ϕ. It is well defined because the image contains all partitions of block size 2 which connect one lower and one upper vertex, in other words $\phi(\sigma) = \{\{i, \sigma(i)'\} | 1 \leqslant i \leqslant n\}$ and this will not depend on the way we decompose $\sigma \in S_n$ in transitions. We call the partitions in the image of ϕ permutation partitions.

Definition 2.1.3. Let $p \in P(k,l)$ and let $B_1, \dots, B_t$ be its blocks, which are respectively of size $(k_1, l_1), \dots, (k_t, l_t)$. Define the partitions $p_i := \{\{1, \dots, k_i, 1', \dots, l_i'\}\} \in P(k_i, l_i)$ for all $1 \leqslant i \leqslant t$. A non-crossing form of p is some horizontal concatenation $p_{i_1} \otimes \dots \otimes p_{i_t}$, where $i_1, \dots, i_t$ is some permutation of $\{1, \dots, t\}$.



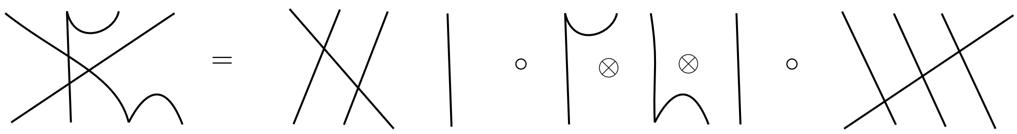
p



Examples of non-crossing forms of p

Proposition 2.1.4. *Every partition $p \in P(k,l)$ can be written as $\phi(\sigma) \circ p' \circ \phi(\rho)$, where $\sigma \in S_l$, $\rho \in S_k$ and p' is some non-crossing form of p. We call $\phi(\sigma) \circ p' \circ \phi(\rho)$ a normal form of p.*

Proof. Let $p' = p_1 \otimes \dots \otimes p_t$ be some non-crossing form. The upper vertices of every block p_j should be uniquely redistributed along $\{1', \dots, l'\}$ according to how the block is positioned in p. There is an unique $\sigma \in S_l$ which achieves this. Similarly we choose a $\rho \in S_k$.



p $\qquad \phi(\sigma) \qquad p' \qquad \phi(\rho)$

□

Definition 2.1.5. Let $p \in P(k,l)$ and p' be some non-crossing form of p. Then by Proposition 2.1.4 we can write $p = \phi(\sigma) \circ p' \circ \phi(\rho)$ for some $\sigma \in S_l$ and $\rho \in S_k$. We call $\phi(\sigma) \circ p' \circ \phi(\rho)$ a normal form of p.

Definition 2.1.6. Let $k, l \in \mathbb{N}$. We call a partition $p \in P(k,l)$ even if all its components contain an even number of vertices. We denote the set of all even partitions in $P(k,l)$ by $P_{even}(k,l)$ and the set of all even partitions by $P_{even} := \sqcup_{k,l \in \mathbb{N}} P_{even}(k,l) \subset P$.

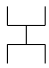
Proposition 2.1.7. *P_{even} is closed under involution, vertical concatenation and horizontal concatenation.*

Proof. It is clear that the involution of an even partition is again even. The horizontal concatenation of two even partitions is even because the the blocks of the individual partitions are kept intact. Let $p \in P_{even}(k,l)$ and $q \in P_{even}(l,m)$. Let $A \in qp$ be a block and let us use the notation from above. Assume that $\#(A \cap \{1, \dots, k\})$ is even. This implies that the number of vertices in the upper row of p connected to $A \cap \{1, \dots, k\}$, is even, because p is even. Assume that $\#(A \cap \{1'', \dots, m''\})$ is odd. Because q is even, the number of vertices in the lower row of q connected to $A \cap \{1'', \dots, m''\}$ is odd. We assume that the number of vertices in the lower row of q connected to $A \cap \{1'', \dots, m''\}$ is less than the number of vertices in the upper row of p connected to $A \cap \{1, \dots, k\}$. Then there is an odd amount of vertices in the middle row of $q \star p$ which are connected to $A \cap \{1, \ldots, k\}$) but not to $A \cap \{1'', \ldots, m''\}$. But these vertices cannot form even blocks in q by itself, because if they are only connected to vertices in the lower row of q which are not connected to the upper row, we still get odd blocks. So some of them have to be connected in q to a vertex in $\{1'', \ldots, m''\}$ which does not lie in A. Which is a contradiction, because by construction this vertex would be connected to the lower row of A in $q \star p$ and thus lie in A. So the number of vertices in the lower row of q connected to $A \cap \{1'', \ldots, m''\}$ is even, which implies that $A \cap \{1'', \ldots, m''\}$ is even, because q is even.

If we assume that the number of vertices in the lower row of q connected to $A \cap \{1'', \ldots, m''\}$ is greater than the number of vertices in the upper row of p connected to $A \cap \{1, \ldots, k\}$, we can proceed the argument analogously.

Similarly we can proof that if $\#(A \cap \{1, \ldots, k\})$ is odd, then also $\#(A \cap \{1'', \ldots, m''\})$ will be odd. We conclude that qp is even.

□

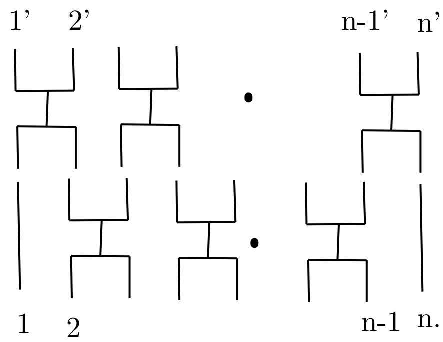
Proposition 2.1.8. *Every even partition $p \in P_{even}$ can be constructed by applying the involution, horizontal and vertical concatenation operations to the partitions*
 $= \{\{1, 2'\}, \{2, 1'\}\}$,  $= \{1, 2, 1', 2'\}$, $=\{1, 1'\}$ *and*  $=\{\{1, 2\}\}$.

Proof. Assume that $p \in P_{even}(k, l)$.

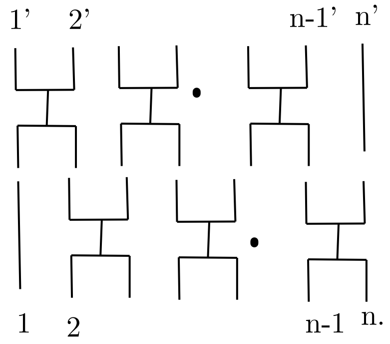
First note that every permutation partition is an even partition, so $\text{im}(\phi) \subset P_{even}(n, n)$, because  and  lie in P_{even}.

We can find $\sigma \in S_k$ and $\rho \in S_l$, such that $\phi(\rho) \circ p \circ \phi(\sigma)$ is a noncrossing partition, meaning that it is a horizontal concatenation of its blocks. The only thing left to prove is that every even block is obtained by the aforementioned partitions and operations.

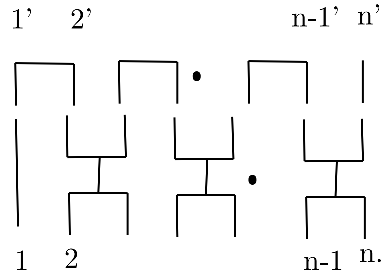
Firstly we show that this hold for every block s_n containing n vertices in the lower and upper row. Assume that $n = 2m$ is even and $m > 0$. Then the block equals the vertical concatenation



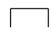
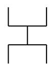
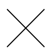
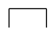
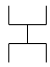
If $n = 2m + 1$ is odd and $m > 0$, then the block equals the vertical concatenation



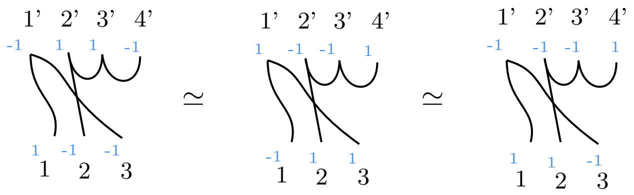
If $n = 1$ we just define $s_1 = \{\{1, 1'\}\}$. Secondly we see that for $m > 0$ every block t_n with $n = 2m + 1$ vertices in the lower row and 1 in the upper row equals the vertical concatenation



Let $k > l \geqslant 1$. Now every even block with k vertices in the lower row and n vertices in the upper row equals the concatenation $s_l(t_{k-l+1} \otimes \underbrace{| \otimes \ldots \otimes |}_{l-1 \text{ times}})$. If $l > k$ we can use the involution and then apply the previous procedure. If $k = l$ every even block with k lower and k upper vertices equals of course s_k.

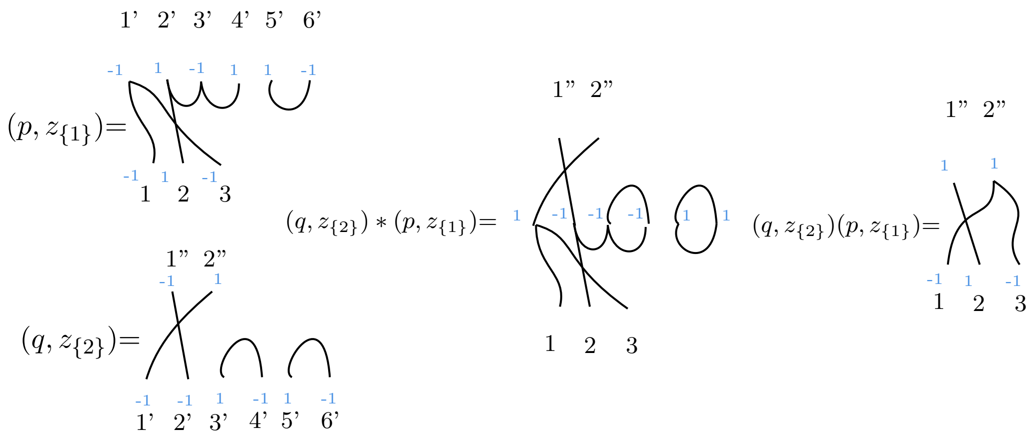
To summarize the proof: Let $\phi(\rho)p\phi(\sigma) = B_1 \otimes \ldots \otimes B_t$, where the B_j are even blocks. The partitions $\phi(\rho)$ and $\phi(\sigma)$ can be constructed using $|$ and . The B_j can be constructed by the partitions , $|$ and . So $p = \phi(\rho^{-1})(B_1 \otimes \ldots \otimes B_t)\phi(\sigma^{-1})$ can be obtained by applying the partition operations to $|$, ,  and . □

Definition 2.1.9. We denote the set of $\mathbb{Z}_2$-coloured partitions by $P_{\mathbb{Z}_2} = \sqcup P_{\mathbb{Z}_2}(k, l)$. An element of $(p, z) \in P_{\mathbb{Z}_2}(k, l)$ is pair consisting of a partition $p \in P(k, l)$ and a vector $z \in (\mathbb{Z}_2)^{k+l}$. Such a $\mathbb{Z}_2$-coloured partition can be visualised by a partition diagram with labeled vertices. Unlabeled vertices will be assumed to be labeled with 1. Two $\mathbb{Z}_2$-coloured partitions are equivalent, denoted by $\simeq$, when the the corresponding partition diagrams are the same and when for each block the labels of one $\mathbb{Z}_2$-coloured partition are obtained by multiplying all labels of the corresesponding block in the other $\mathbb{Z}_2$-coloured partition by the same element of $\mathbb{Z}_2 = \{-1, 1\}$.



Example: $(\{\{1,3,1'\},\{2,2',3',4'\}\},(1,-1,-1,-1,1,1,-1)) \in P_{\{\mathbb{Z}_{\{2\}}\}}(3,4)$

Involution, horizontal concatenation, and loops are defined for the $\mathbb{Z}_2$-coloured partitions as for the non-coloured partitions. For the vertical concatenation we have to be more careful. Let $(p, z_1) \in P_{\mathbb{Z}_2}(k,l)$ and $(q, z_2) \in P_{\mathbb{Z}_2}(l,m)$. As for the non-coloured partitions we let $p \in P(k,l)$ and $q \in P(l,m)$ with the sets of partitions containing partitions of $\{1,\dots,k,1',\dots,l'\}$ and $\{1',\dots,l',1'',\dots,m''\}$ respectively. We label the elements $i' \in \{1',\dots,l'\}$ in the middle row of the stacking $q \star p$ by $z_1^{i'} z_2^{i'}$. We call (p, z_1) and (q, z_2) compatible if for each $i', j' \in \{1',\dots,l'\}$ in $q \star p$ that lie in the same connected component of q, we have that $z_1^{i'} z_2^{i'} = z_1^{j'} z_2^{j'}$. In this case the vertical concatenation $(q, z_2)(p, z_1) := (qp, z_2 z_1)$ is given by the vertical concatenation of the underlying non-coloured partitions, which is then labeled by setting $(z_2 z_1)^i := z_1^i$ for $i \in \{1,\dots,k\}$ and $(z_2 z_1)^{j''} := z_2^{j''} g$ for $j'' \in \{1'',\dots,m''\}$, where g is the labeling of a vertex in the middle row of $q \star p$ which is connected to j'' in q. If there are no such vertices we set $g = 1$.



Example: Vertical Concatenation of compatible diagrams with $l(p,q) = 1$.

Proposition 2.1.10. *Let $(p, z_1), (p, z_3) \in P_{\mathbb{Z}_2}(k,l)$ and $(q, z_2), (q, z_4) \in P_{\mathbb{Z}_2}(l,m)$, so that $((q, z_2), (p, z_1))$ is a compatible pair. If $(q, z_2) \simeq (q, z_4)$, then also $((q, z_4), (p, z_1))$ is a compatible pair. The analogue statement 'If $(p, z_1) \simeq (p, z_3)$, then also $((q, z_2), (p, z_3))$ is a compatible pair' is generally false.*

Proof. Assume that $(q, z_2) \simeq (q, z_4)$. We show that the compatibility of $((q, z_2)(p, z_1))$ implies that $((q, z_4)(p, z_1))$ is a compatible pair. Let $B \in q$ be a connected component. Because (q, z_2) and (p, z_1) are compatible, we know that for each $i', j' \in \{1',\dots,l'\}$ in $q \star p$ that lie in B, the equality $z_1^{i'} z_2^{i'} = z_1^{j'} z_2^{j'}$ holds. Because $(q, z_2) \simeq (q, z_4)$, there exists a $g \in \{-1, 1\}$, so that $g z_2^{i'} = z_4^{i'}$ for all $i' \in B$. But this implies that for each $i', j' \in \{1',\dots,l'\}$

in $q \star p$ that lie in B, the equality

$$z_1^{i'} z_4^{i'} = z_1^{i'} g z_2^{i'} = z_1^{j'} g z_2^{j'} = z_1^{j'} z_4^{j'}$$

holds, which means that $((q, z_4)(p, z_1))$ is compatible.
Now assume that $(p, z_1) \simeq (p, z_3)$. A possible counter example would be $(p, z_1) = (\{\{1, 1'\}, \{2, 2'\}\}, (1, -1, 1, -1)) \in P_{\mathbb{Z}_2}(2, 2)$, $(p, z_3) = (\{\{1, 1'\}, \{2, 2'\}\}, (1, 1, 1, 1))$ and $(q, z_2) = (\{\{1, 2, 1'\}\}, 1, -1, 1) \in P_{\mathbb{Z}_2}(2, 1)$. The pair $((q, z_2)(p, z_1))$ is compatible but $((q, z_2)(p, z_3))$ isn't. □

Proposition 2.1.11. *Let $(p, z_1), (p, z_3) \in P_{\mathbb{Z}_2}(k, l)$ and $(q, z_2), (q, z_4) \in P_{\mathbb{Z}_2}(l, m)$, so that $((q, z_2), (p, z_1))$ and $((q, z_4), (p, z_3))$ are compatible pairs. Let $(p, z_1) \simeq (p, z_3)$ and $(q, z_2) \simeq (q, z_4)$, then $(q, z_2)(p, z_1) \simeq (q, z_4)(p, z_3)$. So equivalences of $\mathbb{Z}_2$-coloured partitions are respected by vertical concatenation.*

Proof. We show that $(q, z_2)(p, z_1) \simeq (q, z_4)(p, z_1)$ and $(q, z_4)(p, z_1) \simeq (q, z_4)(p, z_3)$, this then proves the claim because $\simeq$ is an equivalence relation. Note that these vertical concatenations exist by last proposition. We only prove the first equivalence, the proof for the second one is similar in nature.
Let B be a component of qp, which doesn't only contain vertices in the upper row of the stacking $q \star p$, because otherwise checking the conditions for the equivalence on this component, would follow directly from the equivalence $(q, z_2) \simeq (q, z_4)$.

Let us write $p \in P(k, l)$ and $q \in P(l, m)$ as partitions of the sets $\{1, \ldots, k, 1', \ldots, l'\}$ and $\{1', \ldots, l', 1'', \ldots, m''\}$ respectively. Then qp can be considered as a partition of $\{1, \ldots, k, 1'', \ldots, m''\}$.

We have to prove that there exists a $g \in \{-1, 1\}$ such that $g(z_2 z_1)^t = (z_4 z_1)^t$ for all vertices $t \in B$. If $t \in \{1, \ldots, k\} \cap B$ we see that $z_1^t = g z_1^t$, but because in both cases the lower rows are just labeled by z_1, we get $g = 1$. Let $t \in \{1'', \ldots, m''\}$ and $h \in \{-1, 1\}$ such that $h z_2^t = z_4^t$. By assumption there is a vertex $s \in \{1', \ldots l'\}$ in the middle row of $q \star p$ which is connected to t. It is respectively labeled by $z_1^s z_2^s$ and $z_1^s z_4^s$ in the stackings $(q \star p, z_1 z_2)$ and $(q \star p, z_1 z_4)$. Because $(q, z_2) \simeq (q, z_4)$, and s is connected to t we have that $h z_2^s = z_4^s$. The vertex t is then labeled in $(qp, z_1 z_2)$ by $z_1^s z_2^s z_2^t$ and in $(qp, z_1 z_4)$ by

$$\begin{aligned} z_1^s z_4^s z_4^t &= z_1^s h z_2^s h z_2^t \\ &= h^2 z_1^s z_2^s z_2^t \\ &= 1 z_1^s z_2^s z_2^t \\ &= g z_1^s z_2^s z_2^t \end{aligned}$$

which shows that we have found our g for the component B, therefore $(qp, z_2 z_1) \simeq (qp, z_4 z_1)$. □

Definition 2.1.12. Let $(p, z_1) \in P_{\mathbb{Z}_2}(k, l)$ and $(q, z_2) \in P_{\mathbb{Z}_2}(l, m)$. We call a pair of equivalence classes $([(q, z_2)], [(p, z_1)])$ compatible if there exist representatives which are compatible. We define

$$[(q, z_2)][(p, z_1)] := [(q, z_2)(p, z_1)].$$

Remark 2.1.13. From now on we will consider equivalent $\mathbb{Z}_2$-coloured partitions to be equal.

Remark 2.1.14. We can write every $\mathbb{Z}_2$-coloured partition diagram (p, z) in the following form

$$(p, z) = \overset{z_{1'}}{\vert} \cdots \overset{z_{l-1}}{\vert}\,\overset{z_{l'}}{\vert} \circ p \circ \underset{z_1}{\vert} \cdots \underset{z_{k-1}}{\vert}\,\underset{z_k}{\vert} .$$

This will be useful in reducing problems involving $\mathbb{Z}_2$-coloured partitions to problems containing only non-coloured partitions.

Definition 2.1.15. Let $(p, z) \in P_{\mathbb{Z}_2}(k, l)$. We extend the permutation partitions from Definition 2.1.2 to $\mathbb{Z}_2$-coloured partitions by $\phi : S_n \hookrightarrow P(n, n) \hookrightarrow P_{\mathbb{Z}_2}(n, n)$. If p' is some non-crossing form of p such that $p = \phi(\sigma) \circ p' \circ \phi(\rho)$, then we call

$$(p, z) = \overset{z_{1'}}{\vert} \cdots \overset{z_{l-1}}{\vert}\,\overset{z_{l'}}{\vert} \circ \phi(\sigma) \circ p' \circ \phi(\rho) \circ \underset{z_1}{\vert} \cdots \underset{z_{k-1}}{\vert}\,\underset{z_k}{\vert}$$

a normal form of (p, z).

2.2 Defining the interpolation categories

In this section we define the different interpolation categories and prove their categorical properties using the notions that were introduced in Chapter 1. We state adjustments of Proposition 1.1.27 for the case that the involved categories are tensor categories and the involved functors are tensor functors.

2.2.1 Defining the interpolation categories in the permutation representations for the symmetric groupss

The category $\underline{\mathrm{Rep}}(S_t)$ was originally defined in [Del07], but we use the notation from [CW12, Definition 2.1.1]. We base the proof of Proposition 2.2.2, except for the part about sphericality, mostly on the proof given in [CW12, Section 2.2]. We have to adjust it to our axiomatic setting so that we can apply Proposition 1.1.28 afterwards.

Definition 2.2.1. Define the category $\underline{\mathrm{Rep}_0}(S_t)$ to be the category with objects $\{[n] | n \in \mathbb{N}\}$ and morphism spaces $\mathbb{C}$-linear combinations of partition diagrams, so

$$\mathrm{Hom}_{\underline{\mathrm{Rep}_0}(S_t)}([k], [l]) = \mathbb{C}\, P(k, l)$$

for all $k, l \in \mathbb{N}$. The identity of an object $[n]$ is given by

$$id_{[n]} := \{\{1, 1'\}, \dots, \{n, n'\}\}$$

and the composition of morphisms is defined on the generating morphisms of the morphism spaces by

$$\circ : \mathbb{C}\, P(l, m) \times \mathbb{C}\, P(k, l) \to \mathbb{C}\, P(k, m)$$

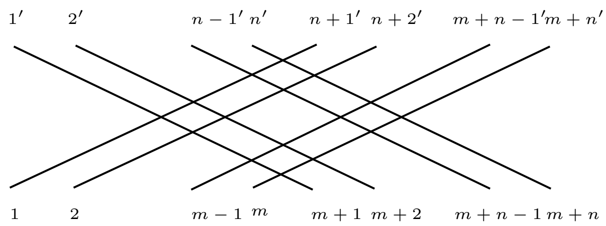
$$(q,p) \mapsto q \circ p := t^{l(q,p)} qp$$

and then extended $\mathbb{C}$-bilinearly. Note that this is well-defined category, the composition is associative, because the composition of partitions is associative and the number of occurring loops doesn't depend on the order of composition. The category has a tensor product, which is given on objects by $[n] \otimes [m] := [n+m]$ and on morphisms by a $\mathbb{C}$-bilinear extension of the horizontal concatenation of partitions.

Proposition 2.2.2. *$\underline{Rep}_0(S_t)$ is a strict $\mathbb{C}$-linear spherical rigid symmetric monoidal category.*

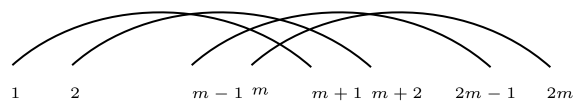
Proof. The category $\underline{\mathrm{Rep}}_0(S_t)$ is $\mathbb{C}$-linear and strict by definition. When we talk about morphisms we will in general only talk about partition diagrams, instead of their sums, knowing that all arguments can be $\mathbb{C}$- linearly extended. Define for all $[m],[n],[k],[l] \in \underline{\mathrm{Rep}}_0(S_t)$

- $[m] \otimes [n] := [m+n]$. For $p \in \mathrm{Hom}([m],[n])$ and $q \in \mathrm{Hom}([k],[l])$ the tensor product $p \otimes q$ is just the horizontal concatenation.
- $\mathbf{1} := [0]$.
- $\alpha_{[m],[n],[k]} := id_{[m+n+k]}$.
- $\lambda_{[m]} := id_{[m]}$
- $\rho_{[m]} := id_{[m]}$.
- $s_{[m],[n]}$ by



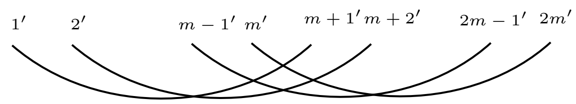
$s_{[m],[n]}$

- $[m]^* := [m]$ and $(p : [m] \to [n])^* := (p^* : [n] \to [m])$ is given by the involution.
- $\tau_{[m]} := id_{[m]}$.
- The evaluation $ev_{[m]} : [2m] \to [0]$ is given by



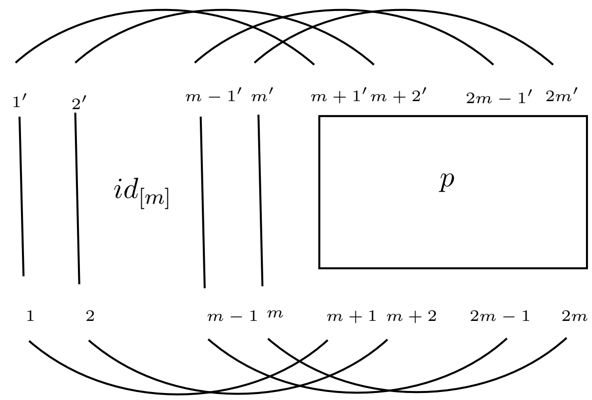
$ev_{[m]}$

- The coevaluation $coev_{[m]} : [0] \to [2m]$ is given by



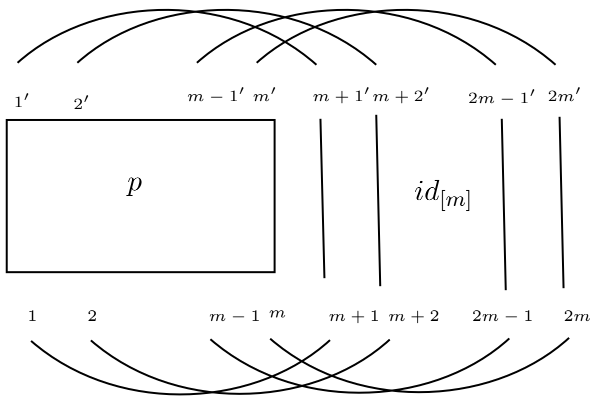
$coev_{[m]}$

- $\nu := id_{[0]} = 1_{\mathbb{C}}$.
- $\gamma_{[m],[n]} := id_{[m+n]}$.
- The right trace of $p \in \mathrm{Hom}([m],[m])$is given by



$Trace_r(p)$

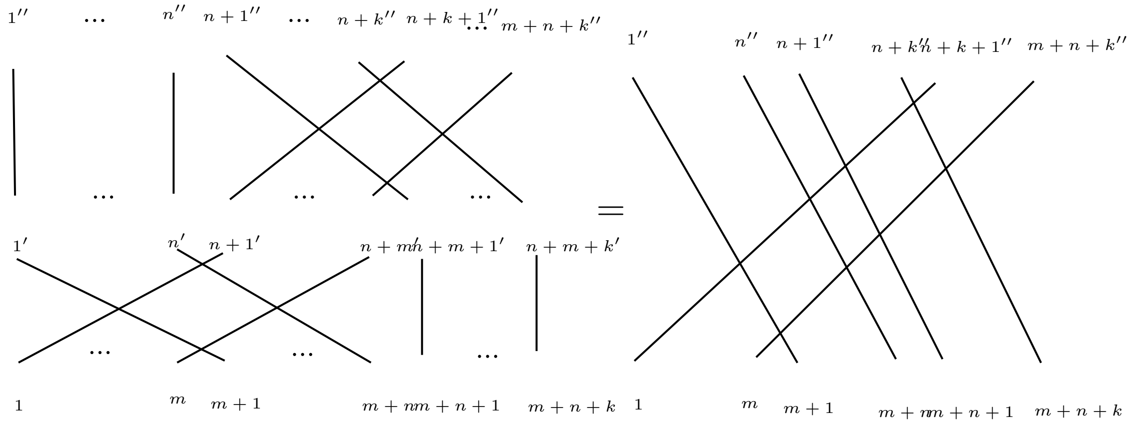
- The left trace of $p \in \mathrm{Hom}([m],[m])$is given by



$Trace_l(p)$

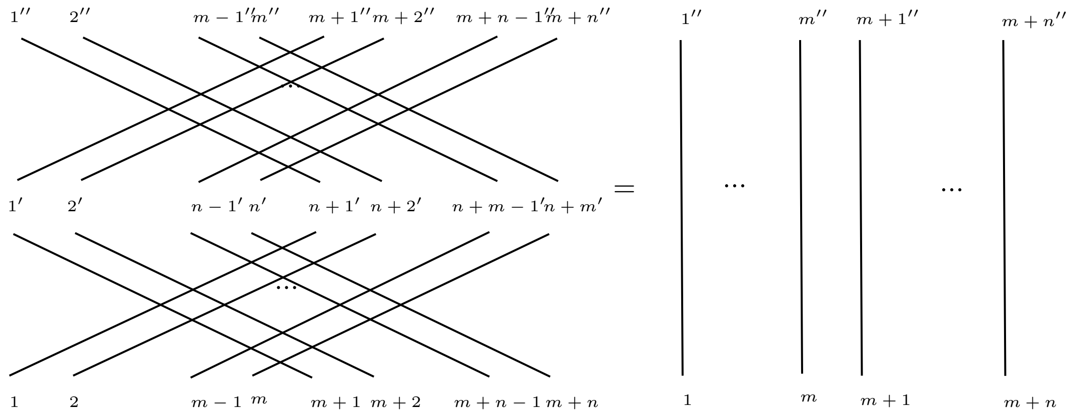
Monoidal category: The triangle and pentagon identities are trivially satisfied.

Symmetric Monoidal category: Unit coherence is trivially satisfied. Associativty coherence holds because



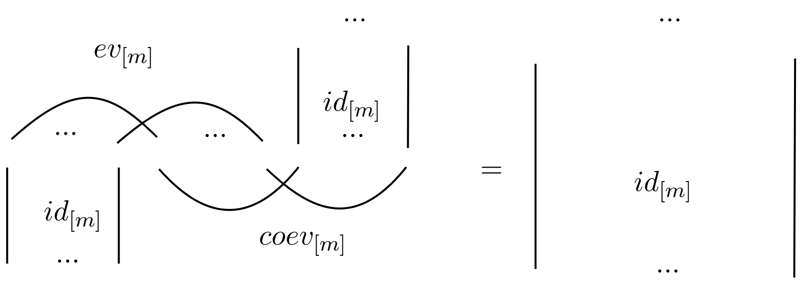
$$s_{[m],[n]\otimes[k]} = \left(id_{[n]} \otimes s_{[m],[k]}\right) \circ \left(s_{[m],[n]} \otimes id_{[k]}\right)$$

The inverse law holds because

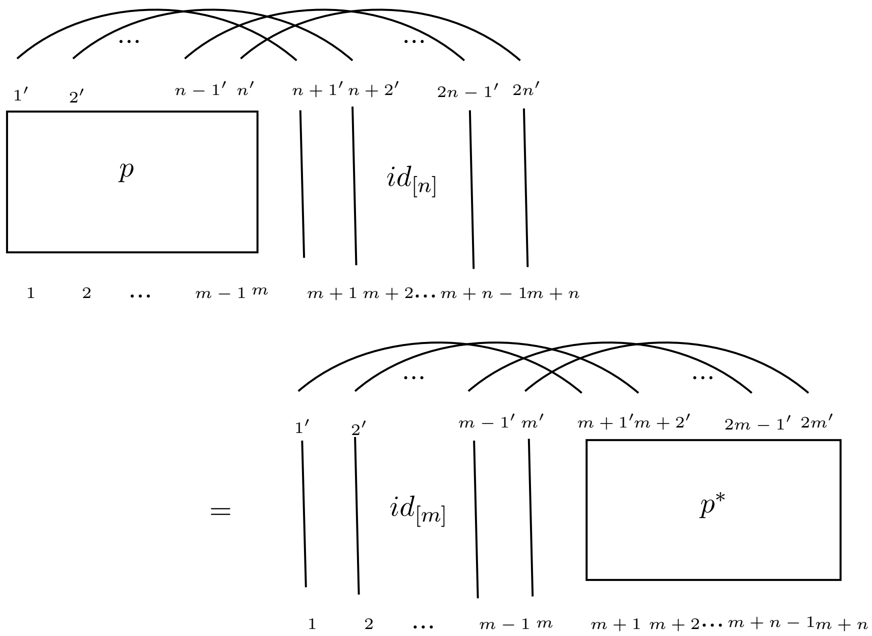


$$s_{[n],[m]} \circ s_{[m],[n]} = id_{[m+n]}$$

Rigid Symmetric Monoidal category: The duality identities follow trivially from the definitions. The first triangle identity holds because



Similarly also the second triangle identity holds. The first morphism identity holds because for $p : [m] \to [n]$ the following diagrams are equal



as can be seen by the fact that in the first diagram the vertex $1 \leqslant i \leqslant m$ is connected to the vertex $m+1 \leqslant j \leqslant m+n$ if and only if i and $j-m'$ lie in the same block of p in that diagram. In the second diagram the vertex $1 \leqslant i \leqslant m$ is connected to the vertex $m+1 \leqslant j \leqslant m+n$ if and only if $m+i'$ and j lie in the same block of p^* in the diagram above. The statement follows from the fact that these conditions are equivalent. One can easily extend this argument to work for all $1 \leqslant i, j \leqslant m+n$. The second morphism identity is proven similarly. Clearly $\text{End}([0]) = \mathbb{C}$.

Spherical Rigid Symmetric Monoidal category: We use the notation from the definition. Let $p : [m] \to [m]$. We first consider the left trace. Let $B_1, \ldots, B_r$ be the blocks of p. For $s, t \in \{1, \ldots, r\}$ we say that blocks B_s and B_t are left related if there exist $i \in B_s$ such that $i' \in B_t$ or $j \in B_t$ such that $j' \in B_s$. We call blocks $B_s \sim_{left} B_t$ left equivalent if there exist $i_1, \ldots, i_u \in \{1, \ldots, r\}$ such that B_s is left related to B_{i_1}, B_{i_u} is left related to B_t and B_{i_j} is left related to $B_{i_{j+1}}$ for all $j \in \{1, \ldots, u-1\}$. This defines an equivalence relation on the blocks of p.

Vertices $1 \leqslant i, j \leqslant m$ lie in the same connected component of $Trace_l(p)$ if there exist left equivalent blocks $B_s \sim B_t$ such that $i \in B_s$ and $j \in B_t$. The number of disconnected loops for the left trace is then the same as the number of left equivalence classes of blocks of p.

Now we consider the right trace. Let $B_1^*, \ldots, B_r^*$ be the blocks of p^*. For $s, t \in \{1, \ldots, r\}$ we say that blocks B_s^* and B_t^* are right related if there exist $m+i \in B_s^*$ such that $m+i' \in B_t^*$ or $m+j \in B_t^*$ such that $m+j' \in B_s^*$. We call blocks $B_s^* \sim_{right} B_t^*$ right equivalent if there exist $i_1, \ldots, i_u \in \{1, \ldots, r\}$ such that B_s^* is right related to $B_{i_1}^*$, $B_{i_u}^*$ is right related to B_t^* and $B_{i_j}^*$ is right related to $B_{i_{j+1}}^*$ for all $j \in \{1, \ldots, u-1\}$. This defines an equivalence relation on the blocks of $p*$. Vertices $m+1 \leqslant i, j \leqslant 2m$ lie in the same connected component of $Trace_r(p)$ if there exist right equivalent blocks $B_s^* \sim B_t^*$ such that $i \in B_s^*$ and $j \in B_t^*$. The number of disconnected loops for the right trace is therfore the same as the number of right equivalence classes of blocks of $p*$. We remark that the right equivalence relation on the blocks of $p*$ induces an equivalence relation on the blocks of p, which coincides with the the left equivalence relation on the blocks p. This implies that $Trace_l(p) = Trace_r(p)$ and that the category is spherical.

□

Definition 2.2.3. We define the Deligne category for $t \in \mathbb{C}$ by $\underline{\mathrm{Rep}}(S_t) := (\underline{\mathrm{Rep}}_0(S_t))^{Kar}$ and denote the corresponding $\mathbb{C}$-linear full embedding by $\iota_F : \underline{\mathrm{Rep}}_0(S_t) \to \underline{\mathrm{Rep}}(S_t)$. By Proposition 2.2.2 and Proposition 1.1.28 we see that $\underline{\mathrm{Rep}}(S_t)$ is a $\mathbb{C}$-linear spherical rigid symmetric monoidal category.

Remark 2.2.4. It is not too difficult to see that that the $\mathbb{C}$-linear functor ι_F is also a tensor functor. As a consequence the following adjustement of the universal property of the Karoubian envelope can be shown. Let $\mathcal{D}$ be a $\mathbb{C}$-linear Karoubi tensor category and $\gamma' : \underline{\mathrm{Rep}}_0(S_t) \to \mathcal{D}$ a strict $\mathbb{C}$-linear tensor functor. Then the universal property of the Karoubian envelope gives us a $\mathbb{C}$-linear tensor functor $\gamma : \underline{\mathrm{Rep}}(S_t) \to \mathcal{D}$ such that $\gamma' = \gamma \circ \iota_F$.

2.2.2 Defining the interpolation categories in the reflection representations for the hyperoctahedral groups

The definition of $\underline{\mathrm{Rep}}(H_t)$ we use, can be found in [FM21, Definition 2.5]. They also discussed the categorical properties of the interpolation categories. We change their definition of the evaluation and coevaluation, so that they become compatible with the definitions we gave for the evaluation and coevaluation in $\underline{\mathrm{Rep}}(S_t)$ and with our notions from Chapter 1.

Definition 2.2.5. Define the category $\underline{\mathrm{Rep}}_0(H_t)$ to be the category with objects $\{[n] | n \in \mathbb{N}\}$ and morphism spaces $\mathbb{C}$-linear combinations of even partition diagrams, so

$$\mathrm{Hom}_{\underline{\mathrm{Rep}}_0(H_t)}([k],[l]) = \mathbb{C}\, P_{even}(k,l).$$

The identity of an object $[n]$ is given by

$$id_{[n]} := \{\{1,1'\},\ldots,\{n,n'\}\}$$

and the composition by extending

$$\begin{aligned}\circ : \mathbb{C}\, P_{even}(l,m) \times \mathbb{C}\, P_{even}(k,l) &\to \mathbb{C}\, P_{even}(k,m)\\ (q,p) &\mapsto q \circ p := t^{l(q,p)} qpn\end{aligned}$$

$\mathbb{C}$-bilinearly. Note that this is a well-defined category because the composition is clearly associative since the composition of partitions is associative and the number of occurring loops doesn't depend on the order of composition. The category has a tensor product, which is given on objects by $[n] \otimes [m] := [n+m]$ and on morphisms by a $\mathbb{C}$-bilinear extension of the horizontal concatenation of partitions.

Proposition 2.2.6. *$\underline{Rep}_0(H_t)$ is strict $\mathbb{C}$-linear spherical rigid symmetric monoidal category.*

Proof. One can make the same choices as we did in Proposition 2.2.2 for $\underline{\mathrm{Rep}}_0(S_t)$, since all the occuring morphisms consist only of even partition diagrams. □

Definition 2.2.7. We define the Deligne category for the hyperoctahedral groups in the reflection representation for $t \in \mathbb{C}$ by $\underline{\mathrm{Rep}}(H_t) := (\underline{\mathrm{Rep}}_0(H_t))^{Kar}$ and denote the corresponding $\mathbb{C}$-linear full embedding by $\iota_G : \underline{\mathrm{Rep}}_0(H_t) \to \underline{\mathrm{Rep}}(H_t)$. As for the case of the symmetric groups, Proposition 2.2.6 and Proposition 1.1.28 show us that $\underline{\mathrm{Rep}}(H_t)$ is a $\mathbb{C}$-linear spherical rigid symmetric monoidal category.

Remark 2.2.8. Similarly as before, it can be seen that the $\mathbb{C}$-linear functor ι_G is also a tensor functor. Now the following adjustement of the universal property of the Karoubian envelope can be shown. Let $\mathcal{D}$ be a $\mathbb{C}$-linear Karoubi tensor category and $\gamma' : \underline{\mathrm{Rep}}_0(H_t) \to \mathcal{D}$ a strict $\mathbb{C}$-linear tensor functor. Then the universal property of the Karoubian envelope gives us a $\mathbb{C}$-linear tensor functor $\gamma : \underline{\mathrm{Rep}}(H_t) \to \mathcal{D}$ such that $\gamma' = \gamma \circ \iota_G$.

2.2.3 Defining the interpolation categories in the permutation representations for the hyperoctahedral groups

We use the definition of $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ that is given in [LS21, Definition 3.6]. Except for the monoidal property, the categorical properties that we state in Proposition 2.2.10, are not mentioned there.

Definition 2.2.9. Define the category $\mathrm{Par}(\mathbb{Z}_2, t)$ to be the category with objects $\{[\tilde{n}] | n \in \mathbb{N}\}$ and morphism spaces $\mathbb{C}$-linear combinations of equivalence classes of $\mathbb{Z}_2$-coloured partition diagrams, so

$$\mathrm{Hom}_{\mathrm{Par}(\mathbb{Z}_2,t)}([\tilde{k}], [\tilde{l}]) = \mathbb{C}\, P_{\mathbb{Z}_2}(k, l),$$

where we identify equivalence classes of Z_2-coloured partitions, according to Remark 2.1.13. The identity of an object $[\tilde{n}]$ is given by

$$id_{[\tilde{n}]} := (\{\{1, 1'\}, \dots, \{n, n'\}\}, (1, \dots, 1))$$

and the composition is given by extending the rule

$$\begin{aligned} \circ : \mathbb{C}\, P_{\mathbb{Z}_2}(l, m) \times \mathbb{C}\, P_{\mathbb{Z}_2}(k, l) &\to \mathbb{C}\, P_{\mathbb{Z}_2}(k, m) \\ (q, p) &\mapsto q \circ p := t^{l(q,p)} qp. \end{aligned}$$

$\mathbb{C}$-bilinearly. Note that this is well-defined category because the composition is clearly associative since the composition of partitions is associative and the number of occurring loops doesn't depend on the order of composition. The category has a tensor product, which is given on objects by $[\tilde{n}] \otimes [\tilde{m}] := [\widetilde{n+m}]$ and on morphisms by a $\mathbb{C}$-bilinear extension of the horizontal concatenation of $\mathbb{Z}_2$-coloured partitions.

Proposition 2.2.10. *$Par(\mathbb{Z}_2, t)$ is strict $\mathbb{C}$-linear spherical rigid symmetric monoidal category.*

Proof. One can again make the same choices as we did in Proposition 2.2.2 for $\underline{\mathrm{Rep}}_0(S_t)$, since every non-coloured partition is also a $\mathbb{Z}_2$- coloured partition if we label all the vertices by 1. These choices are well-defined. The only thing one has to be aware of is that when one composes with the equivalence class of the identity, not all choices of representatives will give a compatible pair. Thus the sign of the blocks of the identity morphism have to be adjusted to the particular case. This also holds for the evaluation and the coevaluation. □

Remark 2.2.11. Note that for $t \in 2\,\mathbb{N}$ the compatibility of certain equivalence classes works remarkable well in calculating the traces in the corresponding representation categories. A correspondence we will dicuss more detailed later on. For example $Trace($ $^{-1}) = 0$, because the signs of the blocks in the evaluation $ev_{[1]}$, the coevaluation $coev_{[m]}$, the identity morphism $id_{[m]}$ and $^{-1}$ can not be chosen in such a way that all occuring compositions are compatible. We will later see that for $t = 2$, this corresponds to the fact that the endomorphism of the permutation representation of H_1 given by

$$\begin{aligned} e_1^1 &\mapsto e_{-1}^1 \\ e_{-1}^1 &\mapsto e_1^1, \end{aligned}$$

has trace zero, as can be seen by looking at its matrix form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 2.2.12. We define the Deligne category for the hyperoctahedral groups in the permutation representation for $t \in \mathbb{C}$ by $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ and denote the corresponding $\mathbb{C}$-linear full embedding by $\iota_H : \mathrm{Par}(\mathbb{Z}_2, t) \to \mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$. Proposition 2.2.10 and Proposition 1.1.28 imply that $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ is a $\mathbb{C}$-linear spherical rigid symmetric monoidal category.

Remark 2.2.13. The $\mathbb{C}$-linear functor ι_H is also a tensor functor. As a consequence the following adjustement of the universal property of the Karoubian envelope can be shown. Let $\mathcal{D}$ be a $\mathbb{C}$-linear Karoubi tensor category and $\gamma' : \mathrm{Par}(\mathbb{Z}_2, t) \to \mathcal{D}$ a strict $\mathbb{C}$-linear tensor functor. Then the universal property of the Karoubian envelope gives us a $\mathbb{C}$-linear tensor functor $\gamma : \mathrm{Par}(\mathbb{Z}_2, t)^{Kar} \to \mathcal{D}$ such that $\gamma' = \gamma \circ \iota_H$.

Remark 2.2.14. The category $\underline{\mathrm{Rep}}_0(S_t)$ can be considered as a subcategory of $\mathrm{Par}(\mathbb{Z}_2, t)$ because every non-coloured partition can be considered as a partition where all vertices are labeled by the unit elements of $\mathbb{Z}_2$. As a consequence $\underline{\mathrm{Rep}}(S_t)$ can be considered as a subcategory of $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$. This will be further discussed in Chapter 3.

2.3 Motivating the definition of the interpolation categories

In this section we discuss the morphism spaces between the tensor products of the representations on which the interpolation categories are modelled. This gives a good motivation for the definition of the interpolation categories and will make it possible to define interesting functors between the interpolation categories and their corresponding representation categories.

2.3.1 Morphism spaces between tensor products of the permutation representations of symmetric groups

In this section we show that the morphism spaces between the tensor products of the reflection representation $u' = \mathbb{C}^n$ of S_n, can be described using the set of partitions P. The structure and ideas of the proof of Proposition 2.3.5 are based on [CO11, Theorem 2.6], but the notation is changed to fit our conventions. The statement is also proven in [BS09, Theorem 1.10], but from this source we mainly take the name for the $\mathbb{C}$-linear map T.

Definition 2.3.1. Let $e_1, \dots, e_n$ be the canonical basis for $u' = \mathbb{C}^n$. For all $k, l \in \mathbb{N}$ we define a $\mathbb{C}$-linear map by its image on the basis elements:

$$\begin{aligned} T : \mathbb{C}\,P(k,l) &\to \operatorname{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}) \\ p &\mapsto T_p \end{aligned}$$

where

$$\begin{aligned} T_p : (\mathbb{C}^n)^{\otimes k} &\to (\mathbb{C}^n)^{\otimes l} \\ e_{i_1} \otimes \dots \otimes e_{i_k} &\mapsto \sum_{1 \leqslant j_1 \dots j_l \leqslant n} \delta_p(i,j)(e_{j_1} \otimes \dots \otimes e_{j_l}). \end{aligned}$$

for all $1 \leqslant i_1, \dots, i_k \leqslant n$. We use the abbreviation $i = (i_1, \dots, i_k)$ and $j = (j_1, \dots, j_l)$. We can label the partition diagram of p by i and j in an obvious way: i labels th lower row from left to right and j the upper row from left to right. By definition $\delta_p(i,j)$ equals 1 if and only if all vertices in the same block of the partition are labeled by the same number. Otherwise it equals 0. We call this a good labeling. We call it a perfect labeling when vertices are labeled by the same number if and only if they are in the same block. Note that a partition $p \in P(k,l)$ with more than n blocks can not have a perfect labeling.

Proposition 2.3.2. *The linear map T is well-defined.*

Proof. We want to see that each morphism T_p commutes with the action of S_n. Define the action from $\sigma \in S_n$ on a k-tuple $i = (i_1, \dots, i_k)$ by $\sigma \cdot i := (\sigma(i_1), \dots, \sigma(i_k))$. Then $\delta_p(i,j) = 1$ if and only if $\delta_p(\sigma(i), \sigma(j)) = 1$ for all $\sigma \in S_n$. The equality

$$\begin{aligned} T_p \circ \sigma(e_{i_1} \otimes \dots \otimes e_{i_k}) &= \sum_{1 \leqslant j_1 \dots j_l \leqslant n} \delta_p(\sigma \cdot i, j)(e_{j_1} \otimes \dots \otimes e_{j_l}) \\ &= \sum_{1 \leqslant j_1 \dots j_l \leqslant n} \delta_p(i, \sigma^{-1} \cdot j)(e_{j_1} \otimes \dots \otimes e_{j_l}) \\ &= \sum_{1 \leqslant j_1 \dots j_l \leqslant n} \delta_p(i,j)(e_{\sigma(j_1)} \otimes \dots \otimes e_{\sigma(j_l)}) \\ &= \sigma \circ T_p(e_{i_1} \otimes \dots \otimes e_{i_k}) \end{aligned}$$

shows that T is well-defined. □

By definition every morphism in $\operatorname{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$ is stable under the S_n action. This motivates the definition of a certain basis for the morphism space.

Definition 2.3.3. Define the action from S_n on $(i,j) \in \{1, \dots, n\}^{k+l}$ by $\sigma(i,j) := (\sigma \cdot i, \sigma, \cdot j)$ for all $\sigma \in S_n$, write $[(i,j)]$ for an equivalence class under this action and let A be the set of such equivalence classes. We define a morphism $f_{[(i,j)]} : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ which sends

$$e_{\sigma(i_1)} \otimes \ldots \otimes e_{\sigma(i_k)} \mapsto e_{\sigma(j_1)} \otimes \ldots \otimes e_{\sigma(j_l)}$$

for all $\sigma \in S_n$ and all other basis elements of $(\mathbb{C}^n)^{\otimes k}$ to 0. This definition is clearly independent of the chosen representative and $\{f_{[(i,j)]} \mid [(i,j)] \in A\}$ is a basis of $\mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$. This basis consists of sums of S_n-orbits of basis elements in $\mathrm{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$.

Similarly we want to define a basis for $\mathbb{C}\,P(k,l)$.

Definition 2.3.4. We recursively define a new set of basiselements

$$x_p := p - \sum_{q \text{ coarser than } p} x_q \text{ for } p \in P(k,l)$$

in $\mathbb{C}\,P(k,l)$. They generate $\mathbb{C}\,P(k,l)$ because $p = x_p + \sum_{q \text{ coarser than } p} x_q$ for all $p \in P(k,l)$ and they form a basis for $\mathbb{C}\,P(k,l)$ because for different $p, q \in P(k,l)$, the sums x_p and x_q are different linear combinations of elements in $P(k,l)$.

Proposition 2.3.5. *The linear map $T : \mathbb{C}\,P(k,l) \to Hom_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$ is surjective and has as kernel $\{x_p \mid p \in P(k,l)$ has more then n parts$\}$. As a consequence T is an isomorphism in the cases that $k + l \leqslant n$.*

Proof. We associate to a basis element $f_{[(i,j)]}$ of the morphism space $\mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$, see Definition 2.3.3, a partition $p_{[(i,j)]} \in P(k,l)$. We label the vertices by (i,j) and let two vertices be in the same block if and only if the they are labeled by the same number. Then the partition $p_{[(i,j)]}$ has a perfect labeling. Note that $T_{p_{[i,j]}}$ does not equal $f_{[(i,j)]}$, but it is easy to check that $T_{x_{p_{[(i,j)]}}} = f_{[(i,j)]}$. This implies that T is surjective.

Now let $p \in P(k,l)$ be a partition with less than or equal to n parts. Then there exists a unique equivalence class of perfect labels $[(i,j)]$ of p, so $p = p_{[(i,j)]}$. We remarked in the first part of the proof that $T_{x_p} = f_{[(i,j)]} \neq 0$. For a different partition $q \in P(k,l)$ with less than or equal to n parts, T_{x_q} will be a different basiselement of $\mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$. So we have that T is injective on the subspace of $\mathbb{C}\,P(k,l)$ spanned by the partitions with less than or equal to n parts with image all of $\mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$.

Now assume that $p \in P(k,l)$ is a partition with more than n parts. Then we can argument inductively as follows. First assume that p has $n+1$ parts. Then

$$T_{x_p} = T_p - \sum_{q \text{ coarser than } p} T_{x_q}$$

where each T_{x_q} in the equation is a linear map $f_{[(i,j]}$ corresponding to some perfect labeling of q. But because p has no perfect labeling, T_p will exactly be the sum of all T_{x_q} for q coarser as p. This shows then that T_{x_p} is zero. One can now apply this argument to inductively prove that $T_{x_p} = 0$ for a partition p with m parts, for all $m > n$.

□

In [BS09] this theorem is proven similarly. They also prove that $P_{S_n(k,l)} = P(k,l)$ for

$$P_{S_n}(k,l) := \{p \in P(k,l) \mid T_p \in \mathrm{Hom}_{S_n}(\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}\}$$

and that its image under T generates $\mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$. This means then again that the morphisms spaces between tensor powers of the reflection representation of S_n can be described completely by linear combinations of the images of partitions under T:

$$< \{T_p \mid p \in P(k,l)\} >= \mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}).$$

Remark 2.3.6. Note that these description of the morphism spaces doesn't depend on the number $n \in \mathbb{N}$. It is this observation that motivated the definition of the interpolation categories. What does depend on n is the composition of the image under T of partitions $p \in P(k,l)$ and $q \in P(l,m)$:

$$T_{qp} = n^{-l(q,p)} T_q \circ T_p \text{ or } n^{l(q,p)} T_{qp} = T_q \circ T_p,$$

which is proven in [BS09, Proposition 1.9].

Remark 2.3.7. To conclude this section we shortly discuss the dimension of the endomorphism algebras in $\underline{\mathrm{Rep}}(S_n)$. In [Blo03] they state the dimension of the $\mathbb{C}$ algebra $P_k(t)$, which equals our algebra $End_{\underline{\mathrm{Rep}}(S_t)}([k])$ for $t \in \mathbb{C}$.

The number of partition diagrams in $P(k,k)$ with l connected components is the Stirling number of the second kind $S(2k,l)$. The dimension of $End_{\underline{\mathrm{Rep}}(S_n)}([k])$ equals the cardinality of $P(k,k)$, which equals the Bell number

$$B(2k) = \sum_{l=1}^{2k} S(2k,l).$$

Because T is an isomorphism for $2k \leqslant n$, we see that in this case $B(2k)$ is also the dimension of the centralizer algebra $End_{\mathrm{Rep}(S_n)}((\mathbb{C}^n)^{\otimes k})$.

2.3.2 Morphism spaces between tensor products of the reflection representations of hyperoctahedral groups

We use and if necessary modify the definitions and proofs of previous section to discuss the morphism spaces between the tensor products of the reflection representation of H_n. They can be described by the set of even partitions P_{even}.

Definition 2.3.8. Let $e_1, \dots, e_n$ be the canonical basis for $u = \mathbb{C}^n$. For all $k, l \in \mathbb{N}$, with $k + l$ even, we define a $\mathbb{C}$-linear map by restricting the map T we defined for the symmetric group case:

$$\begin{aligned} T : \mathbb{C}\, P_{even}(k,l) &\to \mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}) \\ p &\mapsto T_p \end{aligned}$$

where

$$\begin{aligned} T_p : (\mathbb{C}^n)^{\otimes k} &\to (\mathbb{C}^n)^{\otimes l} \\ e_{i_1} \otimes \ldots \otimes e_{i_k} &\mapsto \sum_{1 \leqslant j_1 \ldots j_l \leqslant n} \delta_p(i,j)(e_{j_1} \otimes \ldots \otimes e_{j_l}). \end{aligned}$$

for all $1 \leqslant i_1, \ldots, i_k \leqslant n$. The notation is the same as in the case for the symmetric groups.

Proposition 2.3.9. *The linear map T is well-defined.*

Proof. We want to show that each morphism T_p commutes with the action of H_n for some $p \in P_{even}(k,l)$. Let $a = (a_1, \ldots, a_n, \sigma) \in H_n$ and $1 \leqslant i_1, \ldots, i_k \leqslant n$. We consider the equality

$$\begin{aligned}
T_p \circ a(e_{i_1} \otimes \ldots \otimes e_{i_k}) &= T_p(a_{\sigma(i_1)} e_{\sigma(i_1)} \otimes \ldots \otimes a_{\sigma(i_k)} e_{\sigma(i_k)}) \\
&= a_{\sigma(i_1)} \cdots a_{\sigma(i_k)} T_p(e_{\sigma(i_1)} \otimes \ldots \otimes e_{\sigma(i_k)}) \\
&= a_{\sigma(i_1)} \cdots a_{\sigma(i_k)} \sum_{1 \leqslant j_1 \ldots j_l \leqslant n} \delta_p(\sigma \cdot i, j)(e_{j_1} \otimes \ldots \otimes e_{j_l}) \\
&= a_{\sigma(i_1)} \cdots a_{\sigma(i_k)} \sum_{1 \leqslant j_1 \ldots j_l \leqslant n} \delta_p(i, \sigma^{-1} \cdot j)(e_{j_1} \otimes \ldots \otimes e_{j_l}) \\
&= a_{\sigma(i_1)} \cdots a_{\sigma(i_k)} \sum_{1 \leqslant j_1 \ldots j_l \leqslant n} \delta_p(i, j)(e_{\sigma(j_1)} \otimes \ldots \otimes e_{\sigma(j_l)}) \\
&= \sum_{1 \leqslant j_1 \ldots j_l \leqslant n} \delta_p(i, j)(a_{\sigma(j_1)} e_{\sigma(j_1)} \otimes \ldots \otimes a_{\sigma(j_l)} e_{\sigma(j_l)}) \\
&= a \circ T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}).
\end{aligned}$$

The second last equality can only be true when $a_{\sigma(i_1)} \cdots a_{\sigma(i_k)} = a_{\sigma(j_1)} \cdots a_{\sigma(j_l)}$ for all $i = \{i_1, \ldots, i_k\}$ and $j = \{j_1, \ldots, j_l\}$ with $\delta_p(i,j) = 1$. For this to hold for all $a \in H_n$, it is necessary that every number in $\{i_1, \ldots, i_k, j_1, \ldots, j_l\}$ occurs an even number of times in the labeling of the partition p by (i,j). But this is the case because p is an even partition. This shows that T is well-defined. ☐

Let $k+l$ be even for $k, l \in \mathbb{N}$. Let $(i,j) \in \{1, \ldots, n\}^{k+l}$ such that every number in $\{i_1, \ldots, i_k, j_1, \ldots, j_l\}$ occurs an even number of times in $(i_1, \ldots, i_k, j_1, \ldots, j_l)$ and $[(i,j)]$ its equivalence class under the S_n action. Let B be the set of such equivalence classes. Then we can see by a similar argument as in the last proof that the morphism $f_{[(i,j)]} : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ of Definition 2.3.3 lies in $\mathrm{Hom}_{H_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$ because each of its terms does. The set $\{f_{[(i,j)]} \mid [(i,j)] \in B\}$ then forms a basis of the morphism space.

Definition 2.3.10. We define partitions

$$x_p := p - \sum_{q \text{ coarser than } p} x_q$$

for some $p \in \mathbb{C}\, P_{even}(k,l)$ recursively. Note that when q is coarser than an even partition p, then q is also an even partition, so the x_p form a basis for $\mathbb{C}\, P_{even}(k,l)$.

Proposition 2.3.11. *The linear map $T : \mathbb{C}\, P_{even}(k,l) \to Hom_{H_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$ is surjective and has as a kernel $\{x_p \mid p \in P_{even}(k,l)$ has more then n parts$\}$. As a consequence T is an isomorphism in case $k+l \leqslant n$.*

Proof. We already know that every basis element $f_{[(i,j)]}$ of the morphism space lies in the image of $x_{p_{[(i,j)]}}$ which was constructed in the proof of Proposition 2.3.5. $p_{[(i,j)]}$ is an even partition because $[(i,j)]$ lies in B. This shows that $T : \mathbb{C}\, P_{even}(k,l) \to \mathrm{Hom}_{H_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$ is surjective.

We also showed already that $T_{x_p} = 0$ for an even partition p with more than n parts. Because the T_{x_p} for partitions with n or less parts, are pairwise different, we know that T is injective on the subspace of $\mathbb{C}\,P_{even}(k,l)$ spanned by the partitions with less than or equal to n parts. So the kernel is indeed $\{x_p \mid p \in P_{even}(k,l) \text{ has more then } n \text{ parts}\}$. □

In [BS09] the surjectivity is proven using the results for the symmetric groups, which was proven in the framework of easy quantum groups. They define for all $k, l \in \mathbb{N}$ the set

$$P_{H_n}(k,l) := \{p \in P(k,l) \mid T_p \in \mathrm{Hom}_{H_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})\}.$$

Because S_n is a subgroup of H_n we have

$$\mathrm{Hom}_{H_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}) \subset \mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}) = P(k,l).$$

This implies that all linear maps commuting with the action of the hyperoctahedral groups can be described using partitions and the functor T. A direct calculation, similar as we did above, then shows that $P_{H_n}(k,l) = P_{even}$ or

$$< \{T_p \mid p \in P_{even}(k,l)\} >= \mathrm{Hom}_{H_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}).$$

Remark 2.3.12. Note that the description of the morphism spaces doesn't depend on the number $n \in \mathbb{N}$. It is this observation that motivated the definition of interpolation categories for the reflection representation of H_n. What does depend on n is the composition of the image under T of even partitions $p \in P_{even}(k,l)$ and $q \in P_{even}(l,m)$:

$$T_{qp} = n^{-l(q,p)} T_q \circ t_p \text{ or } n^{l(q,p)} T_{qp} = T_q \circ T_p$$

Remark 2.3.13. In [Ore07] a formula for the dimension of the centralizer algebra $\mathrm{End}_{H_n}((\mathbb{C}^n)^{\otimes k})$ is given. It equals

$$\sum_{i=1}^{k} \binom{2k-1}{2i-1} \dim(\mathrm{End}_{H_n}((\mathbb{C}^n)^{\otimes k-i})).$$

For $2k \leqslant n$, this dimension equals the dimension of $\mathrm{End}_{\underline{\mathrm{Rep}(H_n)}}([k])$. Because one can rotate the even partitions this will also equal the dimension of the morphism spaces $\mathrm{Hom}_{\underline{\mathrm{Rep}}(H_n)}([a],[b])$ for all $a, b \in \mathbb{N}$ with $a + b = 2k \leqslant n$.

2.3.3 Morphism spaces between tensor products of the permutation representations of hyperoctahedral groups

In this section we show that the morphism spaces between the tensor products of the permutation representation $V = \mathbb{C}^{2n}$ of H_n can be described using the set of $\mathbb{Z}_2$-coloured partitions $P_{\mathbb{Z}_2}$. We alter the notation that was given in [LS21, Proposition 5.2] and proceed analogously as in the previous sections to present the proof of Proposition 2.3.19 in a different way as in the literature, although the underlying idea stays the same. Proposition 2.3.19 is also proven in [LS21, Theorem 5.4] and we will shortly discuss this proof and write out the involved isomorphisms more explicitly.

Definition 2.3.14. Recall the set of basis elements $\{e^i_j | 1 \leqslant i \leqslant n, j \in \{-1,1\}\}$ for $\mathbb{C}^{2n}$. For all $k, l \in \mathbb{N}$ we define a $\mathbb{C}$-linear map by its image on the basis elements:

$$\begin{aligned} T : \mathbb{C}\, P_{\mathbb{Z}_2}(k,l) &\to \mathrm{Hom}_{H_n}((\mathbb{C}^{2n})^{\otimes k}, (\mathbb{C}^{2n})^{\otimes l}) \\ (p,z) &\mapsto T_{(p,z)} \end{aligned}$$

where

$$T_{(p,z)}(e^{i_1}_{b_1} \otimes \ldots \otimes e^{i_k}_{b_k}) = \sum_{\substack{1 \leqslant j_1 \ldots j_l \leqslant n \\ c_1 \ldots c_l \in \mathbb{Z}_2}} \delta_{(p,z)}((i,b),(j,c))(e^{j_1}_{c_1} \otimes \ldots \otimes e^{j_l}_{c_l})$$

for all $(i,b) \in \{1,\ldots,n\}^k \times (\mathbb{Z}_2)^k$. We use the notation $(i,b) = ((i_1,\ldots,i_k)(b_1,\ldots,b_k))$ and $(j,c) = ((j_1,\ldots,j_l),(c_1,\ldots,c_l))$. We can label the partition p by (i,j) and (b,c) in an obvious way: i and b label the lower row of p from left to right and j and c label the upper row from left to right. Then $\delta_{(p,z)}((i,b),(j,c))$ equals 1 if and only if for vertices in the same part of p the following two condition hold: the corresponding (i,j)-labels are the same and the corresponding (b,c)-labels multiplied with the corresponding labels of z are the same. Otherwise it equals 0. We call this a good labeling. We call it a perfect labeling when it is good and if the vertices have the same (i,j)-labeling if and only if they are in the same part.

Remark 2.3.15. Note that the definition implies that the image of T doesn't depend on the equivalent classes of the $\mathbb{Z}_2$-coloured partitions. This means that we can still identify equivalence classes of $\mathbb{Z}_2$-coloured partitions, see Remark 2.1.13, and assume that T is defined on equivalence classes of $\mathbb{Z}_2$-coloured partitions.

Proposition 2.3.16. *The linear map T is well-defined.*

Proof. We want to see that each morphism $T_{(p,z)}$ commutes with the action of H_n. Define the action from $a = (a_1,\ldots,a_n,\sigma) \in H_n$ on $(i,b) = ((i_1,\ldots,i_k),(b_1,\ldots,b_k)) \in \{1,\ldots,n\}^k \times (\mathbb{Z}_2)^k$ by $a \cdot (i,b) := ((\sigma(i_1),\ldots,\sigma(i_k)),(a_{\sigma(i_1)}b_1,\ldots,a_{\sigma(i_k)}b_k))$. Then $\delta_{(p,z)}((i,b),(j,c)) = 1$ if and only if $\delta_{(p,z)}(a \cdot (i,b), a \cdot (j,c)) = 1$ for all $a \in H_n$. The equality

$$\begin{aligned} T_{(p,z)} \circ a(e^{i_1}_{b_1} \otimes \ldots \otimes e^{i_k}_{b_k}) &= \sum_{\substack{1 \leqslant j_1 \ldots j_l \leqslant n \\ c_1 \ldots c_l \in \mathbb{Z}_2}} \delta_{(p,z)}(a \cdot (i,b),(j,c))(e^{j_1}_{c_1} \otimes \ldots \otimes e^{j_l}_{c_l}) \\ &= \sum_{\substack{1 \leqslant j_1 \ldots j_l \leqslant n \\ c_1 \ldots c_l \in \mathbb{Z}_2}} \delta_{(p,z)}((i,b), a^{-1} \cdot (j,c))(e^{j_1}_{c_1} \otimes \ldots \otimes e^{j_l}_{c_l}) \\ &= \sum_{\substack{1 \leqslant j_1 \ldots j_l \leqslant n \\ c_1 \ldots c_l \in \mathbb{Z}_2}} \delta_{(p,z)}((i,b),(j,c)) a \cdot (e^{j_1}_{c_1} \otimes \ldots \otimes e^{j_l}_{c_l}) \\ &= a \circ T_{(p,z)}(e^{i_1}_{b_1} \otimes \ldots \otimes e^{i_k}_{b_k}) \end{aligned}$$

shows that T is well-defined.

□

By definition every morphism in $\mathrm{Hom}_{H_n}((\mathbb{C}^{2n})^{\otimes k}, (\mathbb{C}^{2n})^{\otimes l})$ is stable under the H_n action. This motivates the definition of a certain basis for this morphism space.

Definition 2.3.17. Define the action

$$a \cdot ((i,b),(j,c)) := (a \cdot (i,b), a \cdot (j,c))$$

for all $(i,b) \in \{1,\dots,n\}^k \times (\mathbb{Z}_2)^k$, $(j,c) \in \{1,\dots,n\}^l \times (\mathbb{Z}_2)^l$ and $a \in H_n$. We write $[((i,b),(j,c))]$ for an equivalence class under this action and let C denote the set of such equivalence classes. We define a morphism $f_{[((i,b),(j,c))]} : (\mathbb{C}^{2n})^{\otimes k} \to (\mathbb{C}^{2n})^{\otimes l}$ which sends

$$e^{\sigma(i_1)}_{a_{\sigma(i_1)}b_1} \otimes \dots \otimes e^{\sigma(i_k)}_{a_{\sigma(i_k)}b_k} \mapsto e^{\sigma(j_1)}_{a_{\sigma(j_1)}c_1} \otimes \dots \otimes e^{\sigma(j_l)}_{a_{\sigma(j_l)}c_l}$$

for all $a = (a_1,\dots,a_n,\sigma) \in S_n$ and all other basis elements of $(\mathbb{C}^{2n})^{\otimes k}$ to 0. This definition is independent of the representative of the equivalent class and $\{f_{[((i,b),(j,c))]} | [((i,b),(j,c))] \in C\}$ is a basis of $\mathrm{Hom}_{H_n}((\mathbb{C}^{2n})^{\otimes k}, (\mathbb{C}^{2n})^{\otimes l})$. This basis consists of sums over H_n-orbits of basis elements in $\mathrm{Hom}((\mathbb{C}^{2n})^{\otimes k}, (\mathbb{C}^{2n})^{\otimes l})$.

Similarly we want to define a basis for $\mathbb{C}\, P_{\mathbb{Z}_2}(k,l)$.

Definition 2.3.18. We define the partitions

$$x_{(p,z)} := (p,z) - \sum_{q \text{ coarser than } p} \sum_{\substack{z' \in \mathbb{Z}_2^{k+l} \\ \text{with } (p,z') \simeq (p,z)}} x_{(q,z')}$$

in $\mathbb{C}\, P_{\mathbb{Z}_2}(k,l)$ for $(p,z) \in P_{\mathbb{Z}_2}(k,l)$ recursively. So after the substracion we want to sum over all $\mathbb{Z}_2$-coloured partitions (q,z'), where q is can be any partition coarser as p and z' can be any labeling of p such that $(p,z) \simeq (p,z')$.[1]. They form a basis for $\mathbb{C}\, P_{\mathbb{Z}_2}(k,l)$.

Proposition 2.3.19. *The linear map $T : \mathbb{C}\, P_{\mathbb{Z}_2}(k,l) \to Hom_{H_n}((\mathbb{C}^{2n})^{\otimes k}, (\mathbb{C}^{2n})^{\otimes l})$ is surjective and has as kernel $\{x_{(p,z)} \mid (p,z) \in P_{\mathbb{Z}_2}(k,l)$ has more then n parts$\}$. As a consequence T is an isomorphism in case $k + l \leqslant n$.*

Proof. We want to associate to some basis element $f_{[(i,b),(j,c)]}$ of the morphism space a partition $(p,z) \in P_{\mathbb{Z}_2}(k,l)$. We label the $k+l$ vertices by (i,j) and let two vertices be in the same block if and only if the they are labeled by the same number. We colour the vertices by setting $z = (b_1,\dots,b_k,c_1,\dots,c_l)$. Then $((i,b),(j,c))$ is a perfect labeling of the partition (p,z). Note that $T_{(p,z))}$ does not equal $f_{[((i,b),(j,c))]}$, but $T_{x_{(p,z)}} = f_{[(i,b),(j,c)]}$ holds. This shows that T is surjective.

Now let $(p,z) \in P_{\mathbb{Z}_2}(k,l)$ be a partition with less or equal than n parts. Then there exists a unique equivalence class of perfect labels $[(i,b),(j,c)]$ of (p,z) because the colouring can be chosen in each part only up to sign, which doesn't change the equivalence class. We saw in the first part of the proof that $T_{x_{(p,z)}} = f_{[(i,b),(j,c))]} \neq 0$. For a different partition $(q,y) \in P(k,l)$ with less or equal than n parts, $T_{x_{(q,y)}}$ will be a different basiselement of $\mathrm{Hom}_{H_n}((\mathbb{C}^{2n})^{\otimes k}, (\mathbb{C}^{2n})^{\otimes l})$. So we have that T is injective on the subspace of

[1]Note that $(p,z) \simeq (p,z')$ doesn't imply that $(q,z) \simeq (q,z')$ for q coarser as p!

$\mathbb{C}\, P_{\mathbb{Z}_2}(k,l)$ spanned by the partitions with less than or equal to n parts, with image all of $\mathrm{Hom}_{H_n}(\mathbb{C}^{2n})^{\otimes k}, (\mathbb{C}^{2n})^{\otimes l}$.

Now assume that $(p,z) \in P(k,l)$ is a partition with more than n parts. Then we can argument simlarly as in the symmetric group case as follows. First assume that (p,z) has $n+1$ parts. Then

$$T_{x_{(p,z)}} = T_{(p,z)} - \sum_{q \text{ coarser than } p} \sum_{\substack{z' \in \mathbb{Z}_2^{k+l} \\ \text{with } (p,z') \simeq (p,z)}} T_{x_{(q,z')}}$$

where each $T_{x_{(q,z)}}$ in the equation is a linear map $f_{[(i,b),(j,c)]}$ corresponding to some perfect labeling of q. But because (p,z) has no perfect labeling $T_{(p,z)}$ will exactly be a sum of all $T_{x_{(q,z)}}$ for q coarser as p. This shows then that $T_{x_{(p,z)}}$ is zero. One can now apply this argument to inductively prove that $T_{x_{(p,z)}} = 0$ for (p,z) with m parts for all $m > n$.

□

Remark 2.3.20. In [Blo03] they state the dimension of the $\mathbb{C}$-algebra $P_k(t, \mathbb{Z}_2)$, which equals our algebra $End_{\mathrm{Par}(\mathbb{Z}_2,t)}([k])$ for $t \in \mathbb{C}$. To see this, one has to show that the notion of coloured partition diagrams and their composition in [Blo03] corresponds to our notion. This can be easily done if one sees that the orientation of the edges between vertices is irrelevant for the $\mathbb{Z}_2$-case and that one can label the vertices belonging to an edge with label 1 equally and label -1 differently.

The number of partition diagrams in $P_{\mathbb{Z}_2}(k,k)$ with l connected components is $2^{2k-l}S(2k,l)$ because we have to count the non-equivalent labelings. The dimension of $End_{\mathrm{Par}(\mathbb{Z}_2,t)}([k])$ equals the cardinality of $P_{\mathbb{Z}_2}(k,k)$

$$\sum_{l=1}^{2k} 2^{2k-l} S(2k,l).$$

This is also the dimension of the centralizer algebra $End_{\mathrm{Rep}(H_n)}((\mathbb{C}^{2n})^{\otimes k})$ in the case $2k \leqslant n$.

The surjectivity of T can also be derived in a different way, using slightly different notational conventions. In [LS21][Chapter 5] they associate to each partition $(p,z) \in P_{\mathbb{Z}_2}(k,l)$ a linear map $T_{(p,z)} : V^{\otimes k} \to V^{\otimes l}$ between tensor powers of the permutation representation $V = \mathbb{C}^{2n}$. It is defined on basiselements by

$$T_{(p,z)}(e^{i_1}_{j_1} \otimes \ldots \otimes e^{i_k}_{j_k}) = \sum_{\substack{1 \leqslant i_{1'} \ldots i_{l'} \leqslant n \\ j_{1'} \ldots j_{l'} \in \mathbb{Z}_2}} \delta^{(i,j)}_{(p,z)} (e^{i_{1'}}_{j_{1'}} \otimes \ldots \otimes e^{i_{l'}}_{j_{l'}})$$

where $i = (i_1, \ldots, i_k, i_{1'}, \ldots, i_{l'})$ and $j = (j_1, \ldots, j_k, j_{1'}, \ldots, j_{l'})$. We can label the partition p by i and j in an obvious way. Then $\delta^{(i,j)}_{(p,z)}$ equals 1 if and only if for all $c, d \in \{1, \ldots k, 1', \ldots, l'\}$ which lie in the same block of p we have that $i_c = i_d$ and $j_c z_c = j_d z_d$.

The following proof is based on material in [LS21][Lemma 5.3] and [Com20][Chapter 2.3].

Proposition 2.3.21. *The morphisms spaces between tensor products of the permutation representation of H_n can be descibed using $\mathbb{Z}_2$-coloured partitions, more accurately:*

$$< \{T_{(p,z)} \mid (p,z) \in P_{\mathbb{Z}_2}(k,l)\} >= Hom_{H_n}(V^{\otimes k}, V^{\otimes l}).$$

Proof. The inclusion $\subset$ can be verified by checking that the linear maps $T_{(p,z)}$ commute with the action of the hyperoctahedral groups on V. This follows directly from the definitions. Because $\mathbb{C}^{2n}$ is self-dual, there is an isomorphism

$$\begin{aligned} \mathrm{Hom}_{H_n}(V^{\otimes k}, V^{\otimes l}) &\to \mathrm{Hom}_{H_n}(V^{\otimes (k+l)}, \mathbf{1}) \\ g &\mapsto ev_{V^{\otimes l}} \circ (g \otimes id_{V^{\otimes l}}) \end{aligned}$$

with inverse

$$\begin{aligned} \mathrm{Hom}_{H_n}(V^{\otimes (k+l)}, \mathbf{1}) &\to \mathrm{Hom}_{H_n}(V^{\otimes k}, V^{\otimes l}) \\ h &\mapsto (h \otimes id_{V^{\otimes k}}) \circ (id_{V^{\otimes k}} \otimes coev_{V^{\otimes l}}). \end{aligned}$$

Similarly there is a bijection

$$\begin{aligned} P_{\mathbb{Z}_2}(k,l) &\to P_{\mathbb{Z}_2}(k+l,0) \\ (p,z) &\mapsto (\{\{1,l+1\},\{2,l+2\},\dots\{l,2l\}\},1) \circ ((p,z) \otimes (id_l,1)) \end{aligned}$$

with inverse

$$\begin{aligned} P_{\mathbb{Z}_2}(k+l,0) &\to P_{\mathbb{Z}_2}(k,l) \\ (p,z) &\mapsto ((p,z) \otimes (id_l,1)) \circ ((id_k,1) \otimes (\{\{1',l+1'\},\{2',l+2'\},\dots\{l',2l'\}\},1)). \end{aligned}$$

Here 1 indicates the trivial labeling of the partitions and for $m \in \mathbb{N}$ we wrote $id_m := \{\{1,1'\},\dots\{m,m'\}\} \in P(m,m)$. After taking the $\mathbb{C}$-linear span, this bijection becomes an isomorphism. Because the map T commutes with both isomorphisms, we get a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}\,P_{\mathbb{Z}_2}(k,l) & \xrightarrow{\cong} & \mathbb{C}\,P_{\mathbb{Z}_2}(k+l,0) \\ \downarrow T & & \downarrow T \\ \mathrm{Hom}_{H_n}(V^{\otimes k}, V^{\otimes l}) & \xrightarrow{\cong} & \mathrm{Hom}_{H_n}(V^{\otimes (k+l)}, \mathbf{1}). \end{array}$$

$\mathrm{Hom}_{H_n}(V^{\otimes (k+l)}, \mathbf{1})$ has as a basis the different morphisms which each send some H_n-orbit S of a canonical basis element $e_{j_1}^{i_1} \otimes \dots \otimes e_{j_{k+l}}^{i_{k+l}}$ in $V^{\otimes (k+l)}$ to 1 and everything else to 0. Such a morphism lies in the image under T of the inductively defined linear combination of partitions

$$P_{[(p,z)]} := [(p,z)] - \sum_{q \text{ coarser than } p} P_{[(q,z)]}$$

where $(p,z) \in P_{\mathbb{Z}_2}(k+l,0)$ is a representative of an equivalence class of $\mathbb{Z}_2$-coloured partitions representing the orbit S of $V^{\otimes (k+l)}$. For example $(\{\{1,3\},\{2\}\},(-1,1,1))$ represents the orbit of $e_1^2 \otimes e_{-1}^1 \otimes e_{-1}^2 \in (\mathbb{C}^4)^{\otimes 3}$, but also $(\{\{1,3\},\{2\}\},(1,-1,-1))$ represents this orbit. This implies that the vertical right map is surjective, which then implies that the left vertical map is surjective, meaning that the image of the $\mathbb{Z}_2$-coloured partitions under T span the morphism space $\mathrm{Hom}_{H_n}(V^{\otimes k}, V^{\otimes l})$. □

Remark 2.3.22. Again we have the situation where we have described the morphism spaces without distinguishing between different $n \in \mathbb{N}$. This will be essential in constructing our interpolation category of the hyperoctahedral group at the permutation representation. Similarly as before the only difference will occur when one composes the images under T of compatible $\mathbb{Z}_2$-coloured partitions. Let $(p, z_1) \in P_{\mathbb{Z}_2}(k,l)$ and $(q, z_2) \in P_{\mathbb{Z}_2}(l,m)$ be compatible, then

$$T_{(q,z_2)(p,z_1)} = n^{-l(q,p)} T_{(q,z_2)} \circ T_{(p,z_1)} \text{ or } n^{l(q,p)} T_{(q,z_2)(p,z_1)} = T_{(q,z_2)} \circ T_{(p,z_1)}.$$

2.4 Defining functors from the interpolation categories to the representation categories

In this section we use the information obtained about the morphism spaces to relate the interpolation categories to the representation categories. To proof that the interpolation functors are essentially surjective and full, it suffices to put the results of Section 2.3 in a category theoretical framework. Proposition 2.4.2 is also proven in [BS09, Proposition 1.1.9].

2.4.1 Symmetric groups

In this section we define the interpolation functor $F : \underline{\mathrm{Rep}}(S_n) \to \mathrm{Rep}(S_n)$ and prove that it is well-defined. This was done in [CW12, Definition 3.8 and Proposition 2.8], but with different notation.

Definition 2.4.1. We define a strict $\mathbb{C}$-linear tensor functor

$$F' : \underline{\mathrm{Rep}}_0(S_n) \to \mathrm{Rep}(S_n)$$

between $\mathbb{C}$-linear spherical rigid symmetric monoidal categories on objects by $[k] \to (u')^{\otimes k}$ and on the morphisms by the $\mathbb{C}$-linear extension of the rule $p \to T_p$ for some partition $p \in P(k,l)$. If this is well-defined, then Remark 2.2.4 and the fact that $\mathrm{Rep}(S_n)$ is a Karoubi tensor category, give us up to isomorphism a $\mathbb{C}$-linear tensor functor

$$F : \underline{\mathrm{Rep}}(S_n) \to \mathrm{Rep}(S_n),$$

which satisfies $F' = F \circ \iota_F$.

Proposition 2.4.2. *The functor F' is well-defined and a tensor functor.*

Proof. Note that $[k]$ and $(u')^{\otimes k}$ are both self-dual objects of dimension n^k in their respective categories. We will prove that F' is well-defined and monoidal. Checking that the other tensor functor properties are satisfied, is straight-forward but very lengthy. We have to show that T is compatible with the identity morphisms, the composition of morphisms and the tensor product.

First we consider the identity morphism $p = id_{[k]} \in \mathrm{Hom}_{\underline{\mathrm{Rep}}_0(S_n)}([k],[k])$. We see that $T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) = (e_{i_1} \otimes \ldots \otimes e_{i_k})$ for all $1 \leqslant i_1, \ldots, i_k \leqslant n$. This shows that F' respects the identity morphisms.

Now let $p \in P(k,l)$ and $q \in P(l,m)$. Then

$$\begin{aligned} T_q \circ T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) &= T_q(\sum_{1 \leqslant t_1 \ldots t_l \leqslant n} \delta_p(i,t)(e_{t_1} \otimes \ldots \otimes e_{t_l})) \\ &= \sum_{1 \leqslant t_1 \ldots t_l \leqslant n} \delta_p(i,t)(\sum_{1 \leqslant j_1 \ldots j_m \leqslant n} \delta_p(t,j)(e_{j_1} \otimes \ldots \otimes e_{j_m})) \\ &= \sum_{1 \leqslant t_1 \ldots t_l \leqslant n} \sum_{1 \leqslant j_1 \ldots j_m \leqslant n} \delta_p(i,t)\delta_q(t,j)(e_{j_1} \otimes \ldots \otimes e_{j_m}) \\ &= n^{l(q,p)} \sum_{1 \leqslant j_1 \ldots j_m \leqslant n} \delta_{qp}(i,j)(e_{j_1} \otimes \ldots \otimes e_{j_m}) \\ &= n^{l(q,p)} T_{qp}. \end{aligned}$$

This implies the equality

$$T(q \circ p) = T(n^{l(q,p)}qp) = n^{l(q,p)}T_{qp} = n^{l(q,p)}n^{-l(q,p)}T_q \circ T_p = T_q \circ T_p$$

which shows that T, and as a consequence F', is compatible with the composition in both categories.

Let $p \in P(k,l)$ and $q \in P(s,t)$ then

$$\begin{aligned} &T_{p\otimes q}(e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes e_{v_1} \otimes \ldots \otimes e_{v_s}) \\ &\quad = \sum_{\substack{1 \leqslant j_1 \ldots j_l \leqslant n \\ 1 \leqslant w_1 \ldots w_t \leqslant n}} \delta_p(i,j)\delta_q(v,w)(e_{j_1} \otimes \ldots \otimes e_{j_l} \otimes e_{w_1} \otimes \ldots \otimes e_{w_t}) \\ &\quad = (\sum_{1 \leqslant j_1 \ldots j_l \leqslant n} \delta_p(i,j)(e_{j_1} \otimes \ldots \otimes e_{j_l})) \otimes (\sum_{1 \leqslant w_1 \ldots w_t \leqslant n} \delta_p(v,w)(e_{w_1} \otimes \ldots \otimes e_{w_t})) \\ &\quad = T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) \otimes T_q(e_{v_1} \otimes \ldots \otimes e_{v_s}). \end{aligned}$$

for all i and v. This shows that T is compatible with the tensor product, so F' is monoidal. □

Proposition 2.4.3. *The functor F is a full and essentially surjective monoidal functor.*

Proof. The fact that F is full follows from Proposition 2.3.5. Because the permutation representation of S_n is a faithful representation, every irreducible representation is a direct summand of some tensor power of the permutation representation u'. This means that some irreducible representation $A \in \mathrm{Rep}(S_n)$ is isomorphic to the image of some idempotent $e : u'^{\otimes k} \to u'^{\otimes k}$ for some $k \in \mathbb{N}$. Because the functor is full there exists an idempotent $e' : [k] \to [k]$ with $F(e') = e$. As a consequence $F(im(e')) \cong im(e) \cong A$. This shows that F is essentially surjective. □

2.4.2 Hyperoctahedral groups: reflection representations

In this section we define the interpolation functor $G : \underline{\mathrm{Rep}}(H_n) \to \mathrm{Rep}(H_n)$ as it was done in [FM21, Section 2.3], but without using the ideas involving easy quantum groups.

Definition 2.4.4. We define a strict $\mathbb{C}$-linear tensor functor between $\mathbb{C}$-linear spherical rigid symmetric monoidal categories

$$G' : \underline{\mathrm{Rep}}_0(H_n) \to \mathrm{Rep}(H_n)$$

on the objects by $[k] \to u^{\otimes k}$ and on the morphisms by $p \to T_p$ for some partition $p \in P_{even}(k, l)$, after which one linearly extends this rule. Note that $[k]$ and $u^{\otimes k}$ are both self-dual objects of dimension n^k in their respective categories. The fact that this is a well-defined tensor functor was proven in Proposition 2.4.2. By Remark 2.2.8 and the fact that $\mathrm{Rep}(H_n)$ is a Karoubi tensor category, the universal property of the Karoubian envelope in Proposition 1.1.27 gives us a $\mathbb{C}$-linear tensor functor

$$G : \underline{\mathrm{Rep}}(H_n) \to \mathrm{Rep}(H_n),$$

which satisfies $G' = G \circ \iota_G$ and is unique up to isomorphism.

Proposition 2.4.5. *The functor G is a full and essentially surjective monoidal functor.*

Proof. The fullness of G was shown in Proposition 2.3.11. It is essentially surjective because the reflection representation is a faithful representation of H_n and every irreducible representation is a direct summand of some tensor power of a faithful representation. $\square$

We have by definition that

$$\begin{aligned} G(\{1,2,1',2'\}) = G(\sqcup\!\!\!\sqcap) = T(\sqcup\!\!\!\sqcap) : u \otimes u &\to u \otimes u \\ e_i \otimes e_i &\mapsto e_i \otimes e_i \text{ for all } 1 \leqslant i \leqslant n \text{ and} \\ e_i \otimes e_j &\mapsto 0 \text{ for all } 1 \leqslant i \neq j \leqslant n. \end{aligned}$$

This implies that the image of $G(\sqcup\!\!\!\sqcap)$ consists of $\mathbb{C}$-linear combinations of $\{e_i \otimes e_i | 1 \leqslant i \leqslant n\}$ and as a consequence the representation v, see Remark 1.2.15, lies in the essential image of G:

$$G(([1] \otimes [1], \{1,2,1',2'\})) = G(([1] \otimes [1], \sqcup\!\!\!\sqcap)) = im(G(\sqcup\!\!\!\sqcap) : u \otimes u \to u \otimes u)) \cong v.$$

Then also the permutation representation lies in the essential image of G:

$$G([1] \oplus ([1] \otimes [1], \{1,2,1',2'\})) \cong u \oplus v \cong V.$$

2.4.3 Hyperoctahedral groups: permutation representations

In this section we define the interpolation functor $H : \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar} \to \mathrm{Rep}(H_n)$ of [LS21, Theorem 5.1], but without using the presentation of the interpolation category via generators and relations.

Definition 2.4.6. We define a strict $\mathbb{C}$-linear tensor functor between $\mathbb{C}$-linear spherical rigid symmetric monoidal categories

$$H' : \mathrm{Par}(\mathbb{Z}_2, 2n) \to \mathrm{Rep}(H_n)$$

on the objects by $[\tilde{k}] \to V^{\otimes k}$ and on the morphisms by a linear extension of the rule $(p, z) \to T_{(p,z)}$. Note that $[\tilde{k}]$ and $V^{\otimes k}$ are both self-dual objects of dimension $(2n)^k$ in their respective categories. The fact that this is a well-defined tensor functor is proven similarly as in Proposition 2.4.2. By Remark 2.2.13 and the fact that $\mathrm{Rep}(H_n)$ is a Karoubi tensor category, the universal property of the Karoubian envelope in Proposition 1.1.27 gives us a $\mathbb{C}$-linear tensor functor

$$H : \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar} \to \mathrm{Rep}(H_n),$$

which satisfies $F' = F \circ \iota_F$ and is unique up to isomorphism. We will see that some choices of H will be more convenient in particular situations, for example in the proof of Corollary 3.2.18.

Proposition 2.4.7. *The functor H is a full and essentially surjective monoidal functor.*

Proof. The functor is full as is shown in Proposition 2.3.19. It is essentially surjective because the permutation representation is a faithful representation of H_n and every irreducible representation is a direct summand of some tensor power of a faithful representation. □

Corollary 2.4.8. *The reflection representation u and its complement v in V both lie in the essential image of H.*

Remark 2.4.9. In Chapter 3 we will give explicit preimages of the representations u and v for H.

2.5 Semisimplification

We show that the interpolation functors induce equivalences between the interpolation categories and the representation categories. These results follow from the fact that the interpolation functors are full tensor functors and from Proposition 2.5.11, which is proven in [CW12, Proposition 3.23]. We also obtain explicit descriptions of the negligible morphisms in the interpolation categories.

2.5.1 General theory

Our definition of a tensor ideal is equivalent to the one in [AKO02, Definition 6.1.1]. We use the definition of negligible morphisms from [EO22, Definition 2.1]. In Proposition 2.5.5 and Proposition 2.5.11, we write out the proof of [CO11, Proposition 3.23]. The proof that we give for Proposition 2.5.10 is the proof from [AKO02], but the statement also follows from [EO22, Lemma 2.2].

Definition 2.5.1. We define a *tensor ideal $\mathcal{I}$* in a $\mathbb{C}$-linear spherical rigid symmetric monoidal category $\mathcal{C}$ as a collection of morphism spaces $\mathcal{I}(A, B) \subset \mathcal{C}(A, B)$ for all objects

$A, B \in \mathrm{Ob}(\mathcal{C})$. This collection is closed under addition, composition and tensor product with all morphisms, i.e. if $f \in \mathcal{I}(A,B)$, $g \in \mathcal{C}(C,D)$, $h \in \mathcal{C}(B,E)$ and $i \in \mathcal{C}(K,A)$ then $f \otimes g \in \mathcal{I}(A \otimes C, B \otimes D)$, $g \otimes f \in \mathcal{I}(C \otimes A, D \otimes B)$, $h \circ f \in \mathcal{I}(A,C)$ and $f \circ i \in \mathcal{I}(K,A)$ for all $A, B, C, D, K \in \mathrm{Ob}(\mathcal{C})$.

Definition 2.5.2. Let $\mathcal{I}$ be a tensor ideal in a $\mathbb{C}$-linear spherical rigid symmetric monoidal category $\mathcal{C}$. Then the quotient category $\mathcal{C}/\mathcal{I}$ is a $\mathbb{C}$-linear symmetric monoidal category with the same objects as $\mathcal{C}$ and the morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A,B) := \mathrm{Hom}_{\mathcal{C}}(A,B)/\mathcal{I}(A,B).$$

It can be easily checked that this is well-defined.

Remark 2.5.3. If $\mathcal{I}$ is a tensor ideal in a $\mathbb{C}$-linear spherical rigid symmetric monoidal category $\mathcal{C}$ which is invariant under the endofunctor $* : \mathcal{C} \to \mathcal{C}$, i.e. $\mathcal{I}^* = \mathcal{I}$, then the quotient category $\mathcal{C}/\mathcal{I}$ inherits a $\mathbb{C}$-linear spherical rigid symmetric monoidal structure from $\mathcal{C}$. Note that if $ev_A \in \mathcal{I}$ for some $A \in \mathcal{C}$, then the triangle identities imply that $id_A \in \mathcal{I}$ and therefore $\mathcal{I} = \mathcal{C}$. The same holds true if $\mathcal{I}$ contains $coev_A$ or any morphism which can be composed or tensored with to obtain an isomorphism in $\mathcal{C}$.

Definition 2.5.4. Let $\mathcal{C}$ be a $\mathbb{C}$-linear spherical rigid symmetric monoidal category. We call a morphism $f \in Hom_{\mathcal{C}}(A,B)$ for $A, B \in Ob(\mathcal{C})$ *negligible*, if for every $g \in Hom_{\mathcal{C}}(B,A)$, we have $tr(f \circ g) = 0$. We denote the set of all negligible morphism from A to B by $\mathcal{N}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$.

Proposition 2.5.5. *The ideal $\mathcal{N}$ of negligible morphisms in a $\mathbb{C}$-linear spherical rigid symmetric monoidal category $\mathcal{C}$ is a tensor ideal.*

Proof. Assume that $f : A \to B$ is a negligible morphism. Let $g : C \to D$. Let $h \otimes h' : B \otimes D \to A \otimes C$. We want to show that $tr((f \otimes g) \circ (h \otimes h')) = 0$, where $(f \otimes g) \circ (h \otimes h') : B \otimes D \to B \otimes D$. The axioms of a tensor category imply

$$\begin{aligned}
tr((f \otimes g) \circ (h \otimes h')) &= ev_{(B\otimes D)^*} \circ (id_{(B\otimes D)^*} \otimes (f \otimes g) \circ (h \otimes h')) \circ coev_{B\otimes D}. \\
&= ev_{D^*} \circ (id_{D^*} \otimes (g \circ h')) \circ (id_{D^*} \otimes ev_{B^*} \otimes id_D) \circ \\
&\quad (id_{D^*} \otimes id_{B^*} \otimes (f \circ h) \otimes id_D) \circ (id_{D^*} \otimes coev_B \otimes id_D) \circ coev_D \\
&= ev_{D^*} \circ (id_{D^*} \otimes (g \circ h')) \circ (id_{D^*} \otimes tr(f \circ h) \otimes id_D) \circ coev_D \\
&= 0,
\end{aligned}$$

so $f \otimes g$ is negligible. Now let $i : B \to E$ and $j : K \to A$ be morphisms in $\mathcal{C}$. If $i' : E \to A$ and $j' : B \to K$ then we get by Remark 1.1.13 that

$$\begin{aligned}
&tr(i \circ f \circ i') = tr(i' \circ i \circ f) = tr(f \circ i' \circ i) = 0 \text{ and} \\
&tr(f \circ j \circ j') = 0,
\end{aligned}$$

which concludes the proof. ☐

Definition 2.5.6. Let $\mathcal{C}$ be a $\mathbb{C}$-linear spherical rigid symmetric monoidal category and $\mathcal{N}$ its ideal of negligible morphisms. By Proposition 2.5.5 we can define the semisimplification of $\mathcal{C}$ by the quotient $\widehat{\mathcal{C}} := \mathcal{C}/\mathcal{N}$. Its morphism spaces are then of the form $\mathrm{Hom}_{\widehat{\mathcal{C}}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)/\mathcal{N}(A,B)$.

Remark 2.5.7. Note that the ideal $\mathcal{N}$ of negligible morphisms in a $\mathbb{C}$-linear spherical rigid symmetric monoidal category $\mathcal{C}$ satisfies $\mathcal{N}^* = \mathcal{N}$. One can see this as follows. Let $f, g : A \to B$ be a morphisms in $\mathcal{C}$ and f negligible. Then

$$tr(f^* \circ g) = tr(g^* \circ f) = tr(f \circ g^*) = 0,$$

where the first equality follows from the morphism identities and the sphericality, the second from Remark 1.1.13 and the third one form the assumption that f is negligible. So by Remark 2.5.3 the semisimplification $\widehat{\mathcal{C}}$ is a $\mathbb{C}$-linear spherical rigid symmetric monoidal category. The corresponding quotient functor $\mathcal{P} : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ is a $\mathbb{C}$-linear full tensor functor.

Definition 2.5.8. A $\mathbb{C}$-linear category is *semisimple* if each object can be written as a finite direct sum of simple objects and if all those direct sums exists.

Example 2.5.9. Every semisimple category has all finite biproducts and splittings for its idempotents, therefore every semisimple category is Karoubi. The representation categories over the complex numbers of S_n and H_n, $\mathrm{Rep}(S_n)$ and $\mathrm{Rep}(H_n)$ respectively, are $\mathbb{C}$-linear spherical rigid symmetric monoidal semisimple categories for all $n \in \mathbb{N}$, so the following proposition applies.

Proposition 2.5.10. *There are no non-trivial negligible morphisms in a semisimple tensor category.*

Proof. It is proven in [AKO02][Chapter 7 and Appendix A.2] that if $\mathcal{C}$ is a semsimple tensor category then the tensor ideal, called the radical ideal,

$$\mathcal{R}(A, B) := \{f \in \mathcal{C}(A, B) | \forall g \in \mathcal{C}(B, A), 1_A - g \circ f \text{ is invertible.}\}$$

is zero.

The ideal of negligible morphisms is then $\mathcal{N} = 0$ because it equals

$$\mathcal{N}(A, B) = i_{AB}(\mathcal{R}(1, A^* \otimes B))$$

where

$$\begin{aligned} i_{AB} : &\mathcal{C}(\mathbf{1}, A^* \otimes B) \to \mathcal{C}(A, B) \\ &\varphi \to (ev_A \otimes 1_B) \circ (1_A \otimes \varphi) \end{aligned}$$

□

Proposition 2.5.11. *The image of a morphism f under a full tensor functor is negligible if and only if f itself is negligible.*

Proof. Let $\mathcal{K} : \mathcal{C} \to \mathcal{D}$ be a full tensor functor between tensor categories. Let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Assume first that f is negligible. Let $g \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{K}(B), \mathcal{K}(A))$. Then there exists a $g' \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ with $\mathcal{K}(g') = g$. Because tensor functors preserve traces, see Remark 1.1.15, we get

$$tr(\mathcal{K}(f) \circ g) = tr(f \circ g') = 0,$$

which implies that $\mathcal{K}(f)$ is negligible.

Now assume that $\mathcal{K}(f)$ is negligible. Let $h \in \mathrm{Hom}_C(B, A)$. Then

$$tr(f \circ h) = tr(\mathcal{K}(f) \circ \mathcal{K}(h)) = 0$$

and this implies that f is negligible. □

2.5.2 Semisimplification of the interpolation categories for the symmetric groups

We use the results of previous section to show that the functor $F : \underline{\mathrm{Rep}}(S_n) \to \mathrm{Rep}(S_n)$ induces an equivalence $\widehat{F} : \widehat{\underline{\mathrm{Rep}}(S_n)} \to \mathrm{Rep}(S_n)$. This statement is proven in [Del07, Theorem 6.2] and [CO11, Theorem 3.24]. The fact that $\underline{\mathrm{Rep}}(S_t)$ is semisimple for $t \notin \mathbb{N}$ is proven in [Del07, Theorem 2.18].

Proposition 2.5.12. *For all $n \in \mathbb{N}$ the interpolation functor $F : \underline{Rep}(S_n) \to Rep(S_n)$ induces a $\mathbb{C}$-linear functor $\widehat{F} : \widehat{\underline{Rep}(S_n)} \to Rep(S_n)$, which is an equivalence of categories. The semisimplification $\widehat{\underline{Rep}(S_n)}$ is semisimple.*

Proof. Proposition 2.3.5 and Definition 2.4.1 imply that F is a $\mathbb{C}$-linear full tensor functor. Then it reflects and preserves negligible morphisms by Proposition 2.5.11. $\mathrm{Rep}(S_n)$ is semisimple, therefore the only negligible morphisms in this category are the zero morphisms. This implies that the only negligible morphisms in $\underline{\mathrm{Rep}}(S_n)$ are the morphisms that lie in the kernel of

$$F = T : Hom_{\underline{\mathrm{Rep}}(S_n)}([k], [l]) = \mathbb{C}\, P(k, l) \to Hom_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$$

for all $k, l \in \mathbb{N}$. So if $\mathcal{N}$ is the tensor ideal of negligible morphisms in the interpolation category, then it contains all the morphisms in the kernel of F. Therefore we get an induced $\mathbb{C}$-linear functor

$$\widehat{F} : \widehat{\underline{\mathrm{Rep}}(S_n)} \to \mathrm{Rep}(S_n)$$

which is faithful, full and essentially surjective. So it is an equivalence of categories. The semisimplicity of $\mathrm{Rep}(S_n)$ then implies the semisimplicity of $\widehat{\underline{\mathrm{Rep}}(S_n)}$. □

Remark 2.5.13. If we denote the quotient functor corresponding to the semisimplification by $F_{\mathcal{N}} : \underline{\mathrm{Rep}}(S_n) \to \widehat{\underline{\mathrm{Rep}}(S_n)}$, we get that $F = \widehat{F} \circ F_{\mathcal{N}}$.

Remark 2.5.14. Note that the proof shows that the interpolation categories $\underline{\mathrm{Rep}}(S_n)$ are not semisimple for $n \in \mathbb{N}$, because they have nontrivial negligible morphisms.

Remark 2.5.15. Following Remark 2.5.3 and Remark 2.5.7, $\widehat{\underline{\mathrm{Rep}}(S_n)}$ is a $\mathbb{C}$-linear spherical rigid symmetric monoidal category and the quotient functor $F_{\mathcal{N}}$ is a $\mathbb{C}$-linear full tensor functor. Because $F = \widehat{F} \circ F_{\mathcal{N}}$ is also a $\mathbb{C}$-linear tensor functor, so is $\widehat{F}$.

2.5.3 Semisimplification of the interpolation categories in the reflection representations for the hyperoctahedral groups

Analogous as we did in the previous section, we show that the interpolation functor $G : \underline{\mathrm{Rep}}(H_n) \to \mathrm{Rep}(H_n)$ induces an equivalence $\widehat{G} : \widehat{\underline{\mathrm{Rep}}(H_n)} \to \mathrm{Rep}(H_n)$. This statement is proven in [FM21, Proposition 3.17] and we give a short summary of their approach after we prove the statement without relying on the theory of $*$-categories, see 2.5.11. The fact that the interpolation categories $\underline{\mathrm{Rep}}(H_t)$ are semisimple if and only if $t \notin \mathbb{N}$, is proven in [FM21, Chapter 3].

Proposition 2.5.16. *For all $n \in \mathbb{N}$ the interpolation functor $G : \underline{Rep}(H_n) \to Rep(H_n)$ induces a $\mathbb{C}$-linear functor $\widehat{G} : \widehat{\underline{Rep}(H_n)} \to Rep(H_n)$, which is an equivalence of categories. The semisimplification $\widehat{\underline{Rep}(H_n)}$ is semisimple.*

Proof. Proposition 2.3.11 and Definition 2.4.4 tell us that G is a $\mathbb{C}$-linear full tensor functor and as a consequence Proposition 2.5.11 implies that it reflects and preserves negligible properties. Because $\mathrm{Rep}(H_n)$ is semisimple, the only negligible morphisms in this category are the zero morphisms. This implies that the only negligible morphisms in $\underline{\mathrm{Rep}}(H_n)$ are the morphisms that lie in the kernel of

$$G = T : Hom_{\underline{\mathrm{Rep}}(H_n)}([k],[l]) = \mathbb{C}\, P_{even}(k,l) \to Hom_{H_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}).$$

If $\mathcal{N}$ is the tensor ideal of negligible morphisms, then it contains all the morphisms in the kernel of G and we get an induced $\mathbb{C}$-linear functor

$$\widehat{G} : \widehat{\underline{\mathrm{Rep}}(H_n)} \to \mathrm{Rep}(H_n)$$

which is faithful, full and essentially surjective. So we have an equivalence of categories. The semisimplicity of $\mathrm{Rep}(H_n)$ then implies the semisimplicity of $\widehat{\underline{\mathrm{Rep}}(H_n)}$. □

Remark 2.5.17. If we denote the quotient functor corresponding to the semisimplification by $G_{\mathcal{N}} : \underline{\mathrm{Rep}}(H_n) \to \widehat{\underline{\mathrm{Rep}}(H_n)}$, we get that $G = \widehat{G} \circ G_{\mathcal{N}}$.

Remark 2.5.18. Note that the proof shows that the interpolation categories $\underline{\mathrm{Rep}}(H_n)$ are not semisimple for $n \in \mathbb{N}$, because they have nontrivial negligible morphisms.

Remark 2.5.19. Following Remark 2.5.3 and Remark 2.5.7, $\widehat{\underline{\mathrm{Rep}}(H_n)}$ is a $\mathbb{C}$-linear spherical rigid symmetric monoidal category and the quotient functor $G_{\mathcal{N}}$ is a $\mathbb{C}$-linear full tensor functor. Because $G = \widehat{G} \circ G_{\mathcal{N}}$ is also a $\mathbb{C}$-linear tensor functor, so is $\widehat{G}$.

Now we want to take a short look at the approach of [FM21]. First they show that there exists an equivalence between the semisimplification of $\underline{\mathrm{Rep}}(H_n)$ and $\mathrm{Rep}(H_n)$. The following lemma holds in fact for all $\mathbb{C}$-linear rigid symmetric monoidal categories, see [AKO02, Proposition 7.1.4].

Lemma 2.5.20. *For any $t \in \mathbb{C}$, the ideal of negligible morphisms $\mathcal{N}$ is the unique maximal tensor ideal in $\underline{Rep}(H_t)$.*

Proof. Let $f : A \to B$ be a morphism in $\underline{\text{Rep}}(H_t)$ that isn't negligible. Then there exists a $g : B \to A$ such that $tr(f \circ g) \neq 0$. Recall that the trace is a an endomorphism of $[0]$, in this case a non-trivial one. Because it is achieved using tensor products and compositions, any tensor ideal $\mathcal{J}$ in $\underline{\text{Rep}}(H_t)$ containing f, will contain this non-trivial endomorphism and thus the identity on $[0]$. But this shows us that $\mathcal{J}$ will contain all of $\underline{\text{Rep}}(H_t)$, implying the maximality of $\mathcal{N}$. □

Remark 2.5.21. A C^*-algebra is a Banach algebra over $\mathbb{C}$ with an involution. Every finite dimensional C^*-algebra A is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$ with involution the conjugate transposes.

Lemma 2.5.22. *Assume a $\mathbb{C}$-linear Karoubian monoidal category $\mathcal{C}$ has finite-dimensional morphism spaces and admits a positive $*$-operation. Then it is semisimple (and hence, abelian).*

Proof. Because every idempotent splits and the hom-spaces are finite dimensional we know that every object can be written in an unique way as a finite direct sum of indecomposable objects. We need to prove that the indecomposable objects have no subobject, which shows that they are simple. For this it suffices to show that every morphism between indecomposable objects is either zero or an isomorphism. Then it would be impossible to have an inclusion of a proper non-trivial subobject in some indecomposable object. So assume $f : A \to B$ is a non-zero morphism between indecomposable objects. Because the endomorphism algebras of $\mathcal{C}$ are finite dimensional C^*-algebras without non-trivial idempotents we see that $a := f^* \circ f \in \mathcal{C}(A, A) \cong \mathbb{C}$ and $b := f \circ f^* \in \mathcal{C}(B, B) \cong \mathbb{C}$ are scalars. Then $fa = f \circ f^* \circ f = bf$ implies that $a = b$ which implies that $f^{-1} = \frac{1}{a} f^*$ and that f is an isomorphism. □

Remark 2.5.23. The horizontal reflection $*$ of partitions, extended $\mathbb{C}$-antilinearly, induces a $*$-operation on $\underline{\text{Rep}}(H_t)$ which is the identity on objects and is a contravariant involutive antilinear monoidal endofunctor.[2]

Proposition 2.5.24. *For any complex number t, assume $\mathcal{K} : \underline{Rep}(H_t) \to D$ is a (non-zero) monoidal $*$-functor, where D is a $\mathbb{C}$-linear Karoubian monoidal category $\mathcal{C}$ with finite-dimensional morphism spaces and a positive $*$-operation. Then $\mathcal{K}$ induces an equivalence between the semisimplification of $\underline{Rep}(H_t)$ and the image of $\mathcal{K}$.*

Proof. We first observe that the image $\text{Im}(\mathcal{K})$ is a $\mathbb{C}$-linear Karoubian monoidal subcategory of $\mathcal{D}$ with finite-dimensional morphism spaces and a positive $*$-operation. By Lemma 2.5.22 we see that $\text{Im}(\mathcal{K})$ is semisimple, which implies by Proposition 2.5.10 that there are no non-trivial negligible morphisms. As a consequence all negligible morphisms of $\underline{\text{Rep}}(H_t)$ lie in the kernel of $\mathcal{K}$, which is a tensor ideal. Then Lemma 2.5.20 implies that the negligible morphisms are exactly all morphisms in the kernel. Thus the functor $\widehat{\mathcal{K}} : \widehat{\underline{\text{Rep}}(H_t)} \to D$ is faithful and well defined. Because it is also full and essentially surjective, it is an equivalence of the semisimplification and the image of $\mathcal{K}$. □

Proposition 2.5.25. *For any $n \in \mathbb{N}$, the functor G induces an equivalence between the semisimplification of $\underline{Rep}(H_n)$ and $Rep(H_n)$.*

[2]Note that the $*$-operation differs from the endofunctor $(-)^*$, in Proposition 2.2.2, only by the fact that it is antilinear.

Proof. We want to apply Proposition 2.5.24 to G and the representation category $\mathrm{Rep}(H_n)$. It is easy to see that G is a monoidal $*$-functor and that $\mathrm{Rep}(H_n)$ is a $\mathbb{C}$-linear Karoubian monoidal category $\mathcal{C}$ with finite-dimensional morphism spaces and a positive $*$-operation. This shows that the induced functor is essentially surjective and fully faithful. □

2.5.4 Semisimplification of the interpolation categories in the permutation representations for the hyperoctahedral groups

Analogous as we did in the previous sections, we show that the interpolation functor $H : \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar} \to \mathrm{Rep}(H_n)$ induces an equivalence $\widehat{H} : \mathrm{Par}(\widehat{\mathbb{Z}_2, 2n})^{Kar} \to \mathrm{Rep}(H_n)$. This fact was neither mentioned nor proven in [LS21]. The fact that the interpolation categories $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ are semisimple if and only if $t \notin 2\,\mathbb{N}$ is proven in [Kno07][p.596, example 2]. We refer to [LS21, Chapter 9] for more details concerning this statement.

Proposition 2.5.26. *For all $n \in \mathbb{N}$ the interpolation functor $H : Par(\mathbb{Z}_2, 2n)^{Kar} \to Rep(H_n)$ induces a $\mathbb{C}$-linear functor $\widehat{H} : Par(\widehat{\mathbb{Z}_2, 2n})^{Kar} \to Rep(H_n)$, which is an equivalence of categories. The semisimplification $Par(\widehat{\mathbb{Z}_2, 2n})^{Kar}$ is semisimple.*

Proof. Proposition 2.3.19 and Definition 2.4.6 imply that H is a $\mathbb{C}$-linear full tensor functor. Then Proposition 2.5.11 implies that H reflects and preserves negligible properties. Because $\mathrm{Rep}(H_n)$ is semisimple, the only negligible morphisms in this category are the zero morphisms. This implies again that the only negligible morphisms in $\underline{\mathrm{Rep}}(H_n)$ are the morphisms that lie in the kernel of

$$H = T : Hom_{\mathrm{Par}(\mathbb{Z}_2,2n)^{Kar}}([\tilde{k}], [\tilde{l}]) = \mathbb{C}\, P_{\mathbb{Z}_2}(k, l) \to Hom_{H_n}((\mathbb{C}^{2n})^{\otimes k}, (\mathbb{C}^{2n})^{\otimes l}).$$

If $\mathcal{N}$ is the tensor ideal of negligible morphisms, then it contains all the morphisms in the kernel of H and we get an induced $\mathbb{C}$-linear functor

$$\widehat{H} : \mathrm{Par}(\widehat{\mathbb{Z}_2, 2n})^{Kar} \to \mathrm{Rep}(H_n)$$

which is faithful, full and essentially surjective. So we have an equivalence of categories. The semisimplicity of $\mathrm{Rep}(H_n)$ then implies the semisimplicity of $\mathrm{Par}(\widehat{\mathbb{Z}_2, 2n})^{Kar}$. □

Remark 2.5.27. If we denote the quotient functor corresponding to the semisimplification by $H_{\mathcal{N}} : \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar} \to \mathrm{Par}(\widehat{\mathbb{Z}_2, 2n})^{Kar}$, we get that $H = \widehat{H} \circ H_{\mathcal{N}}$.

Remark 2.5.28. Note that the proof shows that the interpolation categories $\mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$ are not semisimple for $n \in \mathbb{N}$, because they have nontrivial negligible morphisms.

Remark 2.5.29. Following Remark 2.5.3 and Remark 2.5.7, $\mathrm{Par}(\widehat{\mathbb{Z}_2, 2n})^{Kar}$ is a $\mathbb{C}$-linear spherical rigid symmetric monoidal category and the quotient functor $H_{\mathcal{N}}$ is a $\mathbb{C}$-linear full tensor functor. Because $H = \widehat{H} \circ H_{\mathcal{N}}$ is also a $\mathbb{C}$-linear tensor functor, so is $\widehat{H}$.

2.6 Universal properties of the interpolation categories

We describe the interpolation categories by giving isomorphisms to categories which are defined by generators and relations. This is done in [LS21, Chapter 4] for the interpolation categories $\underline{\mathrm{Rep}}(S_t)$ and $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$. This makes it possible to discuss the universal properties of the interpolation categories. This was already done in [Del07, Proposition 8.3] for the case $\underline{\mathrm{Rep}}(S_t)$. The proof we give for the universal properties, is in each case analogous to the proof in [EH22, Corollary 8.1.3].

2.6.1 Universal properties of the interpolation categories for the symmetric groups

In this section we describe $\underline{\mathrm{Rep}}_0(S_t)$ using generators and relations and use this result to give a universal property of $\underline{\mathrm{Rep}}(S_t)$.

Definition 2.6.1. $\mathrm{Par}(\{1\})/\sim_t$ is a strict $\mathbb{C}$-linear monoidal category with a single generating object W. The generating morphisms are

 $= id_W : W \to W$

 $: W \otimes W \to W$

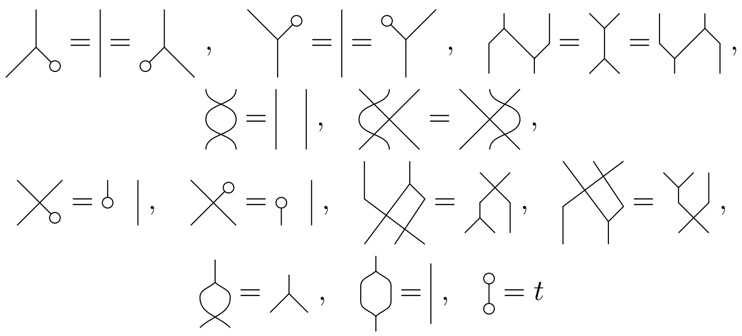
 $: W \to W \otimes W$

 $: W \otimes W \to W \otimes W$

 $: \mathbf{1} \to W$

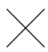
 $: W \to \mathbf{1}$

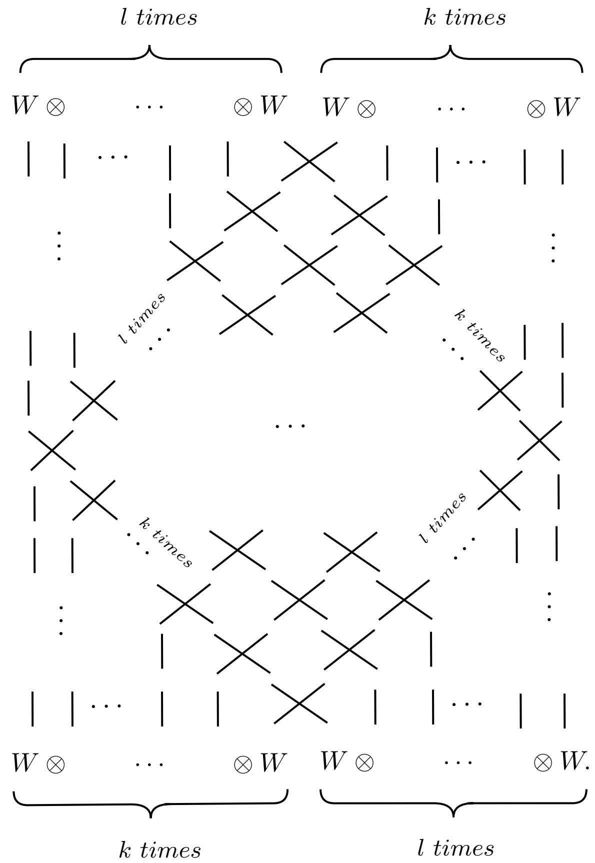
with the following relations:



(2.6.1)
(2.6.2)
(2.6.3)
(2.6.4)

Remark 2.6.2. It can be proven that all the possible horizontal and vertical reflections of the above relations will also hold, as a consequence of the given relations, see [LS21, Proposition 4.3].

Remark 2.6.3. Note that we can use the morphism  to define swap morphisms $s_{W^{\otimes k}, W^{\otimes l}} : W^{\otimes k} \otimes W^{\otimes l} \to W^{\otimes l} \otimes W^{\otimes k}$ for all $k, l \in \mathbb{N}$ by



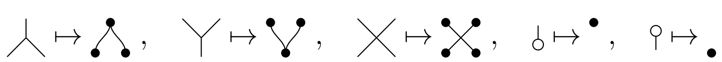
The first relation of 2.6.2 show us that all swap morphisms are isomorphisms. The second relation of (2.6.2) and the relations in (2.6.3) imply that the above swap morphisms are natural with respect to the other generating morphisms.

Altogether these relations are equivalent to the statement that Par($\{1\}$)/ $\sim_t$ is a symmetric monoidal category.

The following statement is already proven in [Com20, Theorem 2.1]. They refer to [Koc03, Chapter 1.4] in which a similar proof is given for the category $2Cob$, a free symmetric monoidal category generated by a commutative Frobenius algebra. We write this proof out in the language of Definition 2.6.1. In our case Par($\{1\}$)/ $\sim_t$ is a free symmetric monoidal category generated by an n-dimensional special commuative Frobenius algebra. So Par($\{1\}$)/ $\sim_t$ is equivalent to $2Cob$ with the second and third relation of (2.6.4) added. The second relation of 2.6.4 tells us that the occuring diagrams will contain no circles. In the source material this corresponds to saying that the cobordisms have genus 0 or that they have no handles.

Theorem 2.6.4. *The categories* $\underline{Rep}_0(S_t)$ *and Par*($\{1\}$)/ $\sim_t$ *are isomorphic as* $\mathbb{C}$*-linear symmetric monoidal categories.*

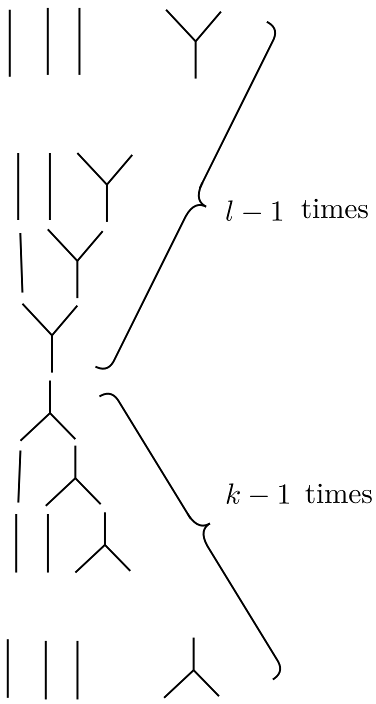
Proof. Define the functor $\tilde{F}$: Par($\{1\}$)/ $\sim_t \to \underline{\text{Rep}}_0(S_t)$ on objects by $W^{\otimes k} \mapsto [k]$. This makes it clearly bijective on the objects. On the morphisms we define it by sending:



and extending this rule $\mathbb{C}$-linearly. The fact that this choice is a well-defined one, can be seen by checking that the relations of Definition 2.6.1 are preserved by $\tilde{F}$ and this amounts to the fact that partition diagrams are the same when the parts are the same.

Let $p \in P(k,l)$ be a partition and $\phi(\sigma) \circ p' \circ \phi(\rho)$ some normal form of p, see Definition 2.1.5. To show that the functor is full, it suffices to show that the permutation partitions and the non-crossing form of the partition p lie in the image. The permutation partitions lie in the image because we can define a monoid homomorphism $\phi' : S_n \to \mathrm{Hom}_{\mathrm{Par}(\{1\})/\sim_t}(W^{\otimes n}, W^{\otimes n})$ for all $n \in \mathbb{N}$ by defining it on the transpositions analogously as we did for ϕ, see Definition 2.1.2. Then $\tilde{F} \circ \phi' = \phi$, showing the claim. To prove that the non-crossing forms of a partition lie in the image, it suffices to show that every one-part partition lies in the image, because the functor is monoidal. So let $q \in P(k,l)$ be a partition of one part. Then it lies in the image of the so-called star diagram $S_k^l \in \mathrm{Hom}_{\mathrm{Par}(\{1\})/\sim_t}(W^{\otimes k}, W^{\otimes l})$ which is defined for $k, l > 1$ by

Star diagram S_k^l

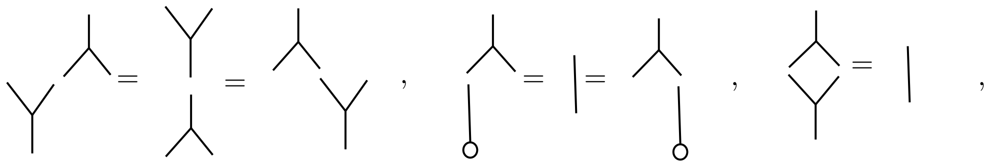


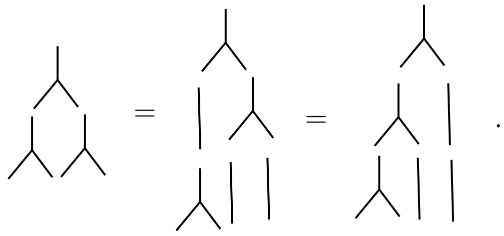
For $k = 1$ or $l = 1$ we replace respectively the lower part or the upper part of the star diagram by the identity partition. In case $k = 0$, we replace the lower part of the star diagram by ⫰. In case $l = 0$, we replace the upper part of the star diagram by ⫯. This defines S_k^l, for all $k, l \in \mathbb{N}$. So for any partition $p \in P(k,l)$ with $\phi(\sigma) \circ p' \circ \phi(\rho)$ some normal form, we see that it is the image of the morphism $f := \phi'(\sigma) \circ S \circ \phi'(\rho)$, where S is a tensor product of star diagrams corresponding to the blocks of p'. We call f a morphism in normal form, similarly as we did for the partition diagrams.

We are left to prove the faithfulness of the functor $\tilde{F}$. Note that we have a notion of parts and the size of parts of a morphism in $\mathrm{Par}(\{1\})/\sim_t$ which corresponds to the notion of parts and the size of parts of its image under $\tilde{F}$. We first show that for a one-part morphism $f \in \mathrm{Hom}_{\mathrm{Par}(\{1\})/\sim_t}(W^{\otimes k}, W^{\otimes l})$ with $f \in P(k,l)$, it is possible to apply the relations to get $f = S_k^l$.

Case 1 f contains no twists:

We want to move all the  down and all the  up, using the relations. We can use the relations (2.6.1) and (2.6.4) to get  past any obstacles on the way down. If all of them are down, we use associativity to get the wanted lower half of the star diagram:

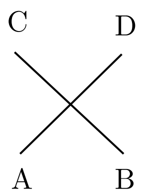




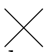
Moving the  down.

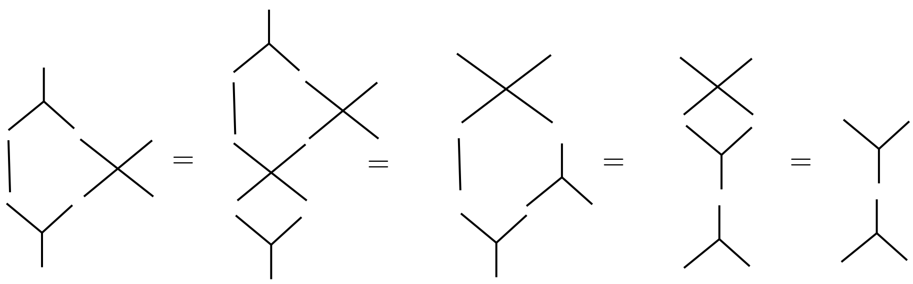
In a similar fashion we can move all  up and get our star diagram.

Case 2 f contains twists: Now we use induction on the number of twists to see how we can remove all twists. Suppose f contains some twist and looks like



where A, B, C, D are morphisms already in the form of a star diagram. We can assume either A and B are connected or A and C are connected, because f consists only of one part.

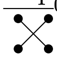
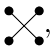
Lets first assume that A and B are connected. Then we can use associativity and locally get something of the form  $\circ$  which equals  by relation (2.6.4). If we assume that A and C are connected we see that we locally get an image, maybe after applying associativity if necessary, that looks like:



so that after applying (2.6.4), (2.6.3), (2.6.1) and (2.6.4) again, the twist is removed and we are back in case 1. So we have proven that a one part morphism f can be transformed to a star diagram.

Now we take an arbitrary $g \in \mathrm{Hom}_{\mathrm{Par}(\{1\})/\sim_t}(W^{\otimes k}, W^{\otimes l})$. We can find permutations σ and ρ such that $\phi(\sigma) \circ \tilde{F}(g) \circ \phi(\rho)$ is some non-crossing form of $\tilde{F}(g)$. Then we can of unknot the interwined parts of $\phi'(\sigma)^{-1} \circ g \circ \phi'(\rho)^{-1}$ using the twist relations in (2.6.2) and (2.6.3) to obtain a morphism $f_1 \otimes \ldots \otimes f_r$, where the f_i have as images the single blocks of the non-crossing form of $\tilde{F}(g)$. By the previous part we can write each f_i as a star diagram $S_{k_i}^{l_i}$ and we obtain the normal form

$$g = \phi'(\sigma) \circ S_{k_1}^{l_1} \otimes \ldots \otimes S_{k_r}^{l_r} \circ \phi'(\rho).$$

This shows that any two morphisms with the same image under $\tilde{F}$ will be equal to the same morphism in normal form. Because a $\mathbb{C}$-linear combination of morphisms in $\mathrm{Par}(\{1\})/\sim_t$ will be linearly independent if and only if its image under $\tilde{F}$ linearly independent, is $\tilde{F}$ is faithful. This concludes the proof that $\tilde{F}$ is an isomorphism between $\mathbb{C}$-linear monoidal categories. We note that the swap morphisms in $\mathrm{Par}(\{1\})/\sim_t$ and $\underline{\mathrm{Rep}}_0(S_t)$ can be constructed as explained in Remark 2.6.3 by the morphisms  and  respectively. Because $\tilde{F}$ is functorial and sends  to , it also preserves the swap morphisms. This shows that $\tilde{F}$ is a $\mathbb{C}$-linear symmetric monoidal functor.

□

Remark 2.6.5. The isomorphism $\tilde{F}$ induces an isomorphism

$$\tilde{F}^{Kar} : (\mathrm{Par}(\{1\})/\sim_t)^{Kar} \to \underline{\mathrm{Rep}}(S_t).$$

The theorem implies that $\mathrm{Par}(\{1\})/\sim_t$ has the structure of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category and that $\underline{\mathrm{Rep}}_0(S_t)$ satisfies the following universal property:

Theorem 2.6.6. *Let $t \in \mathbb{C}$. Let $\mathcal{C}$ be a $\mathbb{C}$-linear symmetric monoidal category with a t-dimensional special commutative Frobenius object $(A, \alpha, \delta, \beta, \epsilon)$ for an object $A \in Ob(\mathcal{C})$, a multiplication $\alpha : A \otimes A \to A$, a comultiplication $\beta : A \to A \otimes A$, a unit $\delta : \mathbf{1} \to A$ and a counit $\epsilon : A \to \mathbf{1}$. Let $\gamma := s^{\mathcal{C}}_{A,A} : A \otimes A \to A \otimes A$ be the swap morphism. These morphisms satisfy:*
1. A is a Frobenius object

$$\alpha \circ (id_A \otimes \delta) = id_A = \alpha \circ (\delta \otimes id_A),$$
$$(id_A \otimes \epsilon) \circ \beta = id_A = (\epsilon \otimes id_A) \circ \beta,$$
$$(\alpha \otimes id_A) \circ (id_A \otimes \beta) = \beta \circ \alpha = (id_A \otimes \alpha) \circ (\beta \otimes id_A).$$

2. A is commutative

$$\alpha \circ \gamma = \alpha.$$

3. A is special

$$\alpha \circ \beta = id_A.$$

4. A has dimension t

$$\epsilon \circ \delta = t.$$

Then there exists a unique strict $\mathbb{C}$-linear symmetric monoidal functor $\mathcal{F}' : \underline{Rep}_0(S_t) \to \mathcal{C}$, with $\mathcal{F}'([k]) = A^{\otimes k}$ for all $k \in \mathbb{N}$ and $\mathcal{F}'(\tilde{F}(\curlywedge)) = \alpha$, $\mathcal{F}'(\tilde{F}(\curlyvee)) = \beta$, $\mathcal{F}'(\tilde{F}(\times)) = \gamma$, $\mathcal{F}'(\tilde{F}(\underset{\circ}{|})) = \delta$ and $\mathcal{F}'(\tilde{F}(\overset{\circ}{|})) = \epsilon$. Furthermore, if $\mathcal{C}$ is Karoubi, there exists an up to isomorphism unique $\mathbb{C}$-linear symmetric monoidal functor $\mathcal{F} : \underline{Rep}(S_t) \to \mathcal{C}$ with $\mathcal{F}' = \mathcal{F} \circ \iota_F$.

Proof. We define the functor $\mathcal{F}'' : \mathrm{Par}(\{1\}/\sim_t \to \mathcal{C}$ on objects by $\mathcal{F}''(W^{\otimes k}) := A^{\otimes k}$ and on morphisms by the $\mathbb{C}$-linear extension of $\mathcal{F}''(\curlywedge) := \alpha, \mathcal{F}''(\curlyvee) := \beta, \mathcal{F}''(\times) := \gamma$, $\mathcal{F}''(\underset{\circ}{|}) := \delta$ and $\mathcal{F}''(\overset{\circ}{|}) := \epsilon$. Because $\mathcal{C}$ is a symmetric monoidal category, the relations (2.6.2) and (2.6.3) hold in this category, see Remark 2.6.7. One checks that all the necessary relations are satisfied and this implies that the functor $\mathcal{F}''$ is indeed a well-defined strict $\mathbb{C}$-linear symmetric monoidal functor. We then use Theorem 2.6.4 to define the strict $\mathbb{C}$-linear symmetric monoidal functor $\mathcal{F}' := \mathcal{F}'' \circ \tilde{F}^{-1}$. If $\mathcal{C}$ is Karoubi, the universal property of the Karoubian envelope implies the existence of the functor $\mathcal{F}$, see Remark 2.2.4, □

Remark 2.6.7. We use the notation of Theorem 2.6.6. Because $\mathcal{C}$ is a symmetric monoidal category, A and the morphisms $\alpha, \beta, \gamma, \delta$ and ϵ satisfy the following relations. The relation

$$\gamma \circ \gamma = id_A \otimes id_A$$

expresses that the swap morphisms, which one can describe using γ in a similar way as we did in Remark 2.6.3, are isomorphisms. The relations

$$\begin{aligned}
&(\gamma \otimes id_A) \circ (id_A \otimes \gamma) \circ (\gamma \otimes id_A) = (id_A \otimes \gamma) \circ (\gamma \otimes id_A) \circ (id_A \otimes \gamma),\\
&\gamma \otimes (id_A \otimes \delta) = \delta \otimes id_A,\\
&(id_A \otimes \epsilon) \circ \gamma = \epsilon \otimes id_A,\\
&(id_A \otimes \alpha) \circ (\gamma \otimes id_A) \circ (id_A \otimes \gamma) = \gamma \circ (\alpha \otimes id_A),\\
&(id_A \otimes \gamma) \circ (\gamma \otimes id_A) \circ (id_A \otimes \beta) = (\beta \otimes id_A) \circ \gamma.
\end{aligned}$$

express the naturality of these swap morphisms with respect to $\alpha, \beta, \gamma, \delta$ and ϵ.

Corollary 2.6.8. *Let $t \in \mathbb{C}$ and $\mathcal{C}$ be a $\mathbb{C}$-linear symmetric monoidal Karoubi category. Then there is an equivalence*

$$Fun_{\mathbb{C}}^{\otimes, Symm}(\underline{Rep}(S_t), \mathcal{C}) \simeq Frob_{\mathbb{C}}^{Spec}(\mathcal{C}, t)$$

between the category $Fun_{\mathbb{C}}^{\otimes, Symm}(\underline{Rep}(S_t), \mathcal{C})$ of $\mathbb{C}$-linear symmetric monoidal functors from $\underline{Rep}(S_t)$ to $\mathcal{C}$ with natural isomorphisms between them, and the subcategory $Frob_{\mathbb{C}}^{Spec}(\mathcal{C}, t)$ of $\mathcal{C}$ consisting of t-dimensional special commutative Frobenius objects and their isomorphisms.

Proof. We want to construct an inverse for the functor

$$\mathrm{Fun}_{\mathbb{C}}^{\otimes, Symm}(\underline{\mathrm{Rep}}(S_t), \mathcal{C}) \to \mathrm{Frob}_{\mathbb{C}}^{Spec}(\mathcal{C}, t)$$

defined by

$$\mathcal{F} \mapsto \mathcal{F}([1]) \text{ for all functors } \mathcal{F} : \underline{\mathrm{Rep}}(S_t) \to \mathcal{C} \text{ and}$$
$$\kappa \mapsto \kappa_A \text{ for all natural isomorphisms } \kappa : \mathcal{F}_1 \to \mathcal{F}_2.$$

Let A be a t-dimensional special commutative Frobenius object in $\mathcal{C}$. Then Theorem 2.6.6 implies the existence of a functor $\mathcal{F}^A : \underline{\mathrm{Rep}}(S_t) \to \mathcal{C}$ with the properties that were stated in the theorem. Let $\kappa : A \to B$ be an isomorphism in $\mathrm{Frob}_{\mathbb{C}}^{Spec}(\mathcal{C}, t)$. We define a natural isomorphism $\mathcal{F}^\kappa : \mathcal{F}^A \to \mathcal{F}^B$ as followed. First we define a natural isomorphism $\mathcal{F}^{\kappa,0}$ between $\mathcal{F}^{A,0} := \mathcal{F}^A \circ \iota_F : \underline{\mathrm{Rep}}_0(S_t) \to \mathcal{C}$ and $\mathcal{F}^{B,0} := \mathcal{F}^B \circ \iota_F : \underline{\mathrm{Rep}}_0(S_t) \to \mathcal{C}$ by

$$\mathcal{F}^{\kappa,0}_{[k]} := \kappa^{\otimes k} : A^{\otimes k} \to B^{\otimes k}$$

for all $k \in \mathbb{N}$. Let $f : [k] \to [l]$ be a morphism in $\underline{\mathrm{Rep}}_0(S_t)$. Because κ respects the structure of A and B, the following square commutes

$$\begin{array}{ccc} A^{\otimes k} & \xrightarrow{\mathcal{F}^{A,0}(f)} & A^{\otimes l} \\ \downarrow{\scriptstyle \kappa^{\otimes k}} & & \downarrow{\scriptstyle \kappa^{\otimes l}} \\ B^{\otimes k} & \xrightarrow{\mathcal{F}^{B,0}(f)} & B^{\otimes l}. \end{array}$$

This shows that $\mathcal{F}^{k,0}$ is a natural isomorphism. By the properties of the universal property of the Karoubian envelope it extends uniquely to a natural isomorphism $\mathcal{F}^\kappa : \mathcal{F}^A \to \mathcal{F}^B$ such that $\mathcal{F}^{\kappa,0} = \mathcal{F}^\kappa \circ \iota_F$. We obtained an inverse functor

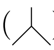
$$\mathrm{Frob}_{\mathbb{C}}^{Spec}(\mathcal{C}, t) \to \mathrm{Fun}_{\mathbb{C}}^{\otimes, Symm}(\underline{\mathrm{Rep}}(S_t), \mathcal{C}),$$

and this shows the equivalence. □

Alternative definition of interpolation functor $F : \underline{\mathbf{Rep}}_0(S_n) \to \mathbf{Rep}(S_n)$

Using the description above we define the interpolation functor $F' : \underline{\mathrm{Rep}}_0(S_n) \to \mathrm{Rep}(S_n)$ in a different way as we did before: by Theorem 2.6.6 we only have to specify the image of the generating morphisms and check that the necessary relations are functorial.

We define the strict $\mathbb{C}$-linear monoidal functor $F'' : \mathrm{Par}(\{1\})/\sim_n \to \mathrm{Rep}(H_n)$ on objects by sending W to $u' \in \mathrm{Rep}(S_n)$. The images of the generating objects are defined by:

$F''($$) : u' \otimes u' \to u',$ $\quad e_i \otimes e_j \mapsto \delta_{i,j} e_i,$

$F''($$) : u' \to u' \otimes u',$ $\quad e_i \mapsto e_i \otimes e_i,$

$F''($$) : u' \otimes u' \to u',$ $\quad v \otimes w \mapsto w \otimes v,$

$F''($$) : \mathbf{1} = \mathbb{C} \to u',$ $\quad 1 \mapsto \sum_{i=1}^n e_i,$

$F''($$) : u' \to \mathbb{C},$ $\quad e_i \mapsto 1.$

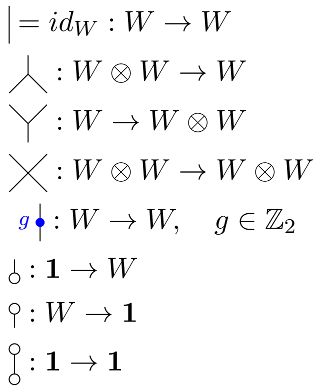
The fact that this yields a well-defined functor is proven in [LS21, Theorem 5.1].

The fact that $F' := F'' \circ \tilde{F}^{-1}$ indeed yields the same functor as before follows from the fact that $F'' \circ \tilde{F}^{-1} = T$ on the basis elements of the morphism spaces of $\underline{\mathrm{Rep}}_0(S_n)$.

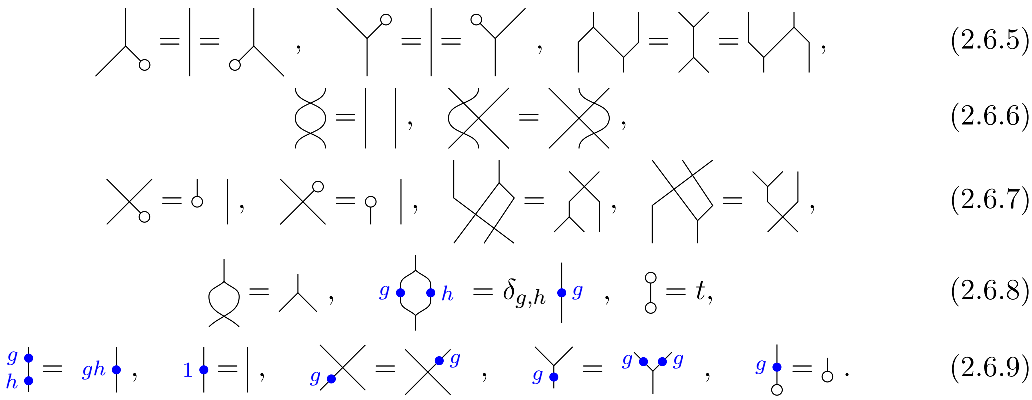
2.6.2 Universal properties of the interpolation categories in the permutation representations for the hyperoctahedral groups

In this section we describe $Par(\mathbb{Z}_2, 2n)$ using generators and relations and use this result to give a universal property of $Par(\mathbb{Z}_2, 2n)^{Kar}$. The description using generators and relations is already stated in [LS21, Theorem 4.4]. We wrote the proof out using the notation we introduced. We also elaborated at some points. The fundamental structure of the proof stays the same. The statement of the universal property in Corollary 2.6.16 is new.

Definition 2.6.9. $Par(\mathbb{Z}_2)/\sim_t$ is a strict $\mathbb{C}$-linear monoidal category with a single generating object W. The generating morphisms are



with the following relations for $g, h \in \mathbb{Z}_2$:

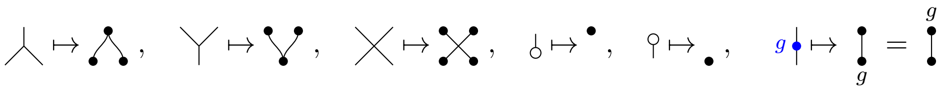


Remark 2.6.10. It can be proven that all the possible horizontal and vertical reflections of the above relations will also hold, as a consequence of the given relations, see [LS21, Proposition 4.3].

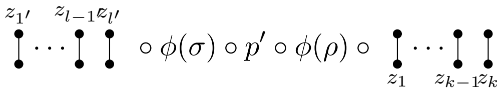
Remark 2.6.11. By the same argument as in Remark 2.6.3, $Par(\mathbb{Z}_2)/\sim_t$ is a symmetric monoidal category.

Theorem 2.6.12. *Let $n \in \mathbb{N}$. The categories $Par(\mathbb{Z}_2, t)$ and $Par(\mathbb{Z}_2)/\sim_t$ are isomorphic as $\mathbb{C}$-linear symmetric monoidal categories.*

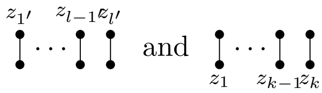
Proof. Define the functor $\tilde{H} : \mathrm{Par}(\mathbb{Z}_2)/\sim_t \to \mathrm{Par}(\mathbb{Z}_2, t)$ on objects by $W^{\otimes k} \mapsto [k]$. This makes it clearly bijective on the objects. On the morphisms we define it by sending:



and extending this rule $\mathbb{C}$-linearly. The fact that this choice is a well-defined one, can be seen by checking that the relations of Definition 2.6.9 are preserved by $\tilde{H}$ and this amounts to the fact that $\mathbb{Z}$-coloured partition diagrams are regarded the same when they are equivalent. Now we want to prove that $\tilde{H}$ is full. Let $(p, z) \in P_{\mathbb{Z}_2}(k, l)$ and let



be a normal form of (p, z), see Definition 2.1.15. It is clear that



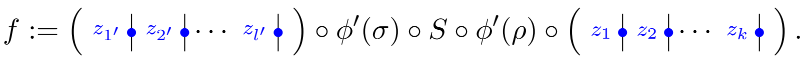
lie in the image of $\tilde{H}$, so it remains to show that $p = \phi(\sigma) \circ p' \circ \phi(\rho)$ lies in the image. We extend the homomorphism of the proof of Theorem 2.6.4

$$\phi' : S_n \hookrightarrow \mathrm{Hom}_{\mathrm{Par}(\{1\})/\sim_t}(W^{\otimes k}, W^{\otimes l}) \hookrightarrow \mathrm{Hom}_{\mathrm{Par}(\mathbb{Z}_2)/\sim_t}(W^{\otimes k}, W^{\otimes l})$$

by considering $Par(1)/\sim_t$ as a subcategory of $\mathrm{Par}(\mathbb{Z}_2)/\sim_t$ and see that $\tilde{H} \circ \phi' = \phi$. We noted in Remark 2.2.14 that $\underline{\mathrm{Rep}}_0(S_t)$ is a subcategory of $\mathrm{Par}(\mathbb{Z}_2, t)$. By combining these facts we get that the restriction

$$\tilde{H}|_{\mathrm{Par}(\{1\})/\sim_t} = \tilde{F} : \mathrm{Par}(\{1\})/\sim_t \to \underline{\mathrm{Rep}}_0(S_t) \subset \mathrm{Par}(\mathbb{Z}_2, t)$$

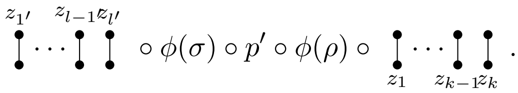
is the full functor of Theorem 2.6.4. This implies that p' lies in the image of $\tilde{H}$. Putting everything together we see that (p, z) lies in the image of



under $\tilde{H}$, where S is a tensor product of star diagrams such that $\tilde{H}(S) = \tilde{F}(S) = p'$. Similarly as before we call f a morphism in normal form.

Now we prove that the functor $\tilde{H}$ is faithful. Let $g \in \mathrm{Hom}_{\mathrm{Par}(\mathbb{Z}_2)/\sim_t}(W^{\otimes k}, W^{\otimes l})$ such that $g \in P_{\mathbb{Z}_2}(k, l)$, meaning that it consists of a single $\mathbb{Z}_2$-coloured partition with coefficient

1. We consider the $\mathbb{Z}_2$-coloured partition $\tilde{H}(g) = (p, z)$ and its normal form

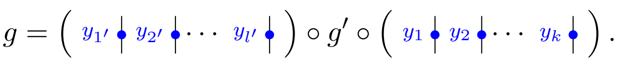


We claim that we can obtain

(2.6.10)

by applying the relations of Definition 2.6.9 to g. To do this we start with moving all the tokens on the diagram of g to the outer ends. This can be done by applying the relations in (2.6.8) and (2.6.9), and the corresponding relations which are obtained by reflecting their diagrams horizontally or vertically. We also need the relations

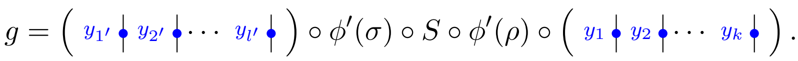
which are proven in [LS21, Proposition4.3] for $g \in \mathbb{Z}_2$. Let



be the obtained diagram with $y = \{y_1, \ldots, y_k, y_{1'}, \ldots, y_{l'}\} \in \mathbb{Z}_2^{k+l}$. Note that the inner part g' has no tokens and therefore lies in $\mathrm{Par}(\{1\})/\sim_t \subset \mathrm{Par}(\mathbb{Z}_2)/\sim_t$. Because $\tilde{F}(g') = \tilde{H}(g') = \phi(\sigma) \circ p' \circ \phi(\rho)$ we can then argument similarly as in the proof of Theorem 2.6.4 and apply the relations (2.6.5),(2.6.6),(2.6.7) and (2.6.8) to get that

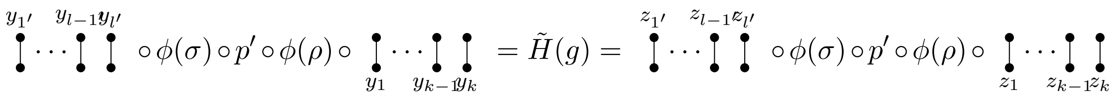
$$g' = \phi'(\sigma) \circ S \circ \phi'(\rho)$$

where S is a tenor product of star diagrams with image the blocks of p'. So we obtained

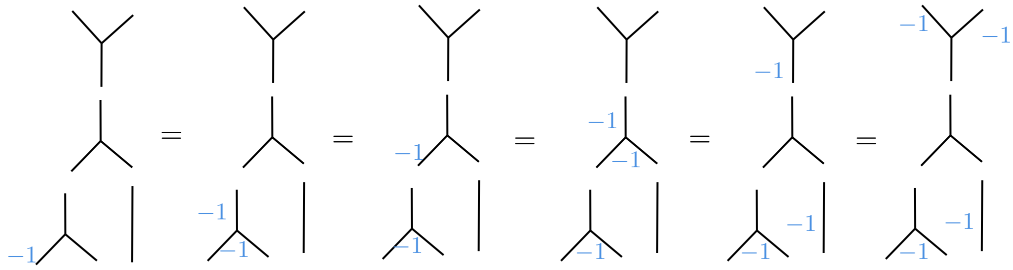


(2.6.11)

Because $\tilde{H}(g') = p = \phi(\sigma) \circ S \circ \phi(\rho)$, we see that

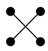
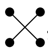


This implies that y and z are equivalent labelings of p, so $(p, z) \simeq (p, y)$. But by definition this means that their labelings for a block of p are either equal or differ up to multiplication by -1 of each label of the block. This means that for some star diagram S' in the tensor product S corresponding to a block of p, the tokens that are connected to its outer ends through the permutation diagrams $\phi'(\sigma)$ and $\phi'(\rho)$, will be either equal in (2.6.11) and (2.6.10) or differ by a factor of -1 for each end of the star diagram S'. But in both cases we can transform one case in the other by applying the first and the fourth relation of (2.6.9) to the star diagram S', as one can see in the following example.



This shows that

$$g = \left(z_{1'} \,\bullet\, z_{2'} \,\bullet \cdots z_{l'} \,\bullet \right) \circ \phi'(\sigma) \circ S \circ \phi'(\rho) \circ \left(z_1 \,\bullet\, z_2 \,\bullet \cdots z_k \,\bullet . \right)$$

This shows that any two morphisms with the same image under $\tilde{H}$ will be equal to the same morphism in normal form. Because a $\mathbb{C}$-linear combination of morphisms in $\mathrm{Par}(\mathbb{Z}_2)/\sim_t$ will be linearly independent if and only if its image under $\tilde{H}$ linearly independent, therefore that $\tilde{H}$ is faithful. This concludes the proof that $\tilde{H}$ is an isomorphism between $\mathbb{C}$-linear monoidal categories. We note that the swap morphisms in $\mathrm{Par}(\mathbb{Z}_2)/\sim_t$ and $\mathrm{Par}(\mathbb{Z}_2, t)$ can be constructed as explained in Remark 2.6.3 by the morphisms  and  respectively. Because $\tilde{H}$ is functorial and sends  to , it also preserves the swap morphisms. This shows that $\tilde{H}$ is a $\mathbb{C}$-linear symmetric monoidal functor.

□

Remark 2.6.13. The isomorphism $\tilde{H}$ induces an isomorphism

$$\tilde{H}^{Kar} : (\mathrm{Par}(\mathbb{Z}_{2)/\sim_t)^{Kar} \to (\mathrm{Par}(\mathbb{Z}_2,t))^{Kar}}.$$

The theorem implies that $\mathrm{Par}(\mathbb{Z}_{2)/\sim_t}$ has the structure of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category and that $\mathrm{Par}(\mathbb{Z}_2, t)$ satisfies the following universal property.

Theorem 2.6.14. *Let $t \in \mathbb{C}$. Let $\mathcal{C}$ be a $\mathbb{C}$-linear symmetric monoidal category with a t-dimensional special commutative Frobenius object with involution $(A, \alpha, \delta, \beta, \epsilon, \zeta)$ for an object $A \in Ob(\mathcal{C})$, a multiplication $\alpha : A \otimes A \to A$, a comultiplication $\beta : A \to A \otimes A$, a unit $\delta : \mathbf{1} \to A$, a counit $\epsilon : A \to \mathbf{1}$ and an involution $\zeta : A \to A$. Let $\gamma := s^{\mathcal{C}}_{A,A} : A \otimes A \to A \otimes A$ be the swap morphism. These morphisms satisfy:*
1. A is a Frobenius object

$$\alpha \circ (id_A \otimes \delta) = id_A = \alpha \circ (\delta \otimes id_A),$$
$$(id_A \otimes \epsilon) \circ \beta = id_A = (\epsilon \otimes id_A) \circ \beta,$$
$$(\alpha \otimes id_A) \circ (id_A \otimes \beta) = \beta \circ \alpha = (id_A \otimes \alpha) \circ (\beta \otimes id_A).$$

2. A is commutative

$$\alpha \circ \gamma = \alpha.$$

3. A is special

$$\alpha \circ \beta = id_A.$$

4. A has dimension t

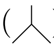
$$\epsilon \circ \delta = t.$$

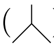
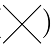
5. ζ is an involution

$$\zeta \circ \zeta = id_A.$$

6. As a special commutative Frobenius object, A is compatible with the involution γ[3]

$$\begin{aligned} \beta \circ \zeta &= (\zeta \otimes \zeta) \circ \beta, \\ \zeta \circ \delta &= \delta, \\ \alpha \circ (\zeta \otimes \zeta) \circ \beta &= \zeta, \\ \alpha \circ (id_A \otimes \zeta) \circ \beta &= 0 = \alpha \circ (\zeta \otimes \mathrm{id}_A) \circ \beta. \end{aligned}$$

Then there exists a unique strict $\mathbb{C}$-linear symmetric monoidal functor $\mathcal{H}' : Par(\mathbb{Z}_2, t) \to \mathcal{C}$, with $\mathcal{H}'([\tilde{k}]) = A^{\otimes k}$ for all $k \in \mathbb{N}$ and $\mathcal{H}'(\tilde{H}($$)) = \alpha$, $\mathcal{H}'(\tilde{H}($$)) = \beta$, $\mathcal{H}'(\tilde{H}($$)) = \gamma$, $\mathcal{H}'(\tilde{H}($$)) = \delta$, $\mathcal{H}'(\tilde{H}($$)) = \epsilon$ and $\mathcal{H}'(\tilde{H}($$)) = \zeta$. Furthermore, if $\mathcal{C}$ is Karoubi, there exists an up to isomorphism uniqe $\mathbb{C}$-linear symmetric monoidal functor $\mathcal{H} : Par(\mathbb{Z}_2, t)^{Kar} \to \mathcal{C}$ with $\mathcal{H}' = \mathcal{H} \circ \iota_H$.

Proof. We define the functor $\mathcal{H}'' : \mathrm{Par}(\{1\}/\sim_t \to \mathcal{C}$ on objects by $\mathcal{H}''(W^{\otimes k}) := A^{\otimes k}$ and on morphisms by the $\mathbb{C}$-linear extension of $\mathcal{H}''($$) := \alpha, \mathcal{H}''($$) := \beta, \mathcal{H}''($$) := \gamma$, $\mathcal{H}''($$) := \delta$, $\mathcal{H}''($$) := \epsilon$ and $\mathcal{H}''($$) := \zeta$. Because $\mathcal{C}$ is a symmetric monoidal category the relations (2.6.6),(2.6.7) and the third relation of (2.6.9) hold, see Remark 2.6.15. A direct verification shows that all the necessary relations are satisfied. This shows that the functor $\mathcal{H}''$ is a well-defined strict $\mathbb{C}$-linear symmetric monoidal functor. We use Theorem 2.6.12 and define $\mathcal{H}' := \mathcal{H}'' \circ \tilde{H}^{-1}$, which is the wanted C-linear strict symmetric monoidal functor. If $\mathcal{C}$ is Karoubi, the universal property of the Karoubian envelope implies the existence of the functor $\mathcal{H}$, see Remark 2.2.13, □

Remark 2.6.15. We use the notation of Theorem 2.6.14. Because $\mathcal{C}$ is a symmetric monoidal category, A and the morphisms $\alpha, \beta, \gamma, \delta, \epsilon$ and ζ satisfy the following relations. The relation

$$\gamma \circ \gamma = id_A \otimes id_A$$

expresses that the swap morphisms, which one can describe using γ in a similar way as we did in Remark 2.6.3, are isomorphisms. The relations

$$\begin{aligned} &(\gamma \otimes id_A) \circ (id_A \otimes \gamma) \circ (\gamma \otimes id_A) = (id_A \otimes \gamma) \circ (\gamma \otimes id_A) \circ (id_A \otimes \gamma), \\ &\gamma \otimes (id_A \otimes \delta) = \delta \otimes id_A, \end{aligned}$$

[3]The first 2 equations together with $\gamma \circ (\zeta \otimes id_A) = (id_A \otimes \zeta) \circ \gamma$ and $(\zeta \otimes id_A) \circ \gamma = \gamma \circ (id_A \otimes \zeta)$ say that the involution is some sort of natural transformation $id_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ which is restricted to the tensor powers of A and the compositions and tensor products of the morphisms $id_A, \alpha, \beta, \delta$ and ϵ. In that case one would define this 'natural transformation' as $\zeta_{A^{\otimes k}} := \zeta^{\otimes k}$.

$$\begin{aligned}
&(id_A \otimes \epsilon) \circ \gamma = \epsilon \otimes id_A,\\
&(id_A \otimes \alpha) \circ (\gamma \otimes id_A) \circ (id_A \otimes \gamma) = \gamma \circ (\alpha \otimes id_A),\\
&(id_A \otimes \gamma) \circ (\gamma \otimes id_A) \circ (id_A \otimes \beta) = (\beta \otimes id_A) \circ \gamma,\\
&\gamma \circ (\zeta \otimes id_A) = (id_A \otimes \zeta) \circ \gamma,\\
&(\zeta \otimes id_A) \circ \gamma = \gamma \circ (id_A \otimes \zeta).
\end{aligned}$$

express the naturality of the swap morphisms with respect to $\alpha, \beta, \gamma, \delta, \epsilon$ and ζ.

Corollary 2.6.16. *Let $t \in \mathbb{C}$ and $\mathcal{C}$ be a $\mathbb{C}$-linear symmetric monoidal Karoubi category. Then there is an equivalence*

$$Fun_{\mathbb{C}}^{\otimes,Symm}(Par(\mathbb{Z}_2,t)^{Kar},\mathcal{C}) \simeq Frob_{\mathbb{C}}^{Spec,inv}(\mathcal{C},t)$$

between the category $Fun_{\mathbb{C}}^{\otimes,Symm}(Par(\mathbb{Z}_2,t)^{Kar},\mathcal{C})$ of $\mathbb{C}$-linear symmetric monoidal functors from $Par(\mathbb{Z}_2,t)^{Kar}$ to $\mathcal{C}$ with natural isomorphisms between them, and the subcategory $Frob_{\mathbb{C}}^{Spec,inv}(\mathcal{C},t)$ of $\mathcal{C}$ consisting of t-dimensional special commutative Frobenius objects with involution and their isomorphisms.

Proof. We want to construct an inverse for the functor

$$\mathrm{Fun}_{\mathbb{C}}^{\otimes,Symm}(\mathrm{Par}(\mathbb{Z}_2,t)^{Kar},\mathcal{C}) \to \mathrm{Frob}_{\mathbb{C}}^{Spec,inv}(\mathcal{C},t)$$

defined by

$$\begin{aligned}
&\mathcal{H} \mapsto \mathcal{G}([\tilde{1}]) \text{ for all functors } \mathcal{H}: \mathrm{Par}(\mathbb{Z}_2,t)^{Kar} \to \mathcal{C} \text{ and}\\
&\kappa \mapsto \kappa_A \text{ for all natural isomorphisms } \kappa: \mathcal{H}_1 \to \mathcal{H}_2.
\end{aligned}$$

Let A be a t-dimensional special commutative Frobenius object with involution in $\mathcal{C}$. Then theorem 2.6.14 implies the existence of a functor $\mathcal{H}^A : \mathrm{Par}(\mathbb{Z}_2,t)^{Kar} \to \mathcal{C}$ with the properties that were stated in the theorem. Let $\kappa : A \to B$ be an isomorphism in $\mathrm{Frob}_{\mathbb{C}}^{Spec,inv}(\mathcal{C},t)$. We define a natural isomorphism $\mathcal{H}^\kappa : \mathcal{H}^A \to \mathcal{H}^B$ as followed. First we define a natural isomorphism $\mathcal{H}^{\kappa,0}$ between $\mathcal{H}^{A,0} := \mathcal{G}^A \circ \iota_H : \mathrm{Par}(\mathbb{Z}_2,t) \to \mathcal{C}$ and $\mathcal{H}^{B,0} := \mathcal{H}^B \circ \iota_H : \mathrm{Par}(\mathbb{Z}_2,t) \to \mathcal{C}$ by

$$\mathcal{H}_{[k]}^{\kappa,0} := \kappa^{\otimes k} : A^{\otimes k} \to B^{\otimes k}$$

for all $k \in \mathbb{N}$. Let $f : [k] \to [l]$ be a morphism in $\mathrm{Par}(\mathbb{Z}_2,t)$. Because κ respects the structure of A and B, the following square commutes

$$\begin{array}{ccc}
A^{\otimes k} & \xrightarrow{\mathcal{H}^{A,0}(f)} & A^{\otimes l}\\
\downarrow{\scriptstyle \kappa^{\otimes k}} & & \downarrow{\scriptstyle \kappa^{\otimes l}}\\
B^{\otimes k} & \xrightarrow{\mathcal{H}^{B,0}(f)} & B^{\otimes l}.
\end{array}$$

This shows that $\mathcal{H}^{k,0}$ is a natural isomorphism. By the properties of the universal property of the Karoubian envelope it extends uniquely to a natural isomorphism $\mathcal{H}^\kappa : \mathcal{H}^A \to \mathcal{G}^B$

such that $\mathcal{H}^{\kappa,0} = \mathcal{H}^{\kappa} \circ \iota_H$. We obtained an inverse functor

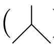
$$\mathrm{Frob}_{\mathbb{C}}^{Spec,inv}(\mathcal{C}, t) \to \mathrm{Fun}_{\mathbb{C}}^{\otimes,Symm}(\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}, \mathcal{C}),$$

and this shows the equivalence. $\square$

Alternative definition of interpolation functor $H' : \mathbf{Par}(\mathbb{Z}_2, 2n) \to \mathbf{Rep}(H_n)$

Using the description above we define the interpolation functor $H' : \mathrm{Par}(\mathbb{Z}_2, 2n) \to \mathrm{Rep}(H_n)$ in a different way as we did before: we only have to specify the image of the generating morphisms and check that the necessary relations are functorial.

We define the strict $\mathbb{C}$-linear monoidal functor $H'' : \mathrm{Par}(\mathbb{Z}_2)/\sim_n \to \mathrm{Rep}(H_n)$ on objects by sending W to $V \in \mathrm{Rep}(H_n)$. The images of the generating objects are defined by:

$H''($$) : V \otimes V \to V, \quad e_g^i \otimes e_h^j \mapsto \delta_{g,h}\delta_{i,j} e_g^i,$

$H''($$) : V \to V \otimes V, \quad e_g^i \mapsto e_g^i \otimes e_g^i,$

$H''($$) : V \otimes V \to V, \quad v \otimes w \mapsto w \otimes v,$

$H''($$) : \mathbb{C} \to V, \quad 1 \mapsto \sum_{g \in G} \sum_{i=1}^{n} e_g^i,$

$H''($$) : V \to \mathbb{C}, \quad e_g^i \mapsto 1,$

$H''($$) : V \to V, \quad e_h^i \mapsto e_{gh}^i$

for $g, h \in \mathbb{Z}_2$. The fact that this is well-defined is proven in [LS21, Theorem 5.1].

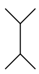
The fact that $H' := H'' \circ \tilde{H}^{-1}$ indeed yields the same functor as before follows from the fact that $H'' \circ \tilde{H}^{-1} = T$ on the generating morphisms and therefore on the basis elements of the morphism spaces.

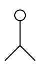
2.6.3 Universal properties of the interpolation categories in the reflection representations for the hyperoctahedral groups

In this section we describe $\underline{\mathrm{Rep}}_0(H_t)$ using generators and relations and use this result to give a universal property of $\underline{\mathrm{Rep}}(H_t)$. The description in Definition 2.6.17 is a new one. The structure on the generating object is a bit different from the previous cases, but it is still possible to discuss the universal property of this category, see Corollary 2.6.23, and it will turn out to be a very useful tool in proving the main theorem of this thesis.

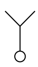
Definition 2.6.17. Par_t is a strict $\mathbb{C}$-linear monoidal category with a single generating object W. The generating morphisms are

$| = id_W : W \to W$

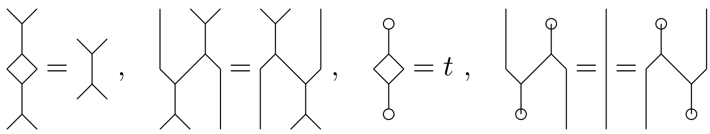
 $: W \otimes W \to W \otimes W$

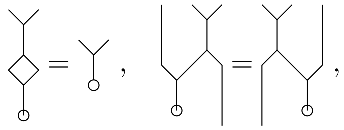
 $: W \otimes W \to \mathbf{1}$

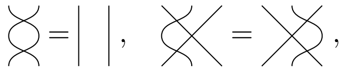
 $: W \otimes W \to W \otimes W$

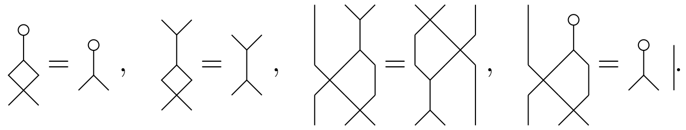
 $: \mathbf{1} \to W \otimes W$

with the following relations and the relations obtained by reflecting them horizontally and/or vertically:


(2.6.12)


(2.6.13)

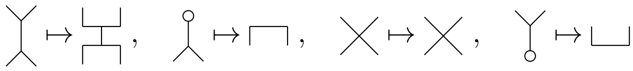

(2.6.14)


(2.6.15)

Remark 2.6.18. By the same argument as in Remark 2.6.3, Par_t is a symmetric monoidal category.

Theorem 2.6.19. *The categories $\underline{\mathrm{Rep}}_0(H_t)$ and Par_t are isomorphic as $\mathbb{C}$-linear symmetric monoidal categories.*

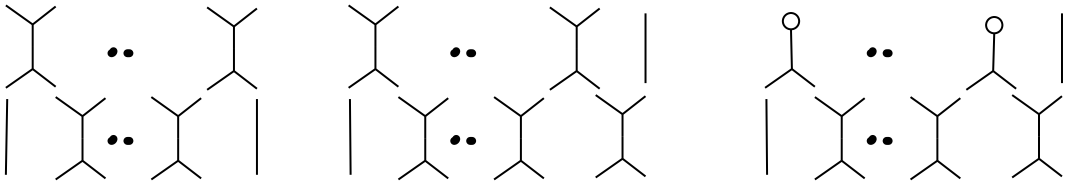
Proof. Define the functor $\tilde{G} : \mathrm{Par}_t \to \underline{\mathrm{Rep}}_0(H_t)$ on objects by $W^{\otimes k} \mapsto [k]$. This makes it bijective on the objects. On the morphisms we define it by sending:



and extending this choice $\mathbb{C}$-linearly. The fact that this choice is well-defined, can be seen by checking that the relations of Definition 2.6.17 are preserved by $\tilde{G}$ and this amounts to the fact that the partition diagrams are the same when the parts are the same.

Let $p \in P_{even}(k,l)$ for $k+l$ even. We note that every permutation partition is an even partition, see Definition 2.1.2, so we can restrict the image of $\phi : S_n \hookrightarrow P_{even}(n,n) \subset P(n,n)$ to the even partitions. Now let $\phi(\sigma) \circ p' \circ \phi(\rho)$ be some normal form of p, see Definition 2.1.5, and note that p' is also an even partition. To show that the functor $\tilde{G}$ is full, it suffices to show that the permutation partitions and the non-crossing form of the partition p lie in the image. The permutation partitions lie in the image because we can define a monoid homomorphism $\phi' : S_n \to \mathrm{Hom}_{\mathrm{Par}_t}(W^{\otimes n}, W^{\otimes n})$ for all $n \in \mathbb{N}$, by defining it on the transpositions analogously as we did in Definition 2.1.2, such that $\tilde{G} \circ \phi' = \phi$. Now we need to prove that the non-crossing form p' lies in the image, but because $\tilde{G}$ is monoidal, we only need to show that any even one-part partition lies in the image. The proof of Proposition 2.1.8 shows that we can write every even block of size (k,l) with $k > l$ in the form $s_k(t_{k-l+1} \otimes \underbrace{| \otimes \ldots \otimes |}_{l-1 \text{ times}})$. So we only have to show that for all $n \in \mathbb{N}$ the partitions s_n

and t_n lie in the image of $\tilde{G}$. Similar arguments apply then to the cases $k \leqslant l$. We define morphisms s'_n and t'_n by



s'_n for n even $\qquad$ s'_n for n odd $\qquad$ t'_n

and see that $\tilde{G}(s'_n) = s_n$ and $\tilde{G}(t'_n) = t_n$. As a consequence

$$\tilde{G}(s'_l(t'_{k-l+1} \otimes \underbrace{| \otimes \ldots \otimes |}_{l-1 \text{ times}})) = s_l(t_{k-l+1} \otimes \underbrace{| \otimes \ldots \otimes |}_{l-1 \text{ times}})$$

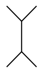
is an even block of size (k, l). We call morphisms of the form $s'_l(t'_{k-l+1} \otimes \underbrace{| \otimes \ldots \otimes |}_{l-1 \text{ times}})$, and their counterparts for $k \leqslant l$, even block diagrams. So for any partition $p \in P(k,l)$ with $\phi(\sigma) \circ p' \circ \phi(\rho)$ some normal form, we see that it is the image of the morphism $f := \phi'(\sigma) \circ B \circ \phi'(\rho)$, where B is a tensor product of even block diagrams corresponding to the blocks of p'. We call f a morphism in normal form.

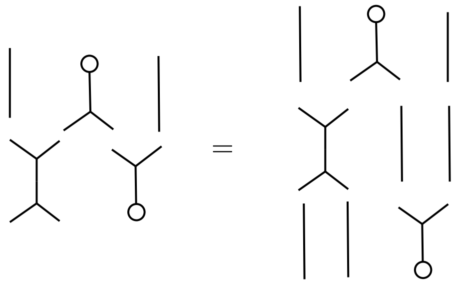
Now we prove that the functor $\tilde{G}$ is faithful. Let $g \in \mathrm{Hom}_{\mathrm{Par}_t}(W^{\otimes k}, W^{\otimes l})$ be an even partition. We can find permutations σ and ρ such that $\phi(\sigma) \circ \tilde{G}(g) \circ \phi(\rho)$ is some non-crossing form of $\tilde{G}(g)$. Then we can unknot the interwined parts of $\phi'(\sigma)^{-1} \circ g \circ \phi'(\rho)^{-1}$ using the twist relations in (2.6.14) and (2.6.15) to obtain a morphism $f_1 \otimes \ldots \otimes f_r$, where the f_i have as images the single blocks of the non-crossing form of $\tilde{G}(g)$. If we can prove that every f_i equals some even block diagram B_i by using the relations, then we have showed that g equals the normal form

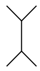
$$g = \phi'(\sigma) \circ B_1 \otimes \ldots \otimes B_r \circ \phi'(\rho).$$

This in its turn shows that any two morphisms with the same image under $\tilde{G}$ will be equal to the same morphism in normal form. Because a $\mathbb{C}$-linear combination of morphisms in Par_t will be linearly independent if and only if its image under $\tilde{G}$ is linearly independent, $\tilde{G}$ is then faithful.

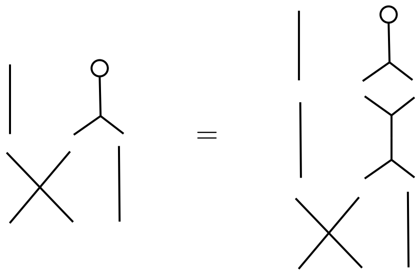
For $k > l$ we show that it is possible to apply the relations to the one-part morphism $f \in \mathrm{Hom}_{\mathrm{Par}_t}(W^{\otimes k}, W^{\otimes l})$ to obtain an even block diagram of the form $s'_l(t'_{k-l+1} \otimes \underbrace{| \otimes \ldots \otimes |}_{l-1 \text{ times}})$. The argument for the cases $k \leqslant l$ is similar.

Step 1: We draw the diagram in such a way that we have a middle part consisting only of compositions of the morphisms $|$,  and . This is done by adding identities, for example

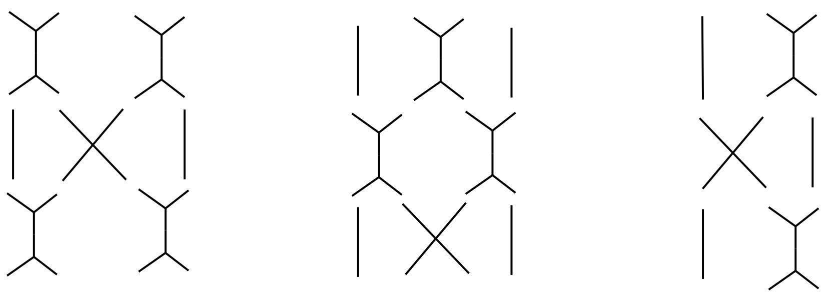


Step 2: We assume the middle part does not contain any crosses and change it in the form s'_n by using the first two relations of (2.6.12) and the first one of (2.6.13). This tells us then that all one-part morphisms of size (k, l) generated only by $|$ and  are the same in $Part_t$.

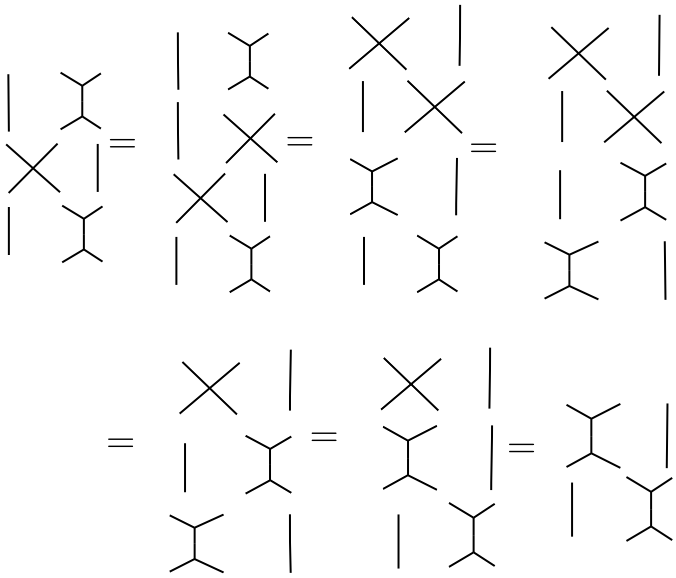
Step 3: Assume there are crosses in the middle part. Firstly we note that the crosses don't touch the outer parts by using that the first relation of (2.6.13) or its reflection, for example

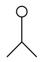
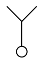
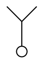


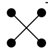
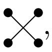
Because the morphism exists in one part we can assume by induction on the number of crosses, as we did in the symmetric group case and after applying the second relation of (2.6.15) to trivially remove crosses, that a cross will appear in one of the following settings:



The cross can be removed in all these settings by using the second and third relations of (2.6.15) and the last fact from the previous case. For example:



Step 4: We move all the  and  to the left using the second relation of (2.6.13) and its reflection after which we can start removing the superfluous ones using the the third and fourth relation of (2.6.12). We end without  because $k > l$.

This concludes the proof that $\tilde{G}$ is an isomorphism between $\mathbb{C}$-linear monoidal categories. We note that the swap morphisms in Par_t and $\underline{\text{Rep}}_0(H_t)$ can be constructed as explained in Remark 2.6.3 by the morphisms  and  respectively. Because $\tilde{G}$ is functorial and sends  to , it also preserves the swap morphisms. This shows that $\tilde{G}$ is a $\mathbb{C}$-linear symmetric monoidal functor.

□

Remark 2.6.20. The isomorphism $\tilde{G}$ induces an isomorphism $\tilde{G}^{Kar} : (\text{Par}_t)^{Kar} \to \underline{\text{Rep}}(H_t)$.

The theorem implies that Par_t has the structure of a $\mathbb{C}$-linear spherical rigid symmetric monoidal category and that $\underline{\text{Rep}}_0(H_t)$ satisfies the following universal property.

Theorem 2.6.21. *Let $t \in \mathbb{C}$. Let $\mathcal{C}$ be a $\mathbb{C}$-linear symmetric monoidal category with a t-dimensional self-dual rigid object with neutralizer $(A, \alpha, \beta, \delta)$ for an object $A \in Ob(\mathcal{C})$, a neutralizer $\alpha : A \otimes A \to A \otimes A$, an evaluation $\beta : A \otimes A \to \mathbf{1}$ and a coevaluation $\delta : \mathbf{1} \to A \otimes A$. Let $\gamma := s^{\mathcal{C}}_{A,A} : A \otimes A \to A \otimes A$ be the swap morphism. These morphisms satisfy:*

1. A is rigid and self-dual

$$(id_A \otimes \beta) \circ (\delta \otimes id_A) = id_A = (\beta \otimes id_A) \circ (id_A \otimes \delta).$$

2. A has dimension t

$$\delta \circ \beta = t.$$

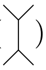
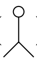
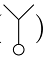
3. α is a neutralizer

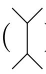
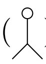
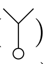
$$\alpha \circ \alpha = \alpha,$$

$$(id_A \otimes \alpha) \circ (\alpha \otimes id_A) = (\alpha \otimes id_A) \circ (id_A \otimes \alpha).$$

4. As a rigid object, A is compatible with the neutralizer

$$\alpha \circ \delta = \delta,$$
$$\beta \circ \alpha = \beta,$$
$$(id_A \otimes \alpha) \circ (\delta \otimes id_A) = (\alpha \otimes id_A) \circ (id_A \otimes \delta),$$
$$(\beta \otimes id_A) \circ (id_A \otimes \alpha) = (id_A \otimes \beta) \circ (\alpha \otimes id_A).$$

Then there exists a unique strict $\mathbb{C}$-linear symmetric monoidal functor $\mathcal{G}' : \underline{Rep}_0(H_n) \to \mathcal{C}$, with $\mathcal{G}'([k]) = A^{\otimes k}$ for all $k \in \mathbb{N}$ and $\mathcal{G}'(\tilde{G}($$)) = \alpha$, $\mathcal{G}'(\tilde{G}($$)) = \beta$, $\mathcal{G}'(\tilde{G}($$)) = \gamma$ and $\mathcal{G}'(\tilde{G}($$)) = \delta$. Furthermore, if $\mathcal{C}$ is Karoubi, there exists an up to isomorphism unique $\mathbb{C}$-linear symmetric monoidal functor $\mathcal{G} : \underline{Rep}(H_t) \to \mathcal{C}$ with $\mathcal{G}' = \mathcal{G} \circ \iota_G$.

Proof. We define the functor $\mathcal{G}'' : \mathrm{Par}_t \to \mathcal{C}$ on objects by $\mathcal{G}''(W^{\otimes k}) := A^{\otimes k}$ and on morphisms by the $\mathbb{C}$-linear extension of $\mathcal{G}''($$) := \alpha, \mathcal{G}''($$) := \beta, \mathcal{G}''($$) := \gamma$ and $\mathcal{G}''($$) := \delta$. Because $\mathcal{C}$ is a symmetric monoidal category the relations (2.6.13) and (2.6.14) hold, see Remark 2.6.22. A direct verification shows that all the necessary relations are satisfied. This shows that the functor $\mathcal{G}''$ is a well-defined strict $\mathbb{C}$-linear symmetric monoidal functor. We use Theorem 2.6.19 to define the strict $\mathbb{C}$-linear symmetric monoidal functor $\mathcal{G}' := \mathcal{G}'' \circ \tilde{G}^{-1}$. If $\mathcal{C}$ is Karoubi, the universal property of the Karoubian envelope implies the existence of the functor $\mathcal{G}$, see Remark 2.2.8, □

Remark 2.6.22. We use the notation of Theorem 2.6.21. Because $\mathcal{C}$ is a symmetric monoidal category, A and the morphisms α, β, γ and δ satisfy the following relations. The relation

$$\gamma \circ \gamma = id_A \otimes id_A$$

expresses that the swap morphisms, which one can describe using γ in a similar way as we did in Remark 2.6.3, are isomorphisms. The relations

$$(\gamma \otimes id_A) \circ (id_A \otimes \gamma) \circ (\gamma \otimes id_A) = (id_A \otimes \gamma) \circ (\gamma \otimes id_A) \circ (id_A \otimes \gamma),$$
$$\beta \circ \gamma = \beta,$$
$$\gamma \circ \delta = \gamma,$$
$$(id_A \otimes \beta) \circ (\gamma \otimes id_A) \circ (id_A \otimes \gamma) = \beta \otimes id_A,$$
$$(\beta \otimes id_A) \circ (id_A \otimes \gamma) \circ (\gamma \otimes id_A) = id_A \otimes \beta,$$
$$(\gamma \otimes id_A) \circ (id_A \otimes \gamma) \circ (\delta \otimes id_A) = id_A \otimes \delta,$$
$$(id_A \otimes \gamma) \circ (\gamma \otimes id_A) \circ (id_A \otimes \delta) = \delta \otimes id_A,$$
$$\alpha \circ \gamma = \alpha = \gamma \circ \alpha,$$
$$(id_A \otimes \alpha) \circ (\gamma \otimes id_A) \circ (id_A \otimes \gamma) = (\gamma \otimes id_A) \circ (id_A \otimes \gamma) \circ (\alpha \otimes id_A),$$
$$(\alpha \otimes id_A) \circ (id_A \otimes \gamma) \circ (\gamma \otimes id_A) = (id_A \otimes \gamma) \circ (\gamma \otimes id_A) \circ (id_A \otimes \alpha)$$

express the naturality of the swap morphisms with respect to α, β, γ and δ.

Corollary 2.6.23. *Let $t \in \mathbb{C}$ and $\mathcal{C}$ be a $\mathbb{C}$-linear symmetric monoidal Karoubi category. Then there is an equivalence*

$$Fun_{\mathbb{C}}^{\otimes,Symm}(\underline{Rep}(H_t),\mathcal{C}) \simeq Rig_{\mathbb{C}}^{sd,neutr}(\mathcal{C},t)$$

between the category $Fun_{\mathbb{C}}^{\otimes,Symm}(\underline{Rep}(H_t),\mathcal{C})$ of $\mathbb{C}$-linear symmetric monoidal functors from $\underline{Rep}(H_t)$ to $\mathcal{C}$ with natural isomorphisms between them, and the subcategory $Rig_{\mathbb{C}}^{sd,neutr}(\mathcal{C},t)$ of $\mathcal{C}$ consisting of t-dimensional rigid self-dual objects with neutralizer and their isomorphisms.

Proof. We want to construct an inverse for the functor

$$\mathrm{Fun}_{\mathbb{C}}^{\otimes,Symm}(\underline{\mathrm{Rep}}(H_t),\mathcal{C}) \to \mathrm{Rig}_{\mathbb{C}}^{sd,neutr}(\mathcal{C},t)$$

defined by

$$\mathcal{G} \mapsto \mathcal{G}([1]) \text{ for all functors } \mathcal{G} : \underline{\mathrm{Rep}}(H_t) \to \mathcal{C} \text{ and}$$
$$\kappa \mapsto \kappa_A \text{ for all natural isomorphisms } \kappa : \mathcal{G}_1 \to \mathcal{G}_2.$$

Let A be a t-dimensional rigid self-dual object with neutralizer in $\mathcal{C}$. Then theorem 2.6.21 implies the existence of a functor $\mathcal{G}^A : \underline{\mathrm{Rep}}(H_t) \to \mathcal{C}$ with the properties that were stated in the theorem. Let $\kappa : A \to B$ be an isomorphism in $\mathrm{Rig}_{\mathbb{C}}^{sd,neutr}(\mathcal{C},t)$. We define a natural isomorphism $\mathcal{G}^\kappa : \mathcal{G}^A \to \mathcal{G}^B$ as followed. First we define a natural isomorphism $\mathcal{G}^{\kappa,0}$ between $\mathcal{G}^{A,0} := \mathcal{G}^A \circ \iota_G : \underline{\mathrm{Rep}}_0(H_t) \to \mathcal{C}$ and $\mathcal{G}^{B,0} := \mathcal{G}^B \circ \iota_G : \underline{\mathrm{Rep}}_0(H_t) \to \mathcal{C}$ by

$$\mathcal{G}^{\kappa,0}_{[k]} := \kappa^{\otimes k} : A^{\otimes k} \to B^{\otimes k}$$

for all $k \in \mathbb{N}$. Let $f : [k] \to [l]$ be a morphism in $\underline{\mathrm{Rep}}_0(H_t)$. Because κ respects the structure of A and B, the following square commutes

$$\begin{array}{ccc} A^{\otimes k} & \xrightarrow{\mathcal{G}^{A,0}(f)} & A^{\otimes l} \\ \downarrow{\kappa^{\otimes k}} & & \downarrow{\kappa^{\otimes l}} \\ B^{\otimes k} & \xrightarrow{\mathcal{G}^{B,0}(f)} & B^{\otimes l}. \end{array}$$

This shows that $\mathcal{G}^{k,0}$ is a natural isomorphism. By the properties of the universal property of the Karoubian envelope it extends uniquely to a natural isomorphism $\mathcal{G}^\kappa : \mathcal{G}^A \to \mathcal{G}^B$ such that $\mathcal{G}^{\kappa,0} = \mathcal{G}^\kappa \circ \iota_G$. We obtained an inverse functor

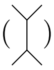
$$\mathrm{Rig}_{\mathbb{C}}^{sd,neutr}(\mathcal{C},t) \to \mathrm{Fun}_{\mathbb{C}}^{\otimes,Symm}(\underline{\mathrm{Rep}}(H_t),\mathcal{C}),$$

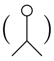
and this shows the equivalence. □

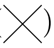
Alternative definition of interpolation functor $G' : \underline{\mathbf{Rep}}_0(H_n) \to \mathbf{Rep}(H_n)$

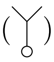
Using the description above we define the interpolation functor $G' : \underline{\mathrm{Rep}}_0(H_n) \to \mathrm{Rep}(H_n)$ in a different way as we did before: we only have to specify the image of the generating morphisms and check that the necessary relations are functorial.

We define the strict $\mathbb{C}$-linear monoidal functor $G'' : \mathrm{Par}_n \to \mathrm{Rep}(H_n)$ on objects by sending W to $u \in \mathrm{Rep}(H_n)$. The images of the generating objects are defined by:

$G''($  $)\colon u \otimes u \to u \otimes u, \qquad e_i \otimes e_j \mapsto \delta_{i,j} e_i \otimes e_i,$

$G''($  $)\colon u \otimes u \to \mathbb{C}, \qquad e_i \otimes e_j \mapsto \delta_{i,j},$

$G''($  $)\colon u \otimes u \to u, \qquad v \otimes w \mapsto w \otimes v,$

$G''($  $)\colon \mathbb{C} \to u \otimes u, \qquad 1 \mapsto \sum_{i=1}^{n} e_i \otimes e_i.$

The fact that this is well-defined is proven similarly as in [LS21, Theorem 5.1].

The fact that $G' := G'' \circ \tilde{G}^{-1}$ indeed yields the same functor as before follows from the fact that $G'' \circ \tilde{G}^{-1} = T$ on the generating morphisms and therefore on the basis elements of the morphism spaces.

Chapter 3

Relating the Interpolation Categories of the Hyperoctahedral Groups

In this chapter we want to discuss how the different interpolation categories relate to each other. We used the sets of partition $P_{even} \subset P \subset P_{\mathbb{Z}_2}$ to define the morphism spaces. This inspires the definition of functors $\Phi : \underline{\mathrm{Rep}}(H_t) \to \underline{\mathrm{Rep}}(S_t)$ and $\Psi : \underline{\mathrm{Rep}}(S_t) \to \mathrm{Par}(\mathbb{Z}_2, t)$. Their composition $\Psi \circ \Phi : \underline{\mathrm{Rep}}(S_t) \to \mathrm{Par}(\mathbb{Z}_2, t)$ however will not be so interesting from a representation theoretical point of view. For integer $t \in \mathbb{N}$ these functors don't commute with the interpolation functors G and H. But inspired by the fact that the reflection representation of the hyperoctahedral group H_n can be considered as a subrepresentation of the permutation representation of H_n, we construct a tensor functor $\Omega : \underline{\mathrm{Rep}}(H_n) \to \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$ which is compatible with the functors G and H and turns out to be an equivalence. We already knew that $\widehat{\underline{\mathrm{Rep}}(H_n)}$ was equivalent to $\mathrm{Par}(\widehat{\mathbb{Z}_2, 2n})^{Kar}$, by the composition $\widehat{H}^{-1} \circ \widehat{G}$ of the equivalences $\widehat{G}$ and $\widehat{H}$. We show that $\widehat{H}^{-1} \circ \widehat{G}$ is isomorphic to the functor that is obtained when Ω descends to a functor between the semisimplifications of the interpolation categories.

3.1 Construction of a functor $\Psi \circ \Phi : \underline{\mathbf{Rep}}(H_t) \to \mathbf{Par}(\mathbb{Z}_2, t)^{Kar}$

Definition 3.1.1. For all $k, l \in \mathbb{N}$ we have an inclusion of sets of partitions $P_{even}(k, l) \subseteq P(k, l)$. So we can define a faithful functor

$$\begin{aligned} \Phi_0 : \underline{\mathrm{Rep}_0}(H_t) &\to \underline{\mathrm{Rep}_0}(S_t) \\ [m] &\mapsto [m] \\ p &\mapsto p \end{aligned}$$

for all $m \in \mathbb{N}$ and for all $p \in P_{even}(k, l)$.

Φ_0 is clearly a well-defined faithful strict $\mathbb{C}$-linear tensor functor. By Definition 1.1.25 and Remark 2.2.4 the composition

$$\iota_F \circ \Phi_0 : \underline{\mathrm{Rep}_0}(H_t) \to \underline{\mathrm{Rep}}(S_t)$$

is also a faithful strict $\mathbb{C}$-linear tensor functor. By the universal property of $\underline{\mathrm{Rep}_0}(H_t)$, see Remark 2.2.8, and Proposition 1.1.29 there exists a faithful tensor functor

$$\Phi : \underline{\mathrm{Rep}}(H_t) \to \underline{\mathrm{Rep}}(S_t)$$

such that $\Phi \circ \iota_G = \iota_F \circ \Phi_0$. We consider for all $n \in \mathbb{N}$ the following square:

Proposition 3.1.2. *For all $n \in \mathbb{N}$ the square*

$$\begin{array}{ccc} \underline{Rep}(H_n) & \xrightarrow{G} & Rep(H_n) \\ \downarrow \Phi & & \downarrow Res^{H_n}_{S_n} \\ \underline{Rep}(S_n) & \xrightarrow{F} & Rep(S_n) \end{array} \tag{3.1.1}$$

is commutative up to isomorphism.

Proof. We precompose the functors G and Φ with ι_G. Then we get

$$F \circ \Phi \circ \iota_G([k]) = F([k]) = (u')^{\otimes k} \text{ and } Res^{H_n}_{S_n} \circ G \circ \iota_G([k]) = Res^{H_n}_{S_n}(u^{\otimes k}) = (u')^{\otimes k}$$

for all $k \in \mathbb{N}$. For a morphism $p \in \mathrm{Hom}_{\underline{\mathrm{Rep}_0}(H_n)}([k],[l]) = \mathbb{C}\, P_{even}(k,l)$ we see that

$$F \circ \Phi \circ \iota_G(p) = F(p) = T_p \text{ and } Res^{H_n}_{S_n} \circ G \circ \iota_G(p) = Res^{H_n}_{S_n} \circ G(p) = Res^{H_n}_{S_n}(T_p) = T_p.$$

This proves that $F \circ \Phi \circ \iota_G = Res^{H_n}_{S_n} \circ G \circ \iota_G$. The universal property of $\underline{\mathrm{Rep}}(H_n)$ then implies by the uniqueness of the induced functor that $F \circ \Phi \cong Res^{H_n}_{S_n} \circ G$. Hence the square (3.1.1) is commutative up to isomorphism. ☐

Definition 3.1.3. For all $k, l \in \mathbb{N}$ we have an inclusion of sets of partitions $P(k,l) \subseteq P_{\mathbb{Z}_2}(k,l)$. So we can define a faithful functor

$$\begin{aligned} \Psi_0 : \underline{\mathrm{Rep}_0}(S_t) &\to \mathrm{Par}(\mathbb{Z}_2, t) \\ [m] &\mapsto [\tilde{m}] \\ p &\mapsto p \end{aligned}$$

for all $m \in \mathbb{N}$ and for all $p \in P(k,l)$.

Ψ_0 is clearly a well-defined faithful strict $\mathbb{C}$-linear tensor functor. By Definition 1.1.25 and Remark 2.2.13 the composition

$$\iota_H \circ \Psi_0 : \underline{\mathrm{Rep}_0}(S_t) \to \mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$$

is a faithful strict tensor funtor. By the universal property of $\underline{\mathrm{Rep}_0}(S_t)$, see Remark 2.2.4, and Proposition 1.1.29 there exists a faithful tensor functor

$$\Psi : \underline{\mathrm{Rep}}(S_t) \to \mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$$

such that $\Psi \circ \iota_F = \iota_H \circ \Psi_0$. Let $g, h : ([k], e) \to ([l], f)$ be morphisms in $\underline{\mathrm{Rep}}(S_t)$, then it follows directly from the definitions that $\Phi(g) = \Phi(h)$ implies $g = h$. So the functor Ψ is faithful. In this case we can not hope to build a similar commutative diagram as we did for Φ. The reason for this is that for an even natural number $n \in \mathbb{N}$, our interpolation functors are

$$F : \underline{\mathrm{Rep}}(S_n) \to \mathrm{Rep}(S_n)$$

with $F([1]) = u$ and

$$H : \mathrm{Par}(\mathbb{Z}_2, n)^{Kar} \to \mathrm{Rep}(H_{\frac{n}{2}})$$

with $H([\tilde{1}]) = V$. There is no sensible way to associate to every S_n representation a $H_{\frac{n}{2}}$ representation. As a consequence the faithful tensor functor

$$\Psi \circ \Phi : \underline{\mathrm{Rep}}(H_t) \to \mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$$

will not be compatible with our interpolation functors G and H. In the next section we give another functor which does satisfy this desired property.

3.2 Construction of a functor $\Omega : \underline{\mathbf{Rep}}(H_t) \to \mathbf{Par}(\mathbb{Z}_2, 2t)^{Kar}$

3.2.1 Preparation and motivation for Ω

We saw in Remark 1.2.15 that the reflection permutation of the hyperoctahedral group H_n is isomorphic to a subrepresentation of the permutation representation of H_n:

$$u \cong \tilde{u} = \bigoplus_{i=1}^{n} \mathbb{C}(e_1^i - e_{-1}^i) \subset V.$$

We want to describe the subrepresentation $\tilde{u} \cong u$ as the image of some idempotent $e : V \to V$. Let $\alpha : \tilde{u} \to V$ be the inclusion and define $\beta : V \to \tilde{u}$ by

$$e_1^i \mapsto \frac{e_1^i - e_{-1}^i}{2} \text{ for all } i \in \{1, \dots, n\}$$
$$e_{-1}^i \mapsto \frac{-e_1^i + e_{-1}^i}{2} \text{ for all } i \in \{1, \dots, n\}.$$

Then the morphism $e := \alpha \circ \beta$ is an idempotent with a splitting given by α, β and $\tilde{u}$. The fact that it is an idempotent follows from $\beta \circ \alpha = id_{\tilde{u}}$, which holds because

$$\begin{aligned}(\beta \circ \alpha)(e_1^i - e_{-1}^i) &= \beta(e_1^i - e_{-1}^i) \\ &= \beta(e_1^i) - \beta(e_{-1}^i) \\ &= \frac{e_1^i - e_{-1}^i}{2} - \frac{-e_1^i + e_{-1}^i}{2} \\ &= \frac{e_1^i - e_{-1}^i + e_1^i - e_{-1}^i}{2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{2e_1^i - 2e_{-1}^i}{2} \\
&= e_1^i - e_{-1}^i.
\end{aligned}
$$

This implies that $e \circ e = \alpha \circ \beta \circ \alpha \circ \beta = \beta \circ \alpha = e$. It has as image $im(e) = im(\alpha \circ \beta) = im(\beta) = \tilde{u}$.

We proved the following lemma already in Proposition 2.4.7, but in the next proof we prove it by giving an explicit preimage.

Lemma 3.2.1. *The reflection representation $\tilde{u}$ of H_n lies in the image of the functor $H : Par(\mathbb{Z}_2, 2n)^{Kar} \to Rep(H_n)$.*

Proof. By definition we know that $H'([\tilde{1}]) = V$ for $H' : \mathrm{Par}(\mathbb{Z}_2, 2n) \to \mathrm{Rep}(H_n)$, the restriction of H to $\mathrm{Par}(\mathbb{Z}_2, 2n) \subset \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$. Define the idempotent

$$
e' := \frac{{}_{1}\!\rvert - {}_{-1}\!\rvert}{2} : [\tilde{1}] \to [\tilde{1}]
$$

in $\mathrm{Par}(\mathbb{Z}_2, 2n)$. It follows directly from the definitions of T that

$$
T({}_{-1}\!\rvert)(e_j^i) = e_{-j}^i
$$

and as a consequence

$$
\begin{aligned}
H'(e')(e_1^i) &= \frac{T({}_{1}\!\rvert)(e_1^i) - T({}_{-1}\!\rvert)(e_1^i)}{2} = \frac{e_1^i - e_{-1}^i}{2} \text{ and} \\
H'(e')(e_{-1}^i) &= \frac{T({}_{1}\!\rvert)e_{-1}^i - T({}_{-1}\!\rvert)e_{-1}^i}{2} = \frac{-e_1^i + e_{-1}^i}{2},
\end{aligned}
$$

for all $i \in \{1, \ldots, n\}$. This implies that $H'(e') = e$.

Let $\alpha' : ([\tilde{1}], e') \to ([\tilde{1}], id_{[\tilde{1}]})$ and $\beta' : ([\tilde{1}], id_{[\tilde{1}]}) \to ([\tilde{1}], e')$ be the splitting of the idempotent $e' : ([\tilde{1}], id_{[\tilde{1}]}) \to ([\tilde{1}], id_{[\tilde{1}]})$ in $\mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$. By the universal property of the Karoubian envelope in Remark 1.1.27, the functor $H : \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar} \to \mathrm{Rep}(H_n)$ is defined by sending a splitting in $\mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$ to some chosen splitting in $\mathrm{Rep}(H_n)$. So we can make the following choices for H:

$$
\begin{aligned}
H(([\tilde{1}], e')) = H(im(e')) &:= im(e) = \tilde{u}, \\
H(\alpha') &:= \alpha \text{ and} \\
H(\beta') &:= \beta.
\end{aligned}
$$

This concludes the proof. □

Remark 3.2.2. The choices at the end of last proof may seem a bit forced, but we are only interested in our interpolation functors and categories up to isomorphism, so no actual problems arise here. A more correct statement of last lemma would be 'The reflection representation $\tilde{u}$ of H_n lies in the essential image of the functor $H : \mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar} \to \mathrm{Rep}(H_n)$. But as we have seen in the proof, for some choices of H, it actually lies in the image.

We now consider the complement of the reflection permutation of the hyperoctahedral group H_n in the permutation representation of H_n:

$$\tilde{v} := \bigoplus_{i=1}^{n} \mathbb{C}(e_1^i + e_{-1}^i) \subset V.$$

Let $\gamma : \tilde{v} \to V$ be the inclusion of representations and

$$\begin{aligned} \delta :& V \to \tilde{v} \\ & e_1^i \mapsto \frac{e_1^i + e_{-1}^i}{2} \text{ for all } i \in \{1, \ldots, n\} \text{ for all} \\ & e_{-1}^i \mapsto \frac{e_1^i + e_{-1}^i}{2} \text{ for all } i \in \{1, \ldots, n\}. \end{aligned}$$

Then the morphism $f := \gamma \circ \delta$ is an idempotent with a splitting given by γ, δ and $\tilde{v}$. The fact that it is an idempotent follows from $\delta \circ \gamma = id_{\tilde{v}}$, which holds because

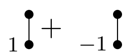
$$\begin{aligned} (\delta \circ \gamma)(e_1^i + e_{-1}^i) &= \delta(e_1^i + e_{-1}^i) \\ &= \delta(e_1^i) + \delta(e_{-1}^i) \\ &= \frac{e_1^i + e_{-1}^i}{2} + \frac{-e_1^i + e_{-1}^i}{2} \\ &= \frac{e_1^i + e_{-1}^i + e_1^i + e_{-1}^i}{2} \\ &= \frac{2e_1^i + 2e_{-1}^i}{2} \\ &= e_1^i + e_{-1}^i \end{aligned}$$

for all $i \in \{1, \ldots, n\}$. It has image $im(f) = im(\gamma \circ \delta) = im(\delta) = \tilde{v}$. Note that $f = id_V - e$.

Simlarly as in Lemma 3.2.1 we prove

Lemma 3.2.3. *The complement of the reflection representation of H_n, $\tilde{v}$, lies in the image of the functor $H : Par(\mathbb{Z}_2, 2n)^{Kar} \to Rep(H_n)$.*

Proof. Let $e' : [\tilde{1}] \to [\tilde{1}]$ be the idempotent defined in Lemma 3.2.1. We consider the idempotent



$$id_{[1]} - e' = \frac{\cdots}{2} : [\tilde{1}] \to [\tilde{1}]$$

in $\mathrm{Par}(\mathbb{Z}_2, 2n)$. Because H' is a $\mathbb{C}$-linear functor, the assumption $H'(e') = e$ implies that $H'(id_{[\tilde{1}]} - e') = id_V - e = f$.

Let $\gamma' : ([\tilde{1}], 1 - e') \to ([\tilde{1}], id_{[\tilde{1}]})$ and $\delta' : ([\tilde{1}], 1 - id_{[\tilde{1}]}) \to ([\tilde{1}], e')$ be the splitting of the idempotent $1 - e : ([\tilde{1}], id_{[\tilde{1}]}) \to ([\tilde{1}], id_{[\tilde{1}]})$ in $\mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$. By a similar argument as in Lemma 3.2.1 we can make the following choices for H:

$$\begin{aligned} H(([\tilde{1}], id_{[\tilde{1}]} - e')) = H(im(id_{[\tilde{1}]} - e')) := im(id_V - e) = im(f) = \tilde{v}, \\ H(\gamma') := \gamma \text{ and} \end{aligned}$$

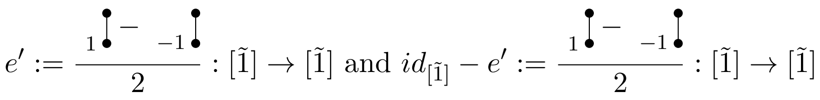
$$H(\delta') := \delta.$$

This concludes the proof. □

Remark 3.2.4. If we summarize the last two lemmas we get

$$\begin{aligned} &H([\tilde{1}]) = V, \\ &H(([\tilde{1}], e')) = \tilde{u} \cong u, \\ &H(([\tilde{1}], id_{[\tilde{1}]} - e')) = \tilde{v} \cong v. \end{aligned}$$

Remark 3.2.5. Let $t \in \mathbb{C}$. We can define the idempotents



in $\mathrm{Par}(\mathbb{Z}_2, 2t)^{Kar}$, similarly as above.

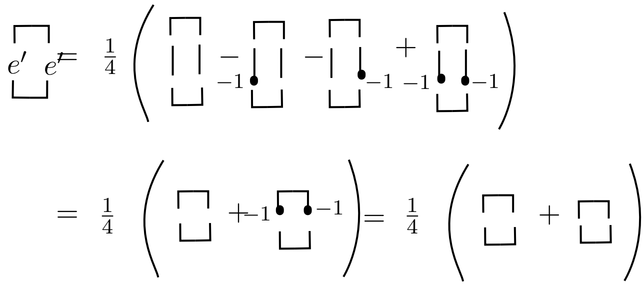
Lemma 3.2.6. *Let $t \in \mathbb{C}$. The object $([\tilde{1}], e') \in Par(\mathbb{Z}_2, 2t)^{Kar}$ is self-dual and has categorical dimension t.*

Proof. The object $([\tilde{1}], e')$ is self-dual because

$$([\tilde{1}], e')^* = ([\tilde{1}]^*, (e')^*) = ([\tilde{1}], e').$$

The first equality follows from the choice of rigid symmetric monoidal structure for the idempotent completion of a rigid symmetric monoidal category, see Proposition 1.1.24. The second equality follows from the self-duality of $[\tilde{1}]$ and the fact that the involution of e' is again e'.

Using the rules of composition for $\mathbb{Z}_2$-coloured partitions, we find that the categorical dimension of $([\tilde{1}], e')$, the trace of $id_{([\tilde{1}], e')} = e'$, equals



$$= \tfrac{1}{4}(4t) = t.$$

□

Remark 3.2.7. Similarly as in Lemma 3.2.6 we can prove that $([\tilde{1}], 1 - e') \in \mathrm{Par}(\mathbb{Z}_2, 2t)^{Kar}$ is also a t-dimensional self-dual object. This makes sense because it is the complement of $([\tilde{1}], e')$ in $[\tilde{1}] = ([\tilde{1}], id_{[\tilde{1}]})$, which is $2t$-dimensional.

Remark 3.2.8. By definition of the monoidal structure of the Karoubian envelope of a category, see proof of Proposition 1.1.24, we see that $([\tilde{1}], e')^{\otimes k} = (([\tilde{k}], (e')^{\otimes k})$. By definition of the Karoubian envelope the morphism spaces between the tensor products of $([\tilde{1}], e')$ are defined by

$$\mathrm{Hom}_{\mathrm{Par}(\mathbb{Z}_2,2t)^{Kar}}(([\tilde{1}], e')^{\otimes k}, ([\tilde{1}], e')^{\otimes l}) = (e')^{\otimes l} \circ \mathrm{Hom}_{\mathrm{Par}(\mathbb{Z}_2,2t)}([\tilde{k}], [\tilde{l}]) \circ (e')^{\otimes k}.$$

By Theorem 2.6.12 we have a full description of these morphisms in terms of generators and relations.

In the following we want to use Theorem 2.6.12 to discuss the morphisms in the Karoubian envelope $\mathrm{Par}(\mathbb{Z}_2, 2t)^{Kar} \cong (\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar}$. The object in $(\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar}$ corresponding to $([\tilde{1}], e')$ is (W, e') where we will set, by abuse of notation,

$$e' := \frac{{}^{1}\!\!\overset{\bullet}{\mid} - {}^{-1}\!\!\overset{\bullet}{\mid}}{2}.$$

There are some interesting relations for the generating morphisms of $\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t}$, see Definition 2.6.9, when we pre- and postcompose them with suitable tensor powers of e'.

Lemma 3.2.9. *The following equalities and inequalities hold in* $(Par(\mathbb{Z}_2)/\sim_{2t})^{Kar}$*:*

$$0 \neq e' \circ \mid \circ e' : (W, e') \to (W, e')$$
$$0 = e' \circ \curlywedge \circ (e' \otimes e') : (W, e') \otimes (W, e') \to (W, e')$$
$$0 = (e' \otimes e') \circ \curlyvee \circ e' : (W, e') \to (W, e') \otimes (W, e')$$
$$0 \neq (e' \otimes e') \circ \times \circ (e' \otimes e') : (W, e') \otimes (W, e') \to (W, e') \otimes (W, e')$$
$$0 \neq e' \circ {}^{g}\!\!\overset{\bullet}{\mid} \circ e' : (W, e') \to (W, e'), \quad g \in \mathbb{Z}_2$$
$$0 = e' \circ ⫰ : \mathbf{1} \to (W, e')$$
$$0 = ⫯ \circ e' : (W, e') \to \mathbf{1}$$

for the compositions of the generating morphisms in $Par(\mathbb{Z}_2)/\sim_{2t}$ *with the suitable tensor products of* e'*. The morphisms*

$$⫯ \circ \curlywedge \circ (e' \otimes e') : (W, e') \otimes (W, e') \to \mathbf{1}$$
$$(e' \otimes e') \circ \curlyvee \circ ⫰ : \mathbf{1} \to (W, e') \otimes (W, e')$$
$$(e' \otimes e') \circ \curlyvee \circ \curlywedge \circ (e' \otimes e') : (W, e') \otimes (W, e') \to (W, e') \otimes (W, e') \text{ and}$$
$$dim((W, e')) = ⫯ \circ \curlywedge \circ (e' \otimes e') \circ (e' \otimes e') \circ \curlyvee \circ ⫰ = t : \mathbf{1} \to \mathbf{1}$$

are non-trivial.

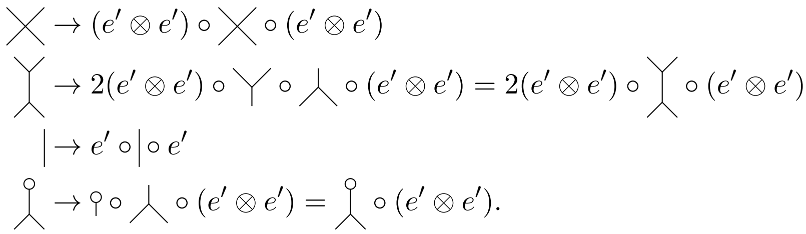
Proof. The equalities and inequalities follow from an easy calculation, using the relations in Definition 2.6.9. Calculating $\dim((W, e'))$ is analogous to the calculation of $\dim(([\tilde{1}], e'))$ we did in Lemma 3.2.6. □

Remark 3.2.10. The emerging pattern of the morphisms $(W, e')^{\otimes k} \to (W, e')^{\otimes l}$ appearing in the lemma, is that they are non-trivial whenever $k + l$ is even. The ones for which $k + l$

is odd, equal zero. This already hints at some correspondence with $\underline{\mathrm{Rep}}(H_t)$. Lemma 3.2.6 and Lemma 3.2.9 suggest the existence of a functor

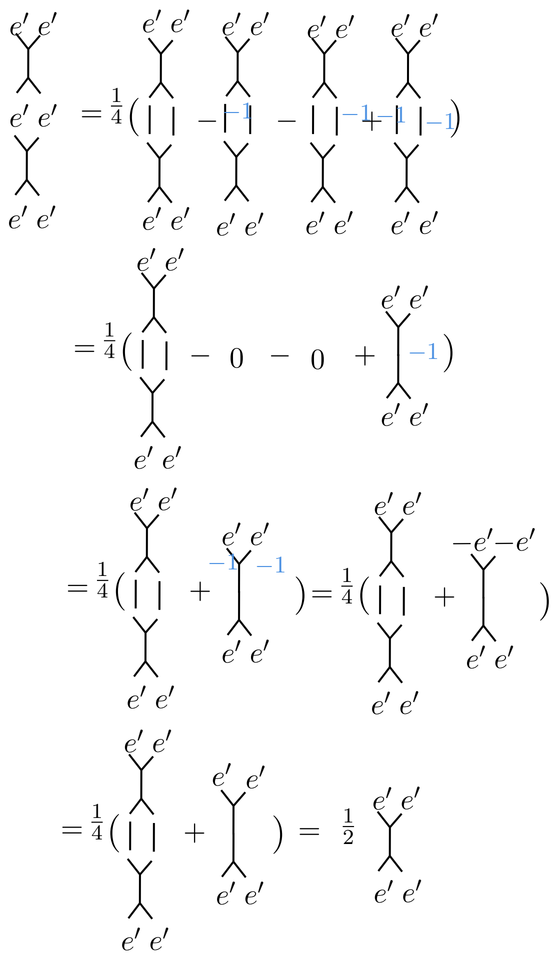
$$\Omega_0 : \underline{\mathrm{Rep}}_0(H_t) \to (\mathrm{Par}(\mathbb{Z}_2, 2t))^{Kar}$$

defined on objects by $[k] \to (W, e')^{\otimes k}$, both self-dual objects of dimension t^k, and on generating morphisms by

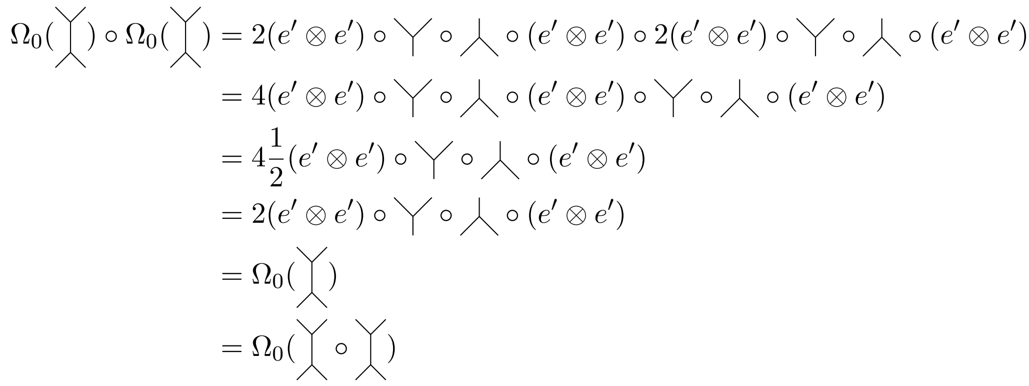


(3.2.1)
(3.2.2)
(3.2.3)
(3.2.4)

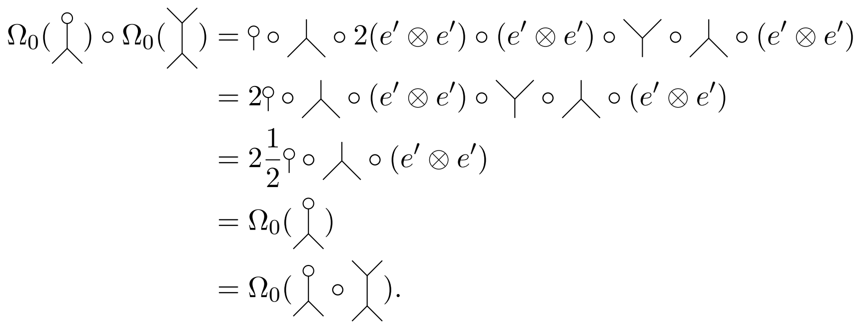
We applied Theorem 2.6.19 to identify $\underline{\mathrm{Rep}}_0(H_t)$ with Par_t, where we have a description of the generating morphisms. Consider the relation



Here we used that $e' \circ -1 = -e' = -1 \circ e'$.This relation explains why the image of the morphism in (3.2.2) gains a factor 2, namely because we would want Ω_0 to be functorial:



and similarly



The lemmata suggest also that Ω_0, which is still to be defined, is injective on objects and fully faithful.

As long as no confusion arises, we will not always mention explicitly when we are using the isomorphisms $\mathrm{Par}_t \cong \underline{\mathrm{Rep}}_0(H_t)$ and $\mathrm{Par}(\mathbb{Z}_2, 2t)) \cong \mathrm{Par}(\mathbb{Z}_2)/\sim_{2t}$ to discuss the morphism spaces of the interpolation categories in terms of generators and relations. The discussion above motivates the following definition.

