Abstracts

Semisimplification of representation categories

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(joint work with Rainer Weissauer)

0.1. **Semisimplification.** Let $\mathcal{C}$ denote a k-linear (k a field) braided rigid monoidal category with unit object $\mathbf{1}$ and $End(\mathbf{1}) \cong k$. For such a category one can define the trace $Tr(f) \in End(\mathbf{1}) \cong k$ of an endomorphism $f \in End(X)$ for any $X \in \mathcal{C}$ and the dimension of X via $dim(X) = Tr(id_X)$. The negligible morphisms

$$\mathcal{N}(X,Y) = \{f : X \to Y \mid Tr(g \circ f) = 0 \ \forall g : Y \to X\}$$

can be seen as an obstruction to the semisimplicity of $\mathcal{C}$. The negligible morphisms form a tensor ideal of $\mathcal{C}$ and the quotient category $\mathcal{C}/\mathcal{N}$ is again a k-linear braided rigid monoidal category. Under some mild assumptions on $\mathcal{C}$ [AK02] the quotient is semisimple. We call this the *semisimplification* of $\mathcal{C}$.

0.2. **Representation categories.** Examples of such categories are often coming from representation theory.

(1) $\mathcal{C} = Rep(G,k)$, the category of finite dimensional representations of an algebraic group over a field k; or $\mathcal{C} = Tilt(G, \mathbb{F}_q) \subset Rep(G, \mathbb{F}_q)$, the category of tilting modules for a semisimple, simply connected algebraic group G.
(2) $\mathcal{C} = Rep(U_q(\mathfrak{g}))$, finite dimensional modules of type 1 for Lusztig's quantum group $U_q(\mathfrak{g})$ for a complex semisimple Lie algebra $\mathfrak{g}$; or $\mathcal{C} = Tilt(U_q(\mathfrak{g})$, the subcategory of tilting modules.
(3) $\mathcal{C} = Del_t$, one of the Deligne categories associated to $GL(n)$, $O(n)$ or S_n for $t \in \mathbb{C}$, or its abelian envelope.

Of particular importance in this list is $Tilt(U_q(\mathfrak{g}))$ (studied e.g. in [AP95]) since the semisimple quotient is a modular tensor category. For other examples see [EO18]. André and Kahn [AK02] studied the case where $\mathcal{C} = Rep(G)$, the category of representations of an algebraic group over a field k of characteristic 0. In this case $\mathcal{C}/\mathcal{N}$ is of the form $Rep(G^{red})$ where G^{red} is a pro-reductive group, the *reductive envelope* of G (this is false in $char(k) > 0$).

0.3. **Representations of supergroups.** The results of [AK02] generalize partially to algebraic supergroups if k is algebraically closed. Using a characterization of super tannakian categories by Deligne [Del02], the quotient $Rep(G)$ of representations of an algebraic supergroup on finite dimensional super vector spaces by the negligible morphisms is of the form $Rep(G^{red}, \varepsilon)$ where G^{red} is an affine supergroup scheme and $\varepsilon : \mathbb{Z}/2\mathbb{Z} \to G^{red}$ such that the operation of $\mathbb{Z}/2\mathbb{Z}$ gives the $\mathbb{Z}_2$-graduation of the representations [He15]. A determination of G^{red} is typically out of reach. More amenable is the full monoidal subcategory $Rep(G)^I$ of direct summands in iterated tensor products of irreducible representations of $Rep(G)$. The irreducible representations of the quotient category $Rep(G)^I/\mathcal{N} \cong Rep(H, \varepsilon')$

correspond to indecomposable direct summands of non-vanishing superdimension in such iterated tensor products. The aim is then to determine H. For an irreducible representation $L(\lambda)$ consider its image in $Rep(H)$ and take the tensor category generated by it. This category is of the form $Rep(H_\lambda, \varepsilon')$ for a reductive group H_λ and $L(\lambda)$ corresponds to an irreducible faithful representation V_λ of H_λ.

0.4. **The category** $Rep(GL(m|n))$**.** Let $\mathcal{T}_{m|n}$ be the category of finite dimensional representations of $GL(m|n)$. The categories $\mathcal{T}_{m|n}$ are not semisimple for $m, n \geq 1$. As above we consider only objects that are retracts of iterated tensor products of irreducible representations $L(\lambda)$. This subcategory is called $\mathcal{T}^I_{m|n}$ and we denote the pro-reductive group of its semisimple quotient by $H_{m|n}$. The crucial tool to determine $H_{m|n}$ is the Duflo-Serganova functor [DS05] [HW14] $DS : \mathcal{T}_{m|n} \to \mathcal{T}_{m-1|n-1}$. It allows us to reduce the determination of $H_{m|n}$ to lower rank.

Theorem [HW18, Theorem 5.15] *a)* $H_{m|n}$ *is a pro-reductive group. b)* DS *restricts to a tensor functor* $DS : \mathcal{T}^I_{m|n} \to \mathcal{T}^I_{m-1|n-1}$ *and gives rise to a functor* $DS : \mathcal{T}^I_{m|n}/\mathcal{N} \to \mathcal{T}^I_{m-1|n-1}/\mathcal{N}$*. c) There is an embedding* $H_{m-1|n-1} \to H_{m|n}$ *and* DS *can be identified with the restriction functor.*

We specialize now to $GL(n|n)$ and use the notation $G_n = (H_{n|n})^0_{der}$ and $G_\lambda = (H_\lambda)^0_{der}$. We also suppose that $sdim(L(\lambda)) > 0$ since we can replace $L(\lambda)$ by its parity shift. We say a representation is weakly selfdual (SD) if it is selfdual after restriction to $SL(n|n)$.

Theorem [HW18, Theorem 6.2] $G_\lambda = SL(V_\lambda)$ *if* $L(\lambda)$ *is not (SD). If* $L(\lambda)$ *is (SD) and* $V_\lambda|_{G_{\lambda'}}$ *is irreducible,* $G_\lambda = SO(V_\lambda)$ *respectively* $G_\lambda = Sp(V_\lambda)$ *according to whether* $L(\lambda)$ *is orthogonal or symplectic selfdual. If* $L(\lambda)$ *is (SD) and* $V_\lambda|_{G_{\lambda'}}$ *decomposes into at least two irreducibe representations, then* $G_\lambda \cong SL(W)$ *for* $V_\lambda|_{G_{\lambda'}} \cong W \oplus W^\vee$.

We conjecture that the last case in the theorem doesn't happen. The ambiguity in the determination of G_λ is only due to the fact that we cannot exclude special elements with 2-torsion in $\pi_0(H_{n|n})$.

Theorem [HW18, Theorem 6.8] *Let* $\lambda \sim \mu$ *if* $L(\lambda) \cong L(\mu)$ *or* $L(\lambda) \cong L(\mu)^\vee$ *after restriction to* $SL(n|n)$*. Then*

$$G_n \cong \prod_{\lambda \in X^+/\sim} G_\lambda.$$

In down to earth terms, these theorems give

- the decomposition law of tensor products of indecomposable modules in $\mathcal{T}^I_{m|n}$ up to indecomposable summands of superdimension 0; and
- a classification (in terms of the highest weights of H_λ and H_μ) of the indecomposable modules of non-vanishing superdimension in iterated tensor products of $L(\lambda)$ and $L(\mu)$.

We remark that the statement about $G_{n|n}$ implies a strange disjointness property of iterated tensor products of irreducible representations of non vanishing superdimension. For the general $\mathcal{T}_{m|n}$-case ((where $m \geq n$) recall that every maximal atypical block in $\mathcal{T}_{m|n}$ is equivalent to the principal block of $\mathcal{T}_{n|n}$. We denote the image of an irreducible representation $L(\lambda)$ under this equivalence by $L(\lambda^0)$.

Conjecture (work in progress) *Suppose that* $\mathrm{sdim}(L(\lambda)) > 0$. *Then* $H_\lambda \cong Rep(GL(m-n)) \times H_{\lambda^0}$ *and* $L(\lambda)$ *corresponds to the representation* $L_\Gamma \otimes V_{\lambda^0}$ *of* H_λ. *Here* L_Γ *is an irreducible representation of* $GL(m-n)$ *which only depends on the block* Γ *(the* core *of* Γ*).*

References

[AP95] Henning Haahr Andersen and Jan Paradowski, *Fusion categories arising from semisimple Lie algebras.*, Commun. Math. Phys., 169, 1995.

[AK02] Yves André and Bruno Kahn, *Nilpotence, radicaux et structures monoïdales.* Rend. Semin. Mat. Univ. Padova, 108, 2002.

[Del02] Deligne, P., *Catégories tensorielles. (Tensor categories).*, Mosc. Math. J. , 2, 2, 2002.

[Del90] Deligne, P., *Catégories tannakiennes. (Tannaka categories).*, The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. II, Prog. Math. 87, 111-195 (1990)..

[DS05] Duflo, M. and Serganova. V., *On associated variety for Lie superalgebras.*, arXiv:math/0507198v1, 2005.

[EO18] Etingof, P. and Ostrik, V., On semisimplification of tensor categories, ArXiv e-prints, 1801.04409, 2018.

[He15] Heidersdorf, T., On supergroups and their semisimplified representation categories, to appear in: Algebr. Represent. Theory ArXiv e-prints, 1512.03420, (2015).

[HW14] Heidersdorf, T. and Weissauer, R., *Cohomological tensor functors on representations of the General Linear Supergroup* to appear in Mem. Am. Math. Soc., ArXiv e-prints, 1406.0321, 2014.

[HW18] Heidersdorf, T. and Weissauer, R., On classical tensor categories attached to the irreducible representations of the General Linear Supergroups $GL(n|n)$, ArXiv e-prints, 1805.00384, (2018).

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