Quantum matrix algebras and their applications. Braided Yangians

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Plan

There are known two ways of introducing quantum algebras.
The most known is based on the QG $U_q(\mathfrak{g})$.
The other is based on a given braiding R.
Consider a vector space V over the ground field $\mathbb{K} = \mathbb{C}$. We call an invertible linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$ *braiding* if it satisfies the so-called *braid relation*

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}, \quad R_{12} = R \otimes I, \;\; R_{23} = I \otimes R.$$

A braiding R is called *involutive symmetry* if $R^2 = I$.
A braiding is called *Hecke symmetry* if it is subject to the Hecke condition

$$(q\,I - R)(q^{-1}\,I + R) = 0,\ \ q \in \mathbb{K}.$$

In particular, such a symmetry comes from the QG $U_q(sl(m))$.

As for the braidings coming from the QG of other series B_n, C_n, D_n, each of them has 3 eigenvalues and it is called BMW symmetry.

The simplest examples are as follows. By fixing a basis $\{x, y\} \in V$ and the corresponding basis $\{x\otimes x, x\otimes y, y\otimes x, y\otimes y\}$ in $V^{\otimes 2}$ we represent Hecke symmetries R by the matrices

$$\begin{pmatrix} q & 0 & 0 & 0\\ 0 & q-q^{-1} & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & q \end{pmatrix}, \quad \begin{pmatrix} q & 0 & 0 & 0\\ 0 & q-q^{-1} & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

They are deformations of the usual flip and the super-flip respectively. However, there is a lot of Hecke symmetries which are deformations neither of flips nor of super-flips.

In order to classify Hecke symmetries, consider "R-symmetric" and "R-skew-symmetric" algebras

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \ \bigwedge\nolimits_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle$$

and the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V)t^k, \ P_-(t) = \sum_k dim \bigwedge\nolimits_R^{(k)}(V)t^k,$$

where the upper index (k) labels homogenous components of these quadratic algebras.

For a generic q the following holds $P_-(-t)P_+(t) = 1$.

Proposition. (Phung Ho Hai)

The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function:

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \ldots + a_r\, t^r}{1 - b_1\, t + \ldots + (-1)^s\, b_s\, t^s} = \frac{\prod_{i=1}^r (1 + x_i t)}{\prod_{j=1}^s (1 - y_j t)},$$

where a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers x_i and y_i are real positive.

We call the couple $(r|s)$ bi-rank. In this sense all Hecke symmetries are similar to super-flips.

Examples. If R comes from the QG $U_q(sl(m))$, then

$$P_-(t) = (1+t)^m.$$

If R is a deformation of the super-flip $P_{m|n}$, then

$$P_-(t) = \frac{(1+t)^m}{(1-t)^n}.$$

Also, there exist "exotic" examples: for any $m \geq 2$ there exits a Hecke symmetry such that

$$P_-(t) = 1 + mt + t^2.$$

If $P_-(t)$ is a polynomial, R is called *even*.

Given an even Hecke symmetry R, how to construct a category, similar to that $Rep - U_q(sl(m))$ of finite dimensional modules? First, it is necessary to extend $R = R^V$ up to a braiding

$$R = R^{V \oplus V^*} : (V \oplus V^*)^{\otimes 2} \to (V \oplus V^*)^{\otimes 2},$$

with the paring $\langle\, , \,\rangle : V \otimes V^* \to \mathbb{K}$ such that

$$\langle\, , \,\rangle_{23} = \langle\, , \,\rangle_{12} R_2 R_1 \ \text{on}\ W \otimes V \otimes V^*$$

$$\langle\, , \,\rangle_{12} = \langle\, , \,\rangle_{23} R_1 R_2 \ \text{on}\ V \otimes V^* \otimes W,$$

where $R = R^{V \oplus V^*}$ and $W = V$ or $W = V^*$.
We say that this pairing is R-invariant.

This extension is possible iff there exists an operator $\Psi : V^{\otimes 2} \to V^{\otimes 2}$ such that

$$R_{ij}^{kl} \Psi_{lp}^{jq} = \delta_i^q \delta_p^k,$$

where $R(x_i \otimes x_j) = R_{ij}^{kl} x_k \otimes x_l$.
If it is so, R is called *skew-invertible*. Then it is possible to construct a quasi-tensor rigid category $SW(V)$ similar to that $Rep - U_q(sl(m))$ (q is generic).
Any two objects of this category U and W are equipped with a braiding $R^{U,W} : U \otimes W \to W \otimes U$.