3.2.2 Definition of Ω

Theorem 3.2.11. *For all $t \in \mathbb{C}$ there is a well-defined strict $\mathbb{C}$-linear tensor functor*

$$\Omega_0 : Par_t \cong \underline{Rep}_0(H_t) \to (Par(\mathbb{Z}_2, 2t))^{Kar} \cong (Par(\mathbb{Z}_2)/\sim_{2t})^{Kar}$$

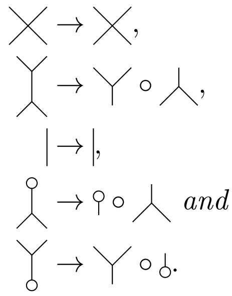
which is defined on objects by

$$\Omega_0([k]) := ([\tilde{k}], (e')^{\otimes k})$$

for all $k \in \mathbb{N}$. On morphisms it is the $\mathbb{C}$-linear extension of the following rule. Let $f : W^{\otimes k} \to W^{\otimes l}$ be a morphism in Par_t which consists of s blocks $B_1, \dots, \mathrm{B}_s$ of size $m_1, \dots, m_s$ respectively. Then we set

$$\Omega_0(f) := 2^{\frac{(\sum_{i=1}^{s} m_i) - 2s}{2}} (e')^{\otimes l} \circ f \circ (e')^{\otimes k},$$

where we identify morphisms in Par_t *with the corresponding morphisms in* $Par(\mathbb{Z}_2)/\sim_{2t}$ *by using the following associations:*



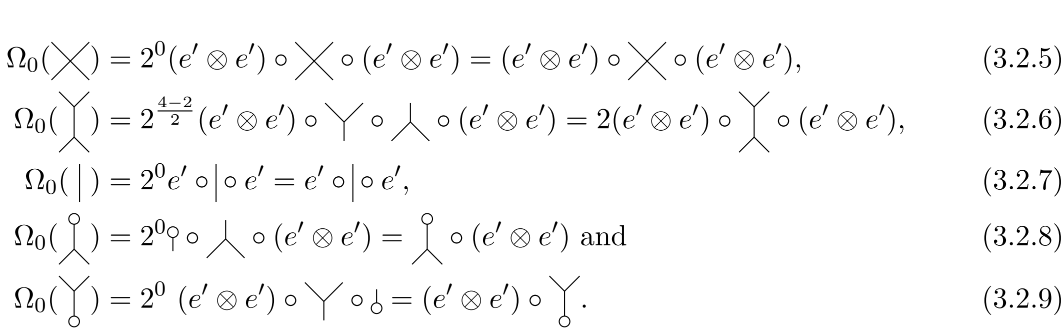
Proof. It is clear that Ω_0 respects the tensor product for objects. For morphisms $f : W^{\otimes k_1} \to W^{\otimes l_1}$ with s_1 blocks $B_{1,1}, \dots, \mathrm{B}_{s,1}$ of size $m_1, \dots, m_{s_1}$ respectively and $g : W^{\otimes k_2} \to W^{\otimes l_2}$ with s_2 blocks $B_{1,2}, \dots, \mathrm{B}_{s_2,2}$ of size $m_{s_1+1}, \dots, m_{s_1+s_2}$ respectively we see that

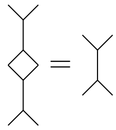
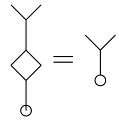
$$\begin{aligned}
\Omega_0(f \otimes g) &= 2^{\frac{(\sum_{i=1}^{s_1+s_2} m_i)-2(s_1+s_2)}{2}} (e')^{\otimes l_1+l_2} \circ f \otimes g \circ (e')^{\otimes k_1+k_2} \\
&= 2^{\frac{(\sum_{i=1}^{s_1} m_i)-2(s_1)}{2}} 2^{\frac{(\sum_{i=s_1+1}^{s_1+s_2} m_i)-2(s_2)}{2}} ((e')^{\otimes l_1} \circ f \circ (e')^{\otimes k_1}) \otimes ((e')^{\otimes l_2} \circ g \circ (e')^{\otimes k_2}) \\
&= (2^{\frac{(\sum_{i=1}^{s_1} m_i)-2(s_1)}{2}} (e')^{\otimes l_1} \circ f \circ (e')^{\otimes k_1}) \otimes (2^{\frac{(\sum_{i=s_1+1}^{s_1+s_2} m_i)-2(s_2)}{2}} (e')^{\otimes l_2} \circ g \circ (e')^{\otimes k_2}) \\
&= \Omega_0(f) \otimes \Omega_0(g).
\end{aligned}$$

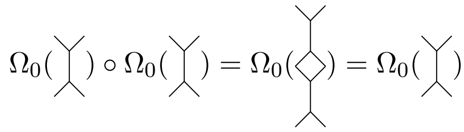
This shows that Ω_0 respects the tensor product of morphisms. We will identify the functor $\Omega_0 : \underline{\mathrm{Rep}}_0(H_t) \to (\mathrm{Par}(\mathbb{Z}_2, 2t))^{Kar}$ with the composition

$$\tilde{H}^{-1} \circ \Omega_0 \circ \tilde{G} : \mathrm{Par}_t \to (\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar},$$

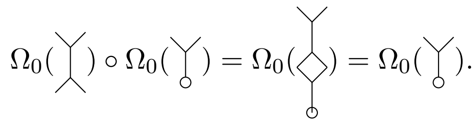
so we can talk easier about the image of the generators and relations of Definition 2.6.17 under the functor Ω_0. The definition implies

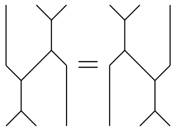


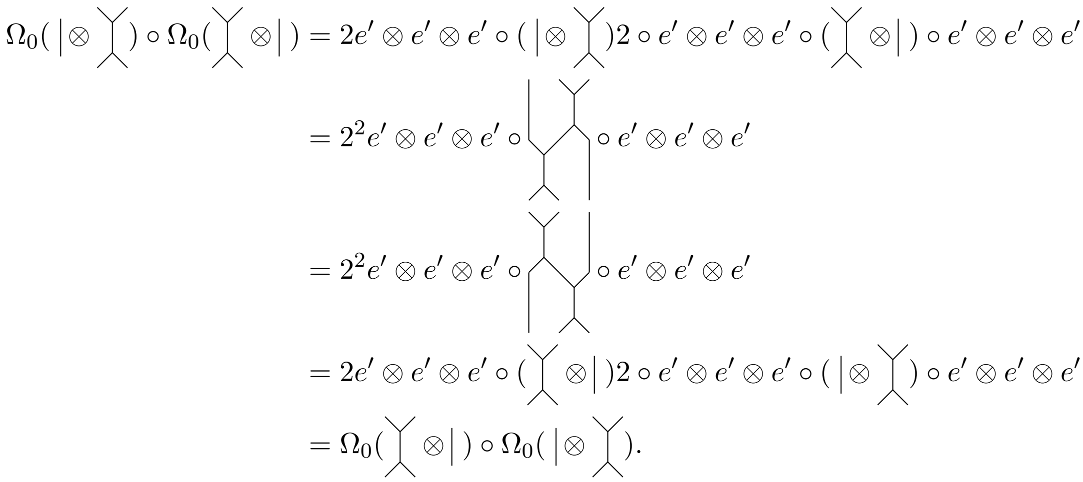
Now we want to show that Ω_0 is functorial. For this we use Theorem 2.6.21. Note that $\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t}$ is a $\mathbb{C}$-linear monoidal category and that the morphisms above will function respectively as the swap morphism, neutralizer, identity, evaluation and coevalution for the object (W, e') of $(\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar}$. We have to show that the object (W, e') and the morphisms in (3.2.5), (3.2.6),(3.2.7),(3.2.8) and (3.2.9) in $(\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar}$ satisfy the relations given in Definition 2.6.17. For the relations  and  saw already that



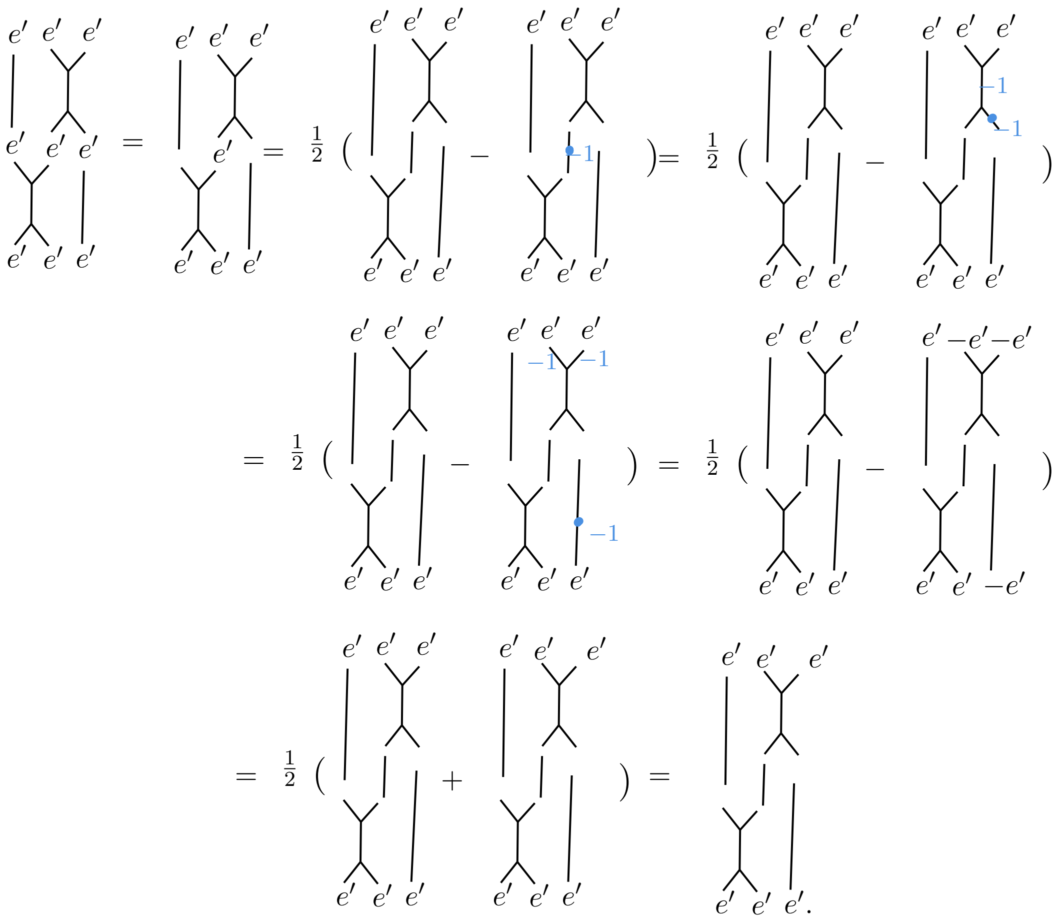
and

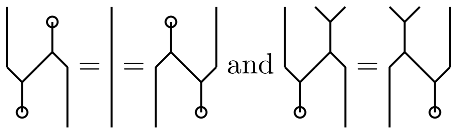


For the relation  we have

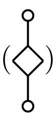


Note that we use the functor Ω_0 to get a shorter expression of the relations but that these parts are not meaningful to prove that the relations hold for the morphisms between the tensor powers of (W, e') in $(\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar}$. The third equality holds because all relations that holds in Par_t, also hold in $\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t}$. The second and fourth equalities follow from



In a similar fashion we can show that the relations  and their reflections hold.

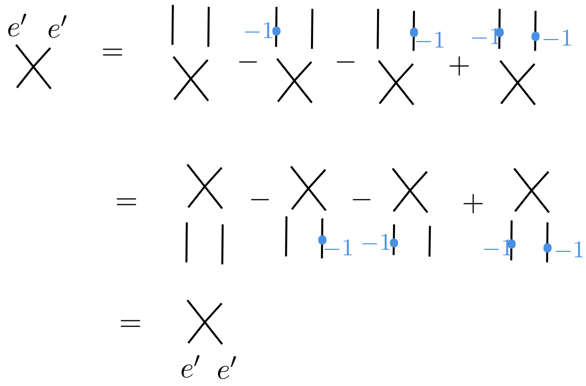
We already saw that

$\Omega_0($$) = \dim((W, e'))$

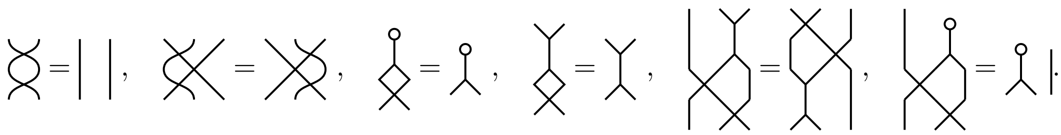
$=$  $\circ$  $\circ (e' \otimes e') \circ (e' \otimes e') \circ$  $\circ$ 

$= t.$

From the relation



and the fact that the corresponding relations hold in $\mathrm{Par}(\mathbb{Z}_2 \,/ \sim_{2t})$ we obtain the remaining relations and their reflections



(3.2.10)

By Theorem 2.6.21 we see that $\Omega_0 : \mathrm{Par}_t \to (\mathrm{Par}(\mathbb{Z}_2 \,/ \sim_{2t}))^{Kar}$ is a well-defined $\mathbb{C}$-linear symmetric monoidal functor between $\mathbb{C}$-linear spherical rigid symmetric monoidal categories. The functor is clearly strict and preserves the different monoidal structures. Therefore it is a strict $\mathbb{C}$-linear tensor functor. □

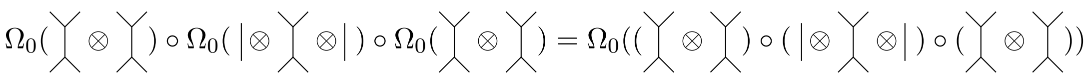
Corollary 3.2.12. *For all $t \in \mathbb{C}$ there is a well-defined $\mathbb{C}$-linear tensor functor*

$$\Omega : \underline{Rep}(H_t) \to (Par(\mathbb{Z}_2, 2t))^{Kar}$$

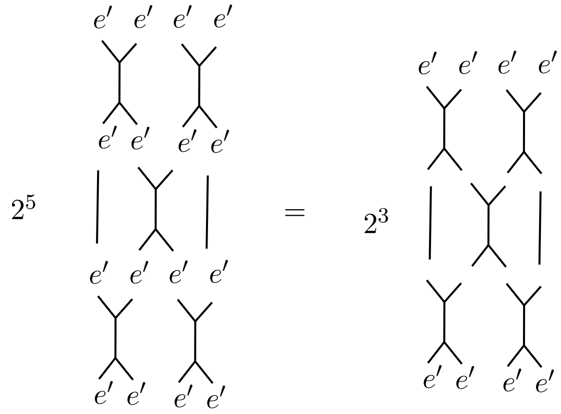
which restricts to $\Omega \circ \iota_G = \Omega_0$.

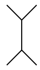
Proof. Apply Remark 2.2.8 to the functor Ω_0 of Theorem 3.2.11. □

Remark 3.2.13. It is not immediately clear why something like



or



holds in general. The main advantage of using the universal property of $\mathrm{Rep}_0(H_t)$ is that we don't have to worry again about these complications. But in this particular case, the diagram on the left side can be modified using the relations of Definition 2.6.17, see the proof of Theorem 3.2.11. In doing this we will compose $e' \otimes e' \circ$  $\circ\, e' \otimes e'$ two times with itself and gain a factor 2^{-2} while changing the diagram of the left side to the one on the right side of the equality.

3.2.3 Properties of Ω

In this section we show that the functor Ω is an equivalence of categories.

Theorem 3.2.14. *The functor $\Omega_0 : \underline{Rep}_0(H_t) \to (Par(\mathbb{Z}_2, 2t))^{Kar}$ is a full embedding.*

Proof. By definition $k \neq l$ implies that $\Omega_0([k]) = ([\tilde{k}], (e')^{\otimes k}) \neq ([\tilde{l}], (e')^{\otimes l}) = \Omega_0([k])$, showing that functor Ω_0 is injective on objects.

Next we want to show that Ω_0 is full. The functor $\Omega_0 : \underline{\mathrm{Rep}}_0(H_t) \to (\mathrm{Par}(\mathbb{Z}_2, 2t))^{Kar}$ is given on morphisms by

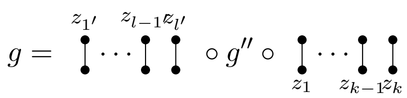
$$\Omega_0(f) = 2^{\frac{(\sum_{i=1}^{s} m_i)-2s}{2}} (e')^{\otimes l} \circ f \circ (e')^{\otimes k},$$

for some partition $f \in P_{even}(k,l)$ with s blocks of size $m_1, \ldots, m_s$ and then extended $\mathbb{C}$-linearly. Remember that f can be considered a $\mathbb{Z}_2$-coloured partition where every vertex is labeled as 1 and that the tensor product of partitions is the horizontal concatenation of the partitions.

We consider the morphism

$$\begin{aligned} f = (e')^{\otimes l} \circ g \circ (e')^{\otimes k} &\in \mathrm{Hom}_{\mathrm{Par}(\mathbb{Z}_2,2t)^{Kar}}(([\tilde{k}], (e')^{\otimes k}), ([\tilde{l}], (e')^{\otimes l})) \\ &= (e')^{\otimes l} \circ \mathrm{Hom}_{\mathrm{Par}(\mathbb{Z}_2,2t)}([\tilde{k}], [\tilde{l}]) \circ (e')^{\otimes k}. \end{aligned}$$

such that $g \in P_{\mathbb{Z}_2}(k,l)$ with labeling $(z_1, \ldots, z_k, z_{1'}, \ldots, z_{l'}) \in \mathbb{Z}_2^{k+l}$. Then



and the relation

$$e' \circ \underset{-1}{|} = -e' = \underset{-1}{|} \circ e'$$

implies

$$(e')^{\otimes l} \circ g \circ (e')^{\otimes k} = \Big(\prod_{j \in \{1,\dots,k,1',\dots,l'\}} (z_j)\Big)(e')^{\otimes l} \circ g'' \circ (e')^{\otimes k}$$

where g'' is the underlying partition of the $\mathbb{Z}_2$-coloured partition g. So we can reduce the problem to the case where g lies in $P(k,l)$. We first assume that g is a non-even partition in $P(k,l)$. By Proposition 2.1.4 we can write $g = \phi(\sigma) \circ g' \circ \phi(\rho)$, where g' is some non-crossing form of g. Then g' is the horizontal concatenation of its blocks and contains an odd block B of size (a,b). For this block we have

$$\begin{aligned}(e')^{\otimes b} \circ B \circ (e')^{\otimes a} &= \Big(\frac{1}{2}\Big)^{a+b} \sum_{\substack{z_1,\dots,z_a \in \mathbb{Z}_2 \\ z_{1'}\dots z_{b'} \in \mathbb{Z}_2}} \Big(\prod_{j \in \{1,\dots,a,1',\dots,b'\}} (z_j)\Big) \overset{z_{1'}}{|} \cdots \overset{z_{n-1}}{|} \overset{z_{b'}}{|} \circ B \circ \underset{z_1}{|} \cdots \underset{z_{a-1}}{|} \underset{z_a}{|} \\ &= 0.\end{aligned}$$

The last equality follows from the fact that

$$\overset{z_{1'}}{|} \cdots \overset{z_{b-1}}{|} \overset{z_{b'}}{|} \circ B \circ \underset{z_1}{|} \cdots \underset{z_{a-1}}{|} \underset{z_a}{|} = \overset{-z_{1'}}{|} \cdots \overset{-z_{b-1}}{|} \overset{-z_{b'}}{|} \circ B \circ \underset{-z_1}{|} \cdots \underset{-z_{a-1}}{|} \underset{-z_a}{|} \quad \text{and}$$

$$\prod_{j \in \{1,\dots,a,1',\dots,b'\}} (z_j) = - \prod_{j \in \{1,\dots,a,1',\dots,b'\}} (-z_j)$$

because there is an odd amount of vertices in the block B. This implies $(e')^{\otimes l} \circ g' \circ (e')^{\otimes k} = 0$ and

$$\begin{aligned} f &= (e')^{\otimes l} \circ g \circ (e')^{\otimes k} \\ &= (e')^{\otimes l} \circ \phi(\sigma) \circ g' \circ \phi(\rho) \circ (e')^{\otimes k} \\ &= \phi(\sigma) \circ (e')^{\otimes l} \circ g' \circ (e')^{\otimes k} \circ \phi(\rho) \\ &= 0, \end{aligned}$$

which lies in the image trivially. The third equality follows from the proven fact that $(e')^2$ commutes with $\times$. Now we can assume that g is an even partition in $P(k,l)$, but then $g \in \mathrm{Hom}_{\underline{\mathrm{Rep}}_0(H_t)}([k],[l])$, so we have that

$$\Omega_0(g) = (e')^{\otimes l} \circ g \circ (e')^{\otimes k} = f.$$

This concludes the proof of the fullness of Ω_0.
Now we want to show that Ω_0 is faithful. Let $f \in \mathrm{Hom}_{\underline{\mathrm{Rep}}_0(H_t)}([k],[l])$, we consider $\Omega_0(f) = (e')^{\otimes l} \circ f \circ (e')^{\otimes k}$. First assume that f contains only one even block $B = f$. Then

similarly as before we see that

$$
\begin{aligned}
(e')^{\otimes l} \circ B \circ (e')^{\otimes k} &= (\frac{1}{2})^{k+l} \sum_{\substack{z_1,\dots,z_k \in \mathbb{Z}_2 \\ z_{1'} \dots z_{l'} \in \mathbb{Z}_2}} (\prod_{j \in \{1,\dots,k,1',\dots,l'\}} (z_j)) \; \overset{z_{1'}}{\rule{0pt}{1em}\mid} \cdots \overset{z_{l-1}}{\mid} \overset{z_{l'}}{\mid} \circ B \circ \underset{z_1}{\mid} \cdots \underset{z_{k-1}}{\mid} \underset{z_k}{\mid} \\
&= 2(\frac{1}{2})^{k+l} \sum_{\substack{z_1 = 1 \\ z_2,\dots,z_k \in \mathbb{Z}_2 \\ z_{1'} \dots z_{l'} \in \mathbb{Z}_2}} (\prod_{j \in \{1,\dots,k,1',\dots,l'\}} (z_j)) \; \overset{z_{1'}}{\mid} \cdots \overset{z_{l-1}}{\mid} \overset{z_{l'}}{\mid} \circ B \circ \underset{z_1}{\mid} \cdots \underset{z_{k-1}}{\mid} \underset{z_k}{\mid} \\
&\neq 0.
\end{aligned}
$$

The inequality follows from the fact that the summands are pairwise different basis elements of $\mathbb{C}\, P_{\mathbb{Z}_2}(k,l)$. For the second equality we identify equivalent $\mathbb{Z}_2$-coloured partitions, so it follows from the fact that

$$
\overset{z_{1'}}{\mid} \cdots \overset{z_{l-1}}{\mid} \overset{z_{l'}}{\mid} \circ B \circ \underset{z_1}{\mid} \cdots \underset{z_{k-1}}{\mid} \underset{z_k}{\mid} = \overset{-z_{1'}}{\mid} \cdots \overset{-z_{l-1}}{\mid} \overset{-z_{l'}}{\mid} \circ B \circ \underset{-z_1}{\mid} \cdots \underset{-z_{k-1}}{\mid} \underset{-z_k}{\mid} \quad \text{and}
$$

$$
\prod_{j \in \{1,\dots,a,1',\dots,b'\}} (z_j) = \prod_{j \in \{1,\dots,a,1',\dots,b'\}} (-z_j)
$$

because we have an even amount of vertices in the block. Now we assume that f is any even partition. Again by Proposition 2.1.4 we can write $f = \phi(\sigma) \circ f' \circ \phi(\rho)$ where f' is a non-crossing form of f. Then

$$
\begin{aligned}
\Omega_0(f) &= (e')^{\otimes l} \circ f \circ (e')^{\otimes k} \\
&= (e')^{\otimes l} \circ \phi(\sigma) \circ f' \circ \phi(\rho) \circ (e')^{\otimes k} \\
&= \phi(\sigma) \circ (e')^{\otimes l} \circ f' \circ (e')^{\otimes k} \circ \phi(\rho) \\
&\neq 0,
\end{aligned}
$$

because f' is a horizontal concatenation of non-zero even blocks. This shows that $\Omega_0(f) \neq 0$ for all $f \in P_{even}$, in other words Ω_0 is non-zero on the generators of $\mathrm{Hom}_{\underline{\mathrm{Rep}}_0(H_t)}([k],[l])$. Now assume that the set of even partitions $\{f_1,\dots,f_m\} \subset P_{even}(k,l)$ consists of pairwise non-equivalent even partitions. Then by definition they form a linearly independent set in $\mathrm{Hom}_{\underline{\mathrm{Rep}}_0(H_t)}([k],[l])$. It follows from the definition of linear independency of the morphism spaces in $\mathrm{Par}(\mathbb{Z}_2, 2t)$ that the set

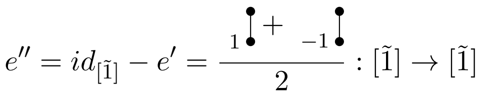
$$
\{ \overset{z_{1'}}{\mid} \cdots \overset{z_{l-1}}{\mid} \overset{z_{l'}}{\mid} \circ f_j \circ \underset{z_1}{\mid} \cdots \underset{z_{k-1}}{\mid} \underset{z_k}{\mid} \mid j \in \{1,\dots,m\}, z_1 = 1, z_2,\dots,z_k \in \mathbb{Z}_2, z_{1'} \dots z_{l'} \in \mathbb{Z}_2 \}
$$

is also linearly independent. This implies in its turn that the set $\{\Omega_0(f_1),\dots,\Omega(f_m)\}$ is linearly independent. This concludes the proof of the faithfulness of Ω_0. □

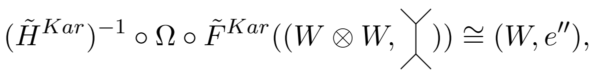
Theorem 3.2.15. *The functor $\Omega : \underline{Rep}(H_t) \to Par(\mathbb{Z}_2, 2t)^{Kar}$ is fully faithful and essen-*

tially surjective, therefore it is an equivalence.

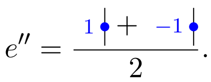
Proof. We apply Proposition 1.1.29 to Theorem 3.2.14 to show that Ω is fully faithful. We are left to prove the essential surjectivity of Ω. We want to show that $[\tilde{1}]$ lies in the essential image of Ω. For this we note that $[\tilde{1}] \cong ([\tilde{1}], e') \oplus ([\tilde{1}], id_{[\tilde{1}]} - e')$ and that $\Omega([1]) = ([\tilde{1}], e')$ by definition. We write



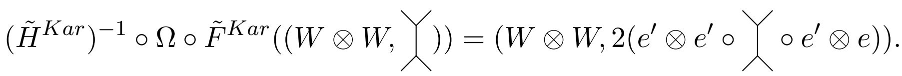
and claim that $([\tilde{1}], e'')$ is isomorphic to the image of ([2], ) under Ω. To prove this it suffices to show the corresponding claim for $(\tilde{H}^{Kar})^{-1} \circ \Omega \circ \tilde{F}^{Kar} : \mathrm{Par}_t^{Kar} \to (\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar}$, namely that



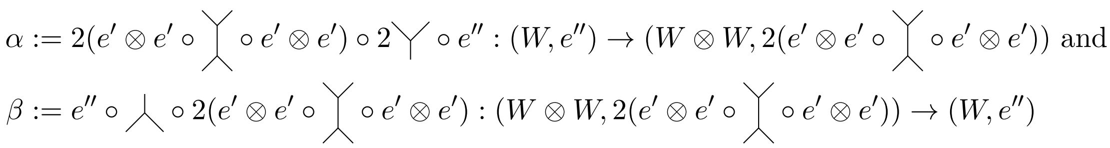
where



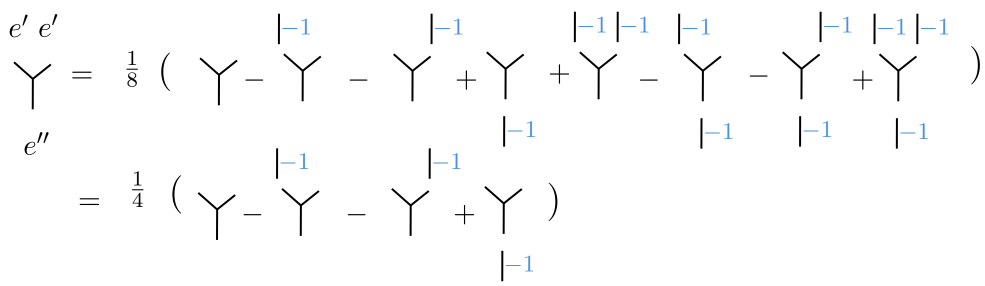
and W in the left side of the equation stands for the generating object in Par_t and in the right side for the generating object in $\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t}$ as was done in Definition 2.6.17 and Definition 2.6.9 respectively. By definition we have that



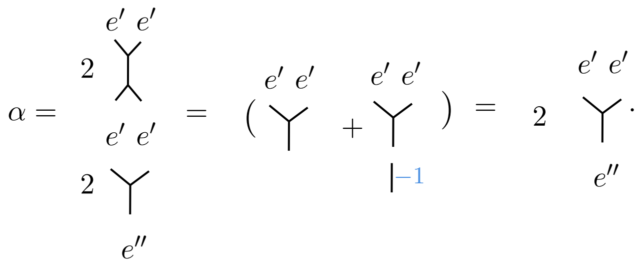
We define the morphisms



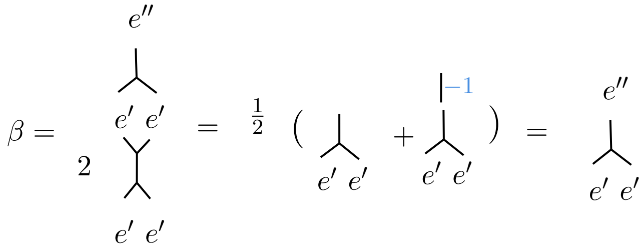
and show that they are isomorphisms. We first use the relations in Definition 2.6.9 to simplify α and β. The equality



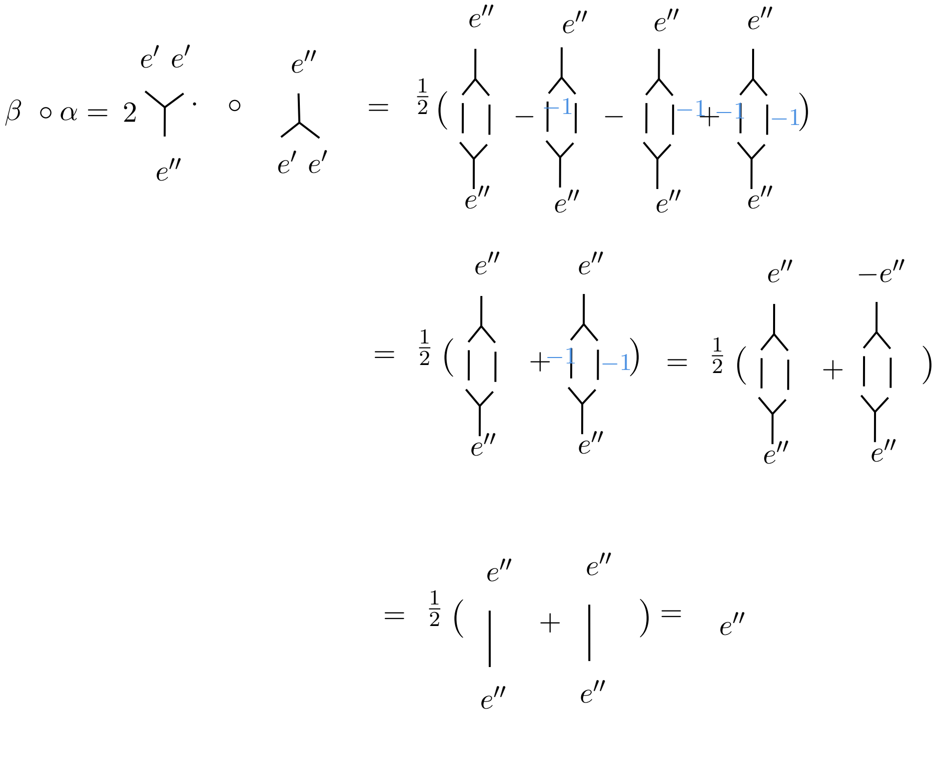
implies that



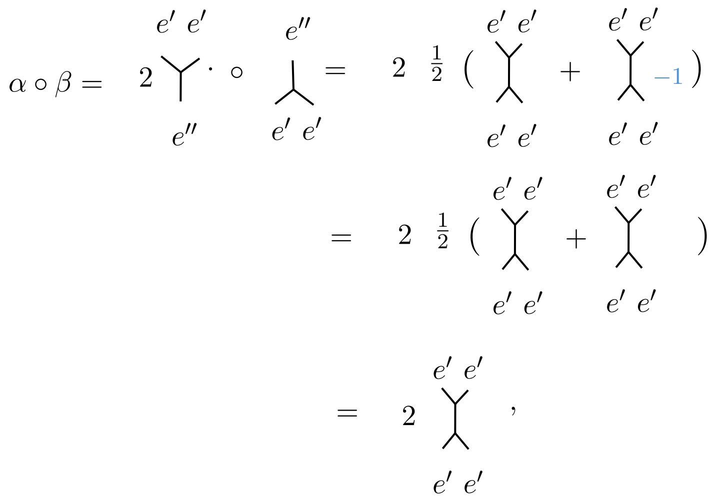
Similarly



Now it is easy to see that



and



which equal $id_{(W,e'')}$ and $id_{(W\otimes W, 2(e'\otimes e' \circ \,\text{[diagram]}\, \circ e'\otimes e'))}$ respectively. This proves the claim. Because Ω is an additive functor we get

$$\Omega([1] \oplus ([2], \sqcap\!\sqcup)) \cong \Omega([1]) \oplus \Omega(([2], \sqcap\!\sqcup)) \cong ([\tilde{1}], e') \oplus ([\tilde{1}], id_{[\tilde{1}]} - e') \cong [\tilde{1}] = ([\tilde{1}], id_{[\tilde{1}]}).$$

Let $([\tilde{k}], f)$ be any element of $\mathrm{Par}(\mathbb{Z}_2, 2t)^{Kar}$, where $k \in \mathbb{N}$ and $f : [\tilde{k}] \to [\tilde{k}]$ is an idempotent. Because $\Omega(([1] \oplus ([2], \sqcap\!\sqcup))^{\otimes k}) \cong [\tilde{k}] = ([\tilde{k}], id_{[\tilde{k}]})$ and Ω is full, there exists a morphism $f' : ([1] \oplus ([2], \sqcap\!\sqcup))^{\otimes k} \to ([1] \oplus ([2], \sqcap\!\sqcup))^{\otimes k}$ with $\Omega(f') \cong f$. Because Ω is a faithful functor and

$$\Omega(f' \circ f') = \Omega(f') \circ \Omega(f') \cong f \circ f = f \cong \Omega(f'),$$

the morphism $f' = f' \circ f'$ is also an idempotent. By definition, see Remark 3.2.16, we get

$$\Omega(([1] \oplus ([2], \sqcap\!\sqcup))^{\otimes k}, f') \cong ([\tilde{k}], f).$$

This concludes the proof. □

3.2.4 Concrete calculation of Ω

In this section we will shortly discuss the special commutative Frobenius algebra with involution-structure of $\Omega([1] \oplus ([2], \sqcap\!\sqcup)) \cong [\tilde{1}]$. By Corollary 2.6.16, this description gives us a $\mathbb{C}$-linear symmetric monoidal functor

$$\begin{aligned} \mathrm{Par}(\mathbb{Z}_2, t)^{Kar} &\to \mathrm{Par}(\mathbb{Z}_2, t)^{Kar} \\ [\tilde{1}] &\mapsto \Omega([1] \oplus ([2], \sqcap\!\sqcup)) \end{aligned}$$

which is isomorphic to the identity functor $id_{\mathrm{Par}(\mathbb{Z}_2,t)^{Kar}}$ in the endomorphism ring

$$\mathrm{End}_{\mathbb{C}}^{\otimes, Symm}(\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}) = \mathrm{Fun}_{\mathbb{C}}^{\otimes, Symm}(\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}, \mathrm{Par}(\mathbb{Z}_2, t)^{Kar}).$$

Remark 3.2.16. By the universal property of the Karoubian envelope, discussed in Remark 1.1.27, the functor Ω is only defined up to isomorphism, so there are multiple choices if we want to define a particular functor. But in this particular case, Ω being a functor from one Karoubian envelope into another, we can make a canonical choice for the images of the objects in $\underline{\mathrm{Rep}}(H_t)$, namely

$$\Omega(([k], f)) = ([\tilde{k}], \Omega_0(f)) = ([\tilde{k}], (e')^{\otimes k} \circ f \circ (e')^{\otimes k}).$$

For this choices it is possible to show that the functor is injective on objects, not just essentially injective which follows from the fact that it is fully faithful, see Lemma 1.1.31. Let

$$\Omega(([k], f)) = (([\tilde{k}], \Omega_0(f)) = ([\tilde{k}], (e')^{\otimes k} \circ f \circ (e')^{\otimes k})$$

equal

$$\Omega(([l], g)) = (([\tilde{l}], \Omega_0(g)) = ([\tilde{l}], (e')^{\otimes l} \circ g \circ (e')^{\otimes l}).$$

By the definition of the Karoubian envelope and the strictness of Ω_0, this will be equal if and only if $k = l$ and $(e')^{\otimes k} \circ f \circ (e')^{\otimes k} = (e')^{\otimes l} \circ g \circ (e')^{\otimes l}$. Because Ω_0 is faithful by Theorem 3.2.14, this holds if and only if $k = l$ and $f = g$. This shows that Ω is injective on objects and thus a full embedding for this particular choice. Note that for this choice the functor is essentially surjective, as was proven in Theorem 3.2.15, but not surjective. The reason for this is that $([\tilde{1}], 1 - e')$ will lie in the image of Ω if and only if we can find a morphism $\lambda : [1] \to [1]$ such that $e' \circ \lambda \circ e' = id_{[1]} - e'$, which is not possible.

Remark 3.2.17. By spelling out concretely some of the inner mechanisms of the proof of Theorem 3.2.15, we can see more clearly how it is possible that the image of the functor Ω has enough morphisms to mimic the behaviour of the morphisms between the tensor powers of $[\tilde{1}]$ in $\mathrm{Par}(\mathbb{Z}_2, 2t)^{Kar}$, even though these morphism spaces contain morphisms given by odd partitions. For this discussion we work in the category $(\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar}$ and use the notation of the proof. We set $\widetilde{W} := (W, 1 - e') \oplus (W \otimes W, 2(e' \otimes e' \circ \underset{\curlywedge}{\curlyvee} \circ e' \otimes e'))$.

The isomorphism $W \cong (W, e') \oplus (W, 1 - e')$, given by

$$\begin{bmatrix} e' \\ 1 - e' \end{bmatrix} : \ W \ \to (W, e') \oplus (W, 1 - e'), \text{ and}$$
$$\begin{bmatrix} e' & 1 - e' \end{bmatrix} : \ (W, e') \oplus (W, 1 - e') \ \to W,$$

and the isomorphism $(W, e') \oplus (W, 1 - e') \cong \widetilde{W}$ given by

$$\begin{bmatrix} e' & 0 \\ 0 & \alpha \end{bmatrix} : (W, e') \oplus (W, 1 - e') \to (W, e') \oplus (W \otimes W, 2(e' \otimes e' \circ \underset{\curlywedge}{\curlyvee} \circ e' \otimes e'))$$
$$\begin{bmatrix} e' & 0 \\ 0 & \beta \end{bmatrix} : (W, e') \oplus (W \otimes W, 2(e' \otimes e' \circ \underset{\curlywedge}{\curlyvee} \circ e' \otimes e')) \to (W, e') \oplus (W, 1 - e'),$$

compose and tensor to an isomorphism $W^{\otimes k} \cong (\widetilde{W})^{\otimes k}$, given by

$$\widetilde{\alpha_k} := \begin{bmatrix} e' \\ \alpha \circ (1-e') \end{bmatrix}^{\otimes k} = \begin{bmatrix} e' \\ \alpha \end{bmatrix}^{\otimes k} : W^{\otimes k} \to (\widetilde{W})^{\otimes k} \text{ and}$$

$$\widetilde{\beta_k} := \begin{bmatrix} e' & (1-e') \circ \beta \end{bmatrix}^{\otimes k} = \begin{bmatrix} e' & \beta \end{bmatrix}^{\otimes k} : (\widetilde{W})^{\otimes k} \to W^{\otimes k}.$$

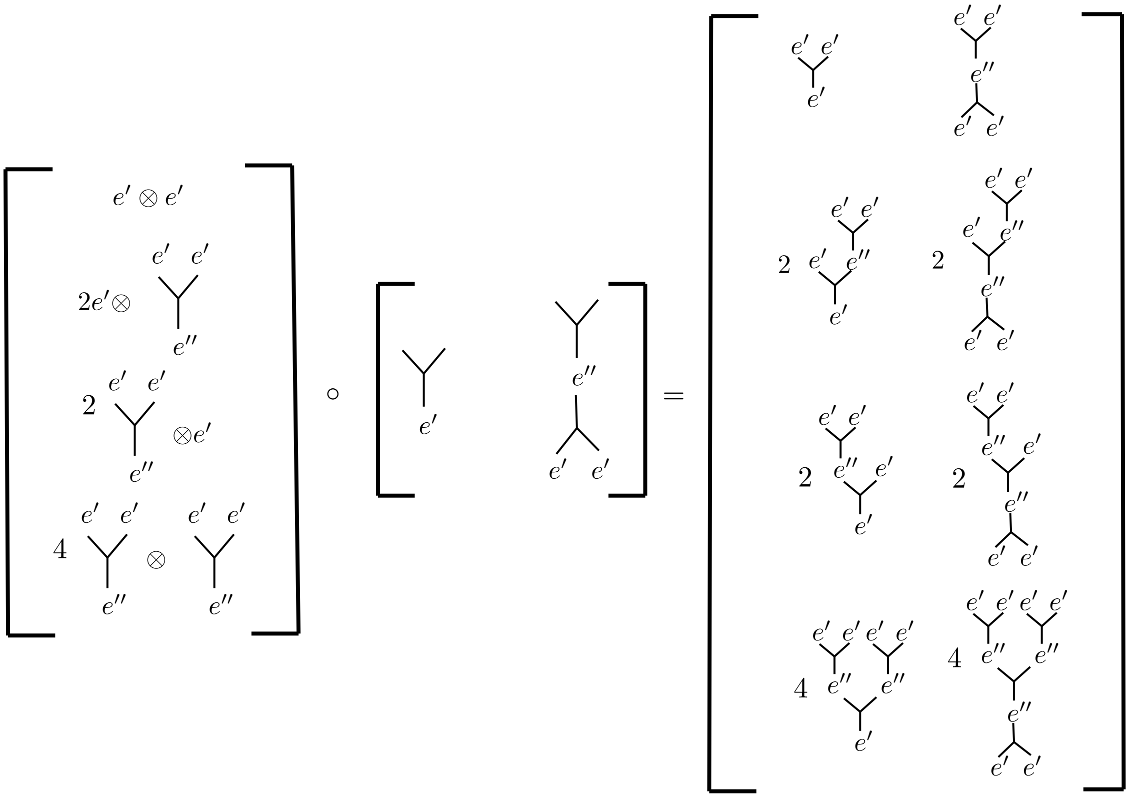
For every morphism $\phi \in \mathrm{Hom}_{(\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar}}(W^{\otimes k}, W^{\otimes l})$ there is corresponding morphism $\widetilde{\phi} := \widetilde{\alpha_k} \circ \phi \circ \widetilde{\beta_k} \in \mathrm{Hom}_{(\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})^{Kar}}(\widetilde{W}^{\otimes k}, \widetilde{W}^{\otimes l})$. For $\psi : W^{\otimes l} \to W^{\otimes m}$, we see that $\widetilde{\psi} \circ \widetilde{\phi} = \widetilde{\psi \circ \phi}$. So the morphisms, which correspond to the generating morphisms $|$, $\curlywedge$, $\curlyvee$, $\times$, $\overset{|}{\circ}$, $\underset{|}{\circ}$ and $-1\,\bullet$ of $(\mathrm{Par}(\mathbb{Z}_2)/\sim_{2t})$, behave exactly the same under composition. We write these out as an example. For compatibility of the matrix notation with the different morphisms and their compositions, we stick to the following order of the direct sumands

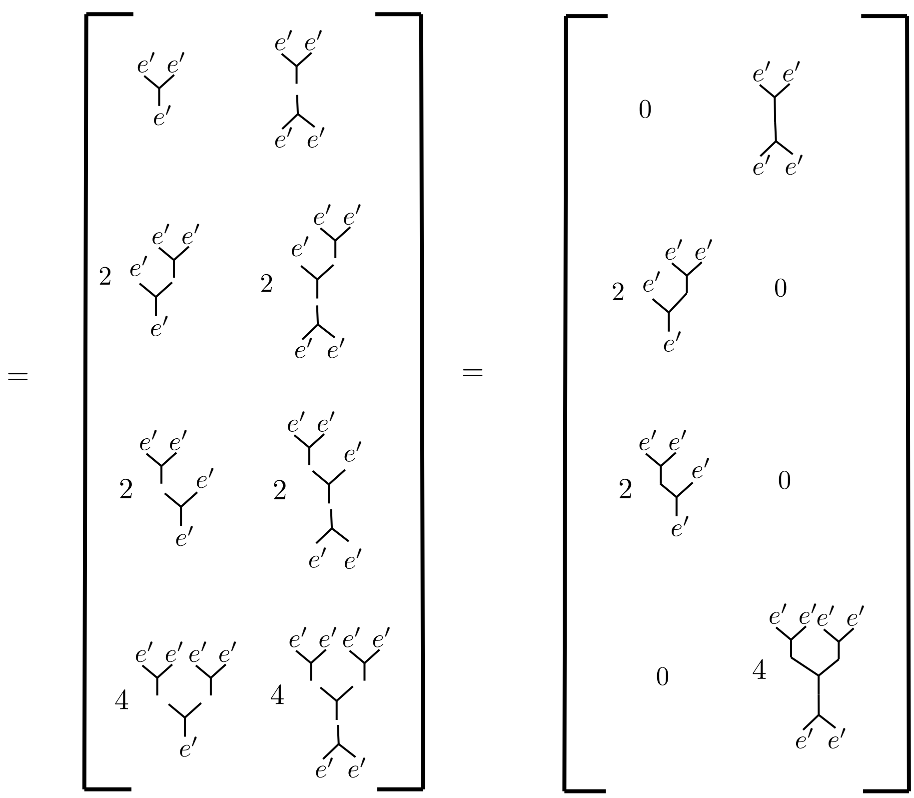
$$\begin{aligned}\widetilde{W} \otimes \widetilde{W} =& (W, e') \otimes (W, e') \\ &\oplus (W, e') \otimes (W \otimes W, 2(e' \otimes e' \circ \overset{\curlyvee}{\curlywedge} \circ e' \otimes e')) \\ &\oplus (W \otimes W, 2(e' \otimes e' \circ \overset{\curlyvee}{\curlywedge} \circ e' \otimes e')) \otimes (W, e') \\ &\oplus (W \otimes W, 2(e' \otimes e' \circ \overset{\curlyvee}{\curlywedge} \circ e' \otimes e')) \otimes (W \otimes W, 2(e' \otimes e' \circ \overset{\curlyvee}{\curlywedge} \circ e' \otimes e')).\end{aligned}$$

We calculate $\widetilde{\curlyvee}$ as follows

$$\begin{aligned}\widetilde{\curlyvee} &= \begin{bmatrix} e' \\ \alpha \end{bmatrix}^{\otimes 2} \circ \curlyvee \circ \begin{bmatrix} e' & \beta \end{bmatrix} \\ &= \begin{bmatrix} e' \otimes e' \\ e' \otimes \alpha \\ \alpha \otimes e' \\ \alpha \otimes \alpha \end{bmatrix} \circ \begin{bmatrix} \curlyvee \circ e' & \curlyvee \circ \beta \end{bmatrix}.\end{aligned}$$

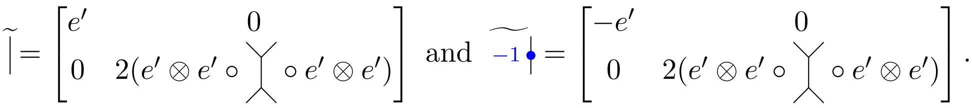
By writing out these diagrams, we obtain



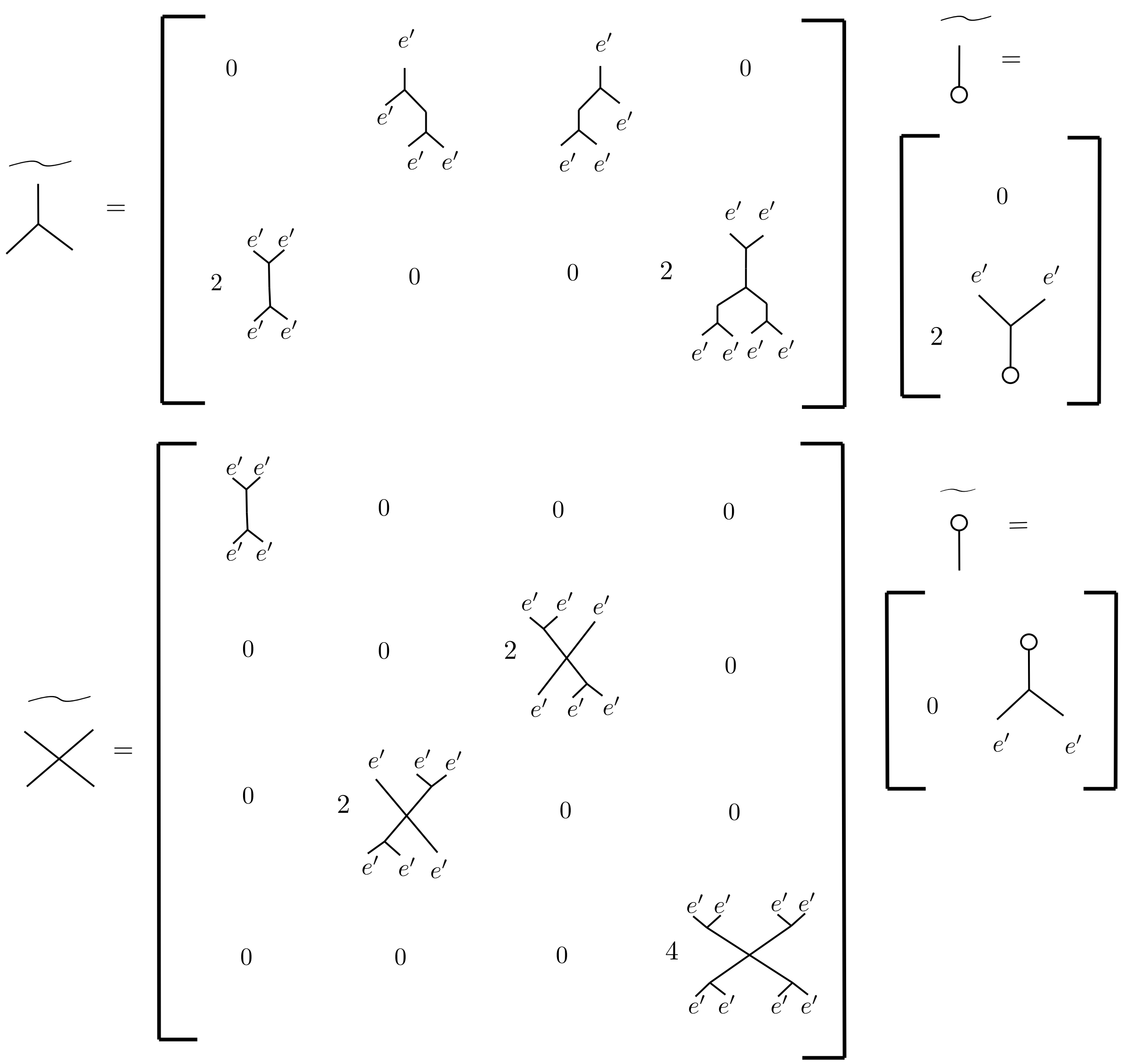


The proof of Theorem 2.6.4 shows that unlabeled diagrams in $Par([\mathbb{Z}_2])/\sim_{2t}$ with only one part and of the same size, are the same. This implies for example that the non-zero morphisms in the left column of the above matrix form for $\widetilde{Y}$, are the same.

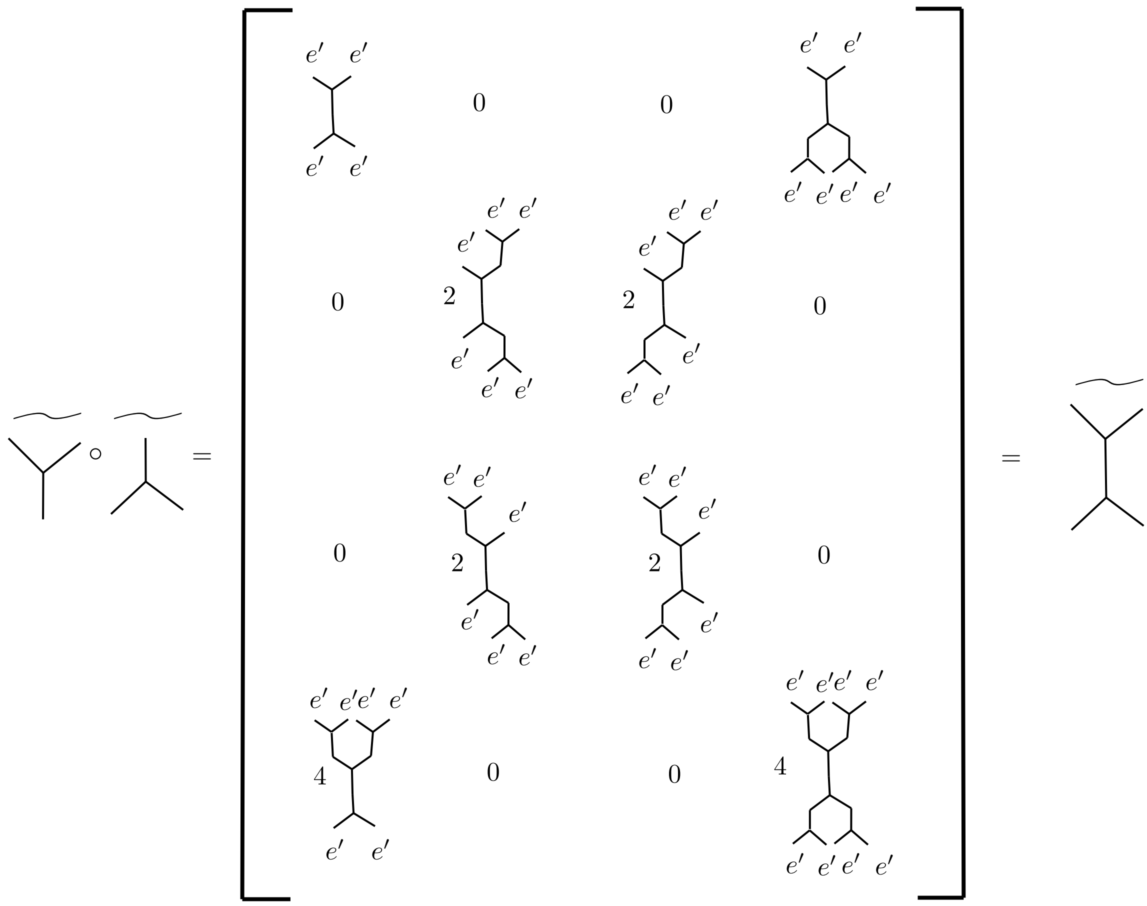
It is relatively easy to see that to



The others can be calculated in a similarly as we did for $\widetilde{Y}$, they equal



Now we can see using matrix multiplication that these morphisms indeed behave as we expected. For example



3.2.5 Ω and semisimplification

Corollary 3.2.18. *The equivalence $\Omega : \underline{Rep}(H_n) \to Par(\mathbb{Z}_2, 2n)^{Kar}$ makes the following square*

$$\begin{array}{ccc} \underline{Rep}(H_n) & \xrightarrow{G} & Rep(H_n) \\ \downarrow \Omega & & \downarrow = \\ Par(\mathbb{Z}_2, 2n)^{kar} & \xrightarrow{H} & Rep(H_n) \end{array} \tag{3.2.11}$$

commute up to isomorphism for all $n \in \mathbb{N}$.

Proof. By the universal property of the Karoubian envelope, the corollary will follow immediately if we can show that

$$H \circ \Omega_0 = G \circ \iota_G = G' : \underline{\mathrm{Rep}}_0(H_n) \to \mathrm{Rep}(H_n)$$

for some choice of H. We will work with the version of H which sends $([\tilde{k}], (e')^{\otimes k})$ to $u^{\otimes k}$ for all $k \in \mathbb{N}$. This is possible because $u^{\otimes k} \cong im(e^{\otimes k})$, where $e : V \to V$ was defined in the beginning of Section 3.2.

We want to show that the diagram

$$\begin{array}{ccccc} \underline{\mathrm{Rep}}_0(H_n) & \xrightarrow{\iota_G} & \underline{\mathrm{Rep}}(H_n) & \xrightarrow{G} & \mathrm{Rep}(H_n) \\ & \searrow^{\Omega_0} & & & \downarrow = \\ & & (Par(\mathbb{Z}_2, 2n))^{kar} & \xrightarrow{H} & \mathrm{Rep}(H_n) \end{array}$$

commutes strictly for our choice of H. For an object $[k] \in \underline{\mathrm{Rep}}_0(H_n)$ it is clear that

$$H \circ \Omega_0([k]) = H(([\tilde{k}], id_{[\tilde{k}]}) = u^{\otimes k} = G([k]) = G \circ \iota_G([k])$$

for this choice of H. Now we want to prove that the diagram commutes for morphisms. For this we use the alternative descriptions of the functors G' and H' at the end of Section 2.6.3 and Section 2.6.2, using the functors G'' and H'' respectively. The equivalences $\tilde{G}$ and $\tilde{H}$ reduce the problem to showing that the corresponding diagram

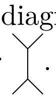
$$\begin{array}{ccccc} Par_n & \xrightarrow{(\tilde{G}^{Kar})^{-1}\circ\iota_G\circ\tilde{G}} & (Par_n)^{Kar} & \xrightarrow{G\circ\tilde{G}^{Kar}} & \mathrm{Rep}(H_n) \\ & \searrow^{(\tilde{H}^{Kar})^{-1}\circ\Omega_0\circ\tilde{G}} & & & \downarrow = \\ & & (Par(\mathbb{Z}_2)/\sim_{2n}))^{kar} & \xrightarrow{H\circ\tilde{H}^{Kar}} & \mathrm{Rep}(H_n) \end{array}$$

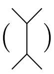
commutes for morphisms. So we have to prove that the images of the generating morphisms in Definition 2.6.17 under the functors

$$H \circ \tilde{H}^{Kar} \circ (\tilde{H}^{Kar})^{-1} \circ \Omega_0 \circ \tilde{G} = H \circ \Omega_0 \circ \tilde{G}$$

and

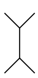
$$G \circ \tilde{G}^{Kar} \circ (\tilde{G}^{Kar})^{-1} \circ \iota_G \circ \tilde{G} = G \circ \iota_G \circ \tilde{G} = G' \circ \tilde{G} = G''$$

are equal. It follows directly from the functoriality of all involved functors that the diagram commutes for the identity . Now we want to show that the diagram commutes for . We first see that

$G''($  $)(e_i \otimes e_j) = \delta_{i,j} e_i \otimes e_i$

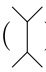
for $i, j \in \{1, \ldots, n\}$ and e_i a canonical basiselement of u. To use the explicit description of H' at the end of Section 2.6.2, we must first consider the case where H sends $([\tilde{k}], (e')^{\otimes k})$ to $(\tilde{u})^{\otimes k}$, a tensor powers of $\tilde{u}$, the subrepresentation of V isomorphic to u, see Remark 1.2.15. After we have done this, we can use the isomorphism

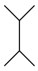
$$\begin{aligned} \tilde{u} &\to u \\ e_1^i - e_{-1}^i &\mapsto e_i \text{ for } i \in \{1, \ldots, n\} \end{aligned}$$

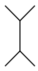
to find out the image of  under the functor $H \circ \tilde{H}^{Kar} \circ (\tilde{H}^{Kar})^{-1} \circ \Omega_0 \circ \tilde{G}$ for our original choice of H. So we will assume now that $H(([\tilde{k}], (e')^{\otimes k})) = (\tilde{u})^{\otimes k}$ until further notice. We already saw that the equality

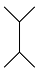
$$\begin{aligned} H(e')((e_1^i - e_{-1}^i)) &= \frac{1}{2}(e_1^i - e_{-1}^i - e_{-1}^i + e_1^i) \\ &= \frac{1}{2}2(e_1^i - e_{-1}^i) \\ &= (e_1^i - e_{-1}^i) \end{aligned}$$

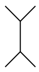
holds, which is obvious because $e_1^i - e_{-1}^i$ lies in the image of $e = H(e')$. Note that $H(e') = H''(e')$ where we use the corresponding definitions of e' in $(\mathrm{Par}(\mathbb{Z}_2, 2n))^{Kar}$ and $\mathrm{Par}(\mathbb{Z}_2)/\sim_{2n}$ respectively. We use $\iota_{\tilde{H}} : \mathrm{Par}(\mathbb{Z}_2)/\sim_{2n} \to (\mathrm{Par}(\mathbb{Z}_2)/\sim_{2n})^{Kar}$ to denote the $\mathbb{C}$-linear full embedding of the category into its Karoubian envelope. Then $H \circ \tilde{H}^{Kar} \circ \iota_{\tilde{H}} = H' \circ \tilde{H} = H''$ implies that

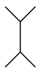
$H \circ \tilde{H}^{Kar} \circ (\tilde{H}^{Kar})^{-1} \circ \Omega_0 \circ \tilde{G}($  $)((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))$

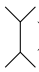
$= H \circ \tilde{H}^{Kar}(2e' \otimes e' \circ$  $\circ\, e' \otimes e')((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))$

$= H \circ \tilde{H}^{Kar} \circ \iota_{\tilde{H}}(2e' \otimes e' \circ$  $\circ\, e' \otimes e')((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))$

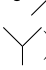
$= H' \circ \tilde{H}(2e' \otimes e' \circ$  $\circ\, e' \otimes e')((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))$

$= H''(2e' \otimes e' \circ$  $\circ\, e' \otimes e')((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))$

$= 2H''(e' \otimes e' \circ$  $\circ\, e' \otimes e')((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))$

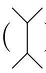
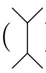
$= 2H''(e' \otimes e' \circ$  $)((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))$

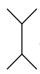
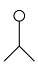
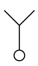
$= 2H''(e' \otimes e' \circ$  $\circ$  $)((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))$

$= 2\delta_{i,j} H''(e' \otimes e' \circ$  $)(e_1^i + e_{-1}^i)$

$$\begin{aligned} &= \delta_{i,j} 2H''(e' \otimes e')(e_1^i \otimes e_1^i + e_{-1}^i \otimes e_{-1}^i) \\ &= \delta_{i,j} 2\frac{1}{4}(2e_1^i \otimes e_1^i - 2e_{-1}^i \otimes e_1^i - 2e_1^i \otimes e_{-1}^i + 2e_{-1}^i \otimes e_{-1}^i) \\ &= \delta_{i,j} \frac{1}{2}2(e_1^i \otimes e_1^i - e_{-1}^i \otimes e_1^i - e_1^i \otimes e_{-1}^i + e_{-1}^i \otimes e_{-1}^i) \\ &= \delta_{i,j}(e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j), \end{aligned}$$

for $i, j \in \{1, \ldots, n\}$. By the previous remarks this shows that

$H \circ \tilde{H}^{Kar} \circ (\tilde{H}^{Kar})^{-1} \circ \Omega_0 \circ \tilde{G}($  $)(e_i \otimes e_j) = \delta_{i,j}(e_i \otimes e_i) = G''($  $)(e_i \otimes e_j)$

for our original choice of H. This implies the commutativity for . The commutativity for  and  is proven similarly. Lastly we want to show the commutativity of the diagram for $\times$. We see that

$$G''(\times)(e_i \otimes e_j) = e_j \otimes e_i$$

and for the choice $H(([\tilde{k}], (e')^{\otimes k})) = (\tilde{u})^{\otimes k}$ we see that

$$\begin{aligned}
H \circ \tilde{H}^{Kar} \circ (\tilde{H}^{Kar})^{-1} \circ \Omega_0 \circ \tilde{G}(\times)((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))& \\
= H \circ \tilde{H}^{Kar}(e' \otimes e' \circ \times \circ e' \otimes e')((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))& \\
= H \circ \tilde{H}^{Kar} \circ \iota_{\tilde{H}}(e' \otimes e' \circ \times \circ e' \otimes e')((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))& \\
= H' \circ \tilde{H}(e' \otimes e' \circ \times \circ e' \otimes e')((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))& \\
= H''(e' \otimes e' \circ \times \circ e' \otimes e')((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))& \\
= H''(e' \otimes e' \circ \times)((e_1^i - e_{-1}^i) \otimes (e_1^j - e_{-1}^j))& \\
= H''(e' \otimes e')((e_1^j - e_{-1}^j) \otimes (e_1^i - e_{-1}^i))& \\
= (e_1^j - e_{-1}^j) \otimes (e_1^i - e_{-1}^i)&
\end{aligned}$$

for all $i, j \in \{1, \ldots, n\}$. By the same arguments that we used to prove the commutativity for $\curlyvee$, this shows the commutativity for $\times$ and we conclude the proof. ☐

Corollary 3.2.19. *The functor Ω induces a $\mathbb{C}$-linear tensor functor*

$$\widehat{\Omega} : \widehat{\underline{Rep}(H_n)} \to \widehat{Par(\mathbb{Z}_2, 2n)}^{Kar}$$

which is an equivalence.

Proof. Ω is a $\mathbb{C}$-linear full tensor functor, so we can apply Proposition 2.5.11. Because the image of a negligible morphism under Ω is again a neglible morphisms the $\mathbb{C}$-linear functor $\widehat{\Omega} : \widehat{\underline{\mathrm{Rep}}(H_n)} \to \widehat{\mathrm{Par}(\mathbb{Z}_2, 2n)}^{Kar}$ exists and is well-defined. It is full because Ω is full. It is faithful because Ω is faithful and because only negligible morphisms have negligible morphisms as their image. The image of the objects under $\widehat{\Omega}$ is the same as for Ω by definition. Because the isomorphisms stay isomorphisms after semisimplification, also the essential image of $\widehat{\Omega}$ stays the same, showing that $\widehat{\Omega}$ is essentially surjective. Because $\widehat{G}$ and $\widehat{H}$ are tensor functors and equivalences and the diagram

$$\begin{array}{ccc}
\widehat{\underline{\mathrm{Rep}}(H_n)} & \xrightarrow{\widehat{G}} & \mathrm{Rep}(H_n) \\
\big\downarrow \widehat{\Omega} & & \big\downarrow = \\
\widehat{Par(\mathbb{Z}_2, 2n)}^{kar} & \xrightarrow{\widehat{H}} & \mathrm{Rep}(H_n)
\end{array}$$

commutes, $\widehat{\Omega}$ is also a tensor functor respecting the given monoidal structures of the semisimplifications of the interpolation categories for the hyperoctahedral group H_n.

☐

3.3 Image of indecomposable objects under Ω

Knop showed in [Kno07, Theorem 6.1] that in the semisimple case $t \neq 2\,\mathbb{N}$, the irreducible objects in $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ are classified by the set of all bipartitions, see also [LS21, Chapter 9]. We extend this result to the classification of the indecomposable objects in $\mathrm{Par}(\mathbb{Z}_2, t)^{Kar}$ for the non-semisimple cases $t \in 2\,\mathbb{N}\setminus\{0\}$.

3.3.1 Classyfying the indecomposables

We will not introduce the language that is used in [FM21] again. We do note that the set of even partitions P_{even} is a so-called category of partitions, see [FM21, Section 2.1], and that the partitions that we use to classify the indecomposable objects are so-called integer partitions. They state the following proposition, see [FM21, Proposition 5.12].

Proposition 3.3.1. *Let $t \in \mathbb{C}\setminus\{0\}$. Then there is a bijection between the set of bipartitions $\lambda = (\lambda_1, \lambda_2)$ of arbitrary size and the set of isomorphism classes of non-zero indecomposable objects in $\underline{\mathrm{Rep}}(H_t)$.*

Theorem 3.3.2. *Let $t \in \mathbb{C}\setminus\{0\}$. Then there is a bijection between the set of of bipartitions $\lambda = (\lambda_1, \lambda_2)$ of arbitrary size and the set of isomorphism classes of non-zero indecomposable objects in $Par(\mathbb{Z}_2, t)^{Kar}$.*

Proof. Any equivalence between categories induces a bijection between the isomorphism classes of the indecomposable objects. The statement follows therefore immediately from Proposition 3.3.1 and Theorem 3.2.15. □

3.3.2 Concrete examples of indecomposable objects

Instead of presenting a revised proof of Proposition 3.3.1, tailored to the particular case for the even partitions P_{even}, we explain how the bijection works by associating an indecomposable object in $\mathrm{Par}(\mathbb{Z}_2, 2t)^{Kar}$ to an arbitrary bipartition $\lambda = (\lambda_1, \lambda_2)$ for $t \in \mathbb{C}\setminus\{0\}$. It is proven in [FM21, Chapter 4 and 5] why this indeed yields a bijection.

It is a well-known fact that there is a bijection between partitions μ of size d and irreducible representations of the d-th symmetric group S_d. The reason for this is that we can associate to every partition μ a primitive idempotent $\frac{1}{n_\mu} c_\mu \in \mathbb{C}\,S_d$ and an irreducible representation $\mathbb{C}\,S_d \frac{1}{n_\mu} c_\mu$. Let $\{e_g|\ g \in S_d\}$ be a basis of the group algebra $\mathbb{C}\,S_d$. The Young symmetrizer is defined by $c_\mu := a_\mu b_\mu$, where $a_\mu = \sum_{g\in P} e_g$ and $b_\mu = \sum_{g\in Q} sgn(g) e_g$. Here P is the set of all elements in S_d of which the action preserves the rows for some Young Tableaux corresponding to μ and Q is the set of all elements in S_d of which the action preserves the columns of the same Young Tableaux.

Let $t \in \mathbb{C}\setminus\{0\}$ and $\lambda = (\lambda_1, \lambda_2)$ an arbitrary bipartition of size (k_1, k_2). Let $\frac{1}{n_{\lambda_1}} c_{\lambda_1} \in \mathbb{C}\,S_{k_1}$ and $\frac{1}{n_{\lambda_2}} c_{\lambda_2} \in \mathbb{C}\,S_{k_2}$ be the corresponding idempotents. We define the morphisms

$$p_1 := |^{\otimes k_1} \in \mathrm{End}_{\underline{\mathrm{Rep}}(H_t)}([k_1]) \text{ and}$$
$$p_2 := \begin{smallmatrix}\sqcup\\ \sqcap\end{smallmatrix}^{\otimes k_2} \in \mathrm{End}_{\underline{\mathrm{Rep}}(H_t)}([2k_2]).$$

Let $e_\sigma \in \mathbb{C}\, S_{k_1}$ and $e_\rho \in \mathbb{C}\, S_{k_2}$. Then we set

$$\begin{aligned} e_\sigma \cdot p_1 &:= \phi(\sigma) : [k_1] \to [k_1] \text{ and} \\ e_\rho \cdot p_2 &:= \sqcup^{\otimes k_2} \circ \phi(\rho) \circ \sqcap^{\otimes k_2} : [k_2] \to [k_2], \end{aligned}$$

and extend this definition $\mathbb{C}$-linearly. The morphism $\phi : S_d \to P_{even}(d, d)$ was defined in Definition 2.1.2 and as we noted already, the image of ϕ consists of even partitions only. Let $k := k_1 + 2k_2$. Then the object

$$([k], (\frac{1}{n_{\lambda_1}} c_{\lambda_1} \cdot p_1) \otimes (\frac{1}{n_{\lambda_2}} c_{\lambda_2} \cdot p_2))$$

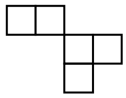
is indecomposable in $\underline{\mathrm{Rep}}(H_t)$. By Remark 3.2.16, the indecomposable object in $\mathrm{Par}(\mathbb{Z}_2, 2n)^{Kar}$ corresponding to the bipartition λ is given by

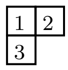
$$\begin{aligned} \Omega(([k], (\frac{1}{n_{\lambda_1}} c_{\lambda_1} \cdot p_1) \otimes (\frac{1}{n_{\lambda_2}} c_{\lambda_2} \cdot p_2))) &= ([\tilde{k}], \Omega_0((\frac{1}{n_{\lambda_1}} c_{\lambda_1} \cdot p_1) \otimes (\frac{1}{n_{\lambda_2}} c_{\lambda_2} \cdot p_2))) \\ &= ([\tilde{k}], (e')^{\otimes k} \circ ((\frac{1}{n_{\lambda_1}} c_{\lambda_1} \cdot p_1) \otimes (\frac{1}{n_{\lambda_2}} c_{\lambda_2} \cdot p_2)) \circ (e')^{\otimes k}). \end{aligned}$$

The objects which correspond to the bipartitions $(1, 0)$ and $(0, 1)$ are

$$\begin{aligned} \Omega(([1], |\,)) = \Omega_0([k]) &= ([\tilde{1}], e') \text{ and} \\ \Omega(([2], \substack{\sqcup \\ \sqcap})) = ([\tilde{2}], 2e' \otimes e' \circ \substack{\sqcup \\ \sqcap} \circ e' \otimes e') &\cong ([\tilde{1}], id_{[\tilde{1}]} - e') \end{aligned}$$

respectively. This shows that $([\tilde{1}], e')$ and $([\tilde{1}], id_{[\tilde{1}]} - e')$ are indecomposable objects in $\mathrm{Par}(Z_2, 2t)^{Kar}$. As a last example we give a description of the indecomposable object corresponding to the bipartition $\lambda = (\lambda_1, \lambda_2)$ of size $(3, 2)$, where $\lambda_1 = \{\{1, 2\}, \{3\}\}$ and $\lambda_2 = \{\{1, 2\}\}$.
The following preparations

$\lambda =$ 

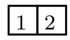
$\lambda_1 =$ 

$a_{\lambda_1} = (12) + id$

$b_{\lambda_1} = -(13) + id$

$c_{\lambda_1} = a_\lambda b_\lambda = id\ + (12) - (13) - (123)$

$n_{\lambda_1} = 3$

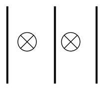
$\lambda_2 =$ 

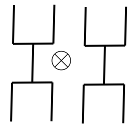
$a_{\lambda_2} = (12) + id$

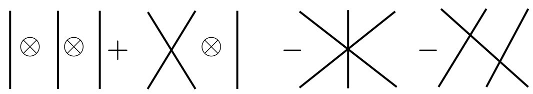
$b_{\lambda_2} = id$

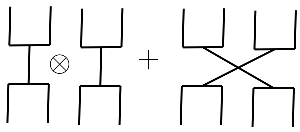
$c_{\lambda_2} = id + (12)$

$n_{\lambda_2} = 2$

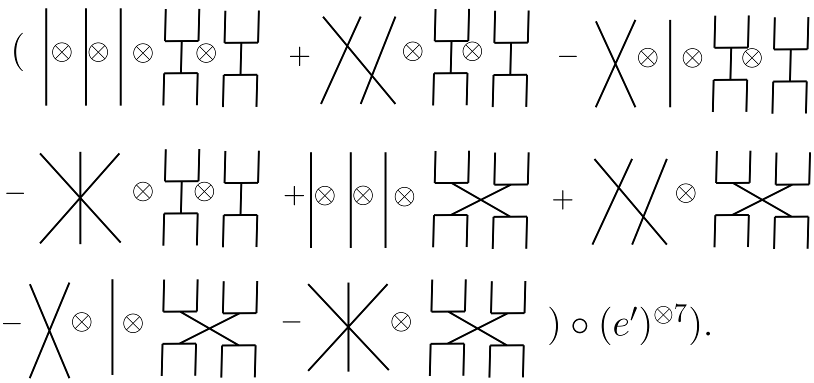
$p_1 =$ 

$p_2 =$ 

$\frac{1}{n_{\lambda_1}} c_{\lambda_1} p_1 = \frac{1}{3}($  $)$

$\frac{1}{n_{\lambda_2}} c_{\lambda_2} p_2 = \frac{1}{2}($  $)$

give us the indecomposable object in $Par(Z_2, 2t)^{Kar}$ corresponding to the bipartition $\lambda = (\lambda_1, \lambda_2)$:

$(\widetilde{[7]}, \frac{1}{6}(e')^{\otimes 7} \circ ($  $) \circ (e')^{\otimes 7}).$

Indecomposable object corresponding to λ.

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