For a given object U of the category $SW(V)$ consider the object $End(U) \cong U \otimes U^*$. By using the operator Ψ, it is possible to define the so-called *R-trace* $Tr_R : End(U) \to \mathbb{K}$ and *R-dimension* of U.

Example. If $R : V^{\otimes 2} \to V^{\otimes 2}$ has bi-rank $(m|n)$, then

$$dim_R\, V = q^{n-m}(m-n)_q, \;\; k_q = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

Thus, for a standard R, i.e. coming from $U_q(sl(m))$

$$dim_R V = q^{-m}\, m_q.$$

Also, it is tempting to define R-analog $gl(V_R)$ of the Lie algebra $gl(m)$ by putting

$$[\, ,\,]_R : X \otimes Y \to X \circ Y - \circ R^{\mathrm{End}(V)}(X \otimes Y), \;\; X, Y \in End(V)$$

and its enveloping algebra

$$U(gl(V_R)) = T(End(V))/\langle X \otimes Y - R^{\mathrm{End}(V)}(X \otimes Y) - [X, Y]_R\rangle.$$

If R is involutive [Gurevich 1983], the bracket $[\,,\,]_R$ meets 3 axioms:

1. R-skew-symmetry,
2. R-Jacobi,
3. compatibility with $R^{\mathrm{End}(V)}$: the bracket is R-invariant.

However, if R is Hecke, there is no "natural" R-Jacobi, but an analog of the Lie algebra $gl(V_R)$ and its enveloping algebra $U(gl(V_R))$ exists.

By QM algebras I mean the RTT algebra

$$RT_1T_2 - T_1T_2R, \ T = (t_i^j), 1 \le i,j \le m$$

and RE algebra

$$RL_1RL_1 - L_1RL_1R = 0, \ L = (l_i^j), \ 1 \le i,j \le m.$$

Also, we consider the so-called modified RE algebra

$$RL_1RL_1 - L_1RL_1R = h(RL_1 - L_1R),$$

which is quadratic-linear algebra.
If R is standard, this modified RE algebra is a two parameter deformation of $Sym(gl(m))$.

The corresponding Poisson structure is a Poisson pencil.
Example. The first Poisson bracket is $sl(2)$ bracket:

$$\{h,x\}=2x,\ \{h,y\}=-2y,\ \{x,y\}=h.$$

The second one is

$$\{h,x\}=2xh,\ \{h,y\}=-2yh,\ \{x,y\}=h^2.$$

It is not unimodular. But it can be quantized with a R-trace.

If R comes from a QG of other series, the modified RE algebra is defined by the same formula, but is not a deformation of $Sym(\mathfrak{g})$ (even the RE algebra is not). [Donin]

Other properties of the RTT algebra and RE one differ drastically. One of them: the RTT is a bi-algebra with the coproduct

$$\Delta(t_i^j) = \sum t_i^k \otimes t_k^j.$$

The RE is a *braided* bi-algebra [Majid]

$$\Delta(l_i^j) = \sum l_i^k \otimes l_k^j,$$

$$\Delta(l_i^j l_m^n) = \Delta(l_i^j)\Delta(l_m^n) = (\sum_k l_i^k \otimes l_k^j)(\sum_p l_m^p \otimes l_p^l) =$$

$$\sum_{k,p} l_i^k R^{End(V)}(l_k^j \otimes l_m^p) l_p^l$$

Here l_i^j are treated to be elements of $End(V)$.

Note that in the standard case the QG $U_q(sl(m))$ acts on the elements l_i^j by the adjoint action. Whereas the generators t_i^j can be equipped with $U_q(sl(m)) \times U_q(sl(m))^{op}$ (the action of each factor is one-sided).

Another difference in properties is related to the so-called characteristic subalgebra.
To define it introduce the following notation

$$L_{\overline{1}} = L_1,\ \ L_{\overline{2}} = R_1\, L_{\overline{1}}\, R_1^{-1},\ \ L_{\overline{3}} = R_2\, L_{\overline{2}}\, R_2^{-1} = R_2\, R_1\, L_{\overline{1}}\, R_1^{-1}\, R_2^{-1}, \dots$$

In this notation the defining relations of the RE algebra become similar to the RTT ones

$$RL_{\overline{1}}L_{\overline{2}} = L_{\overline{1}}L_{\overline{2}}R.$$

Consider the Hecke algebras $H_k(q)$ and their so-called R-matrix representations (denoted ρ_R) in the spaces $V^{\otimes k}$ realised via a given Hecke symmetry R.

Proposition.

For an element $x(k) \in H_k(q)$ denote

$$ch(x(k)) := Tr_{R\,(1\ldots k)} X(k) L_{\overline{1}} \ldots L_{\overline{k}}, \text{ where } X(k) = \rho_R(x(k)).$$

Consider a linear subspace $Ch[R] \subset \mathcal{L}(R)$ spanned by the unity and elements $ch(x(k))$ for all $k \geq 1$ and $x(k) \in H_k(q)$. The space $Ch[R]$ is a subalgebra of the center of the algebra $\mathcal{L}(R)$. Moreover, it is covariant with respect to the action of the QG $U_q(sl(m))$ in the standard case.

This is also valid for a skew-invertible involutive symmetry, but $H_k(q)$ should be replaced by $\mathbb{K}[S_k]$.

Definition

The algebra $Ch[R]$ is called characteristic.

Remark.

In a similar manner the characteristic subalgebra of an RTT algebra can be defined. The only modification to do is: the product $L_{\overline{1}}...L_{\overline{k}}$ should be replaced by $T_1...T_k$. However, this subalgebra is not central. It is only possible to claim that this subalgebra is commutative.

In what follows we need the only element of the characteristic subalgebra: namely, *determinant*, provided R is even.

Nevertheless, if R has the bi-rank $(m|n)$, analogs of the Berezinian and determinant can be also defined.

Also, note that elements $Tr_R(L^k)$, $k = 0, 1, ...$ are central in the RE algebra. Besides, in this algebra there is an analog of the Cayley-Hamilton identity. If R is even of the bi-rank $(m|0)$, the corresponding CH identity is

$$L^m + a_{m-1} L^{m-1} + ... + a_0 I = 0,$$

where a_k are central in the RE algebra.

Pass now to the first application. It arises from differential calculus on the RE algebra. In a sense a deferential calculus on an RE algebra is more interesting than that on an RTT algebra (Woronowicz and others). By passing to the limit $q \to 1$ we obtained a differential calculus on the enveloping algebras $U(gl(m)_h)$ and their super-analogs.

Here subscribe h means that it is introduced in the Lie bracket in order to represent the enveloping algebra as result of quantization of the algebra $Sym(gl(m))$.

Below, we define analogs of partial derivatives on the algebra $U(gl(m)_h)$ so that for $h = 0$ we recover the usual partial derivatives in generators of $Sym(gl(m))$.

Also, we define a quantum analog of the differential algebra $\Omega(Sym(gl(m))$ and this of the de Rham operator. All objects are deformations of their classical counterparts.

Given a NC algebra A, the differential algebra $\Omega(A)$ on A is usually defined via the de Rham operator satisfying the classical Leibniz rule $d(a\,b) = da\,b + a\,db$ but without relation $a(db) = (db)a$.

This approach leads to the universal differential algebra which is much bigger than the classical one is, if A is commutative. In our construction we retrieve the classical differential algebra as $h \to 0$.

Fix in the Lie algebra $gl(m)_h$ the standard basis $\{n_i^j\},\ 1 \le i, j \le m$ and the usual Lie bracket

$$[n_i^j, n_k^l] = h(n_i^l \delta_k^j - n_k^j \delta_i^l), 1 \le i, j, k, l, \le m.$$

Now, define "partial derivatives" on the algebra $U(gl(m)_h)$.

Observe that in the algebra $Sym(gl(m))$ the partial de derivatives $\partial_k^l = \partial_{n_l^k}$ are defined via the action on the generators $\partial_k^l(n_i^j) = \delta_i^l\, \delta_k^j$ (i.e. the partial derivatives span the space dual to that $span(n_i^j)$) and the coproduct

$$\Delta(\partial_k^l) = \partial_k^l \otimes 1 + 1 \otimes \partial_k^l.$$

Thus, we have the Leibniz rule

$$\partial_k^l(ab) = \partial_k^l(a)b + a\partial_k^l(b).$$

By passing to the algebra $U(gl(m)_h)$ we do not change the first property (the pairing) and define the new Leibniz rule by means of the following coproduct

$$\Delta(\partial_i^j) = \partial_i^j \otimes 1 + 1 \otimes \partial_i^j + h \sum_k \partial_k^j \otimes \partial_i^k.$$

So, we have

$$\partial_i^j(ab) = \partial_i^j(a)b + a\partial_i^j(b) + h \sum_k \partial_k^j(a)\partial_i^k(b).$$

Observe that the partial derivatives ∂_i^j commute with each other. Denote $\mathcal{D}$ commutative algebra generated by them.

Our next aim is to define an analog of the Weyl algebra $\mathcal{W}(Sym(gl(m))$. The simplest classical Weyl algebra is $pq - qp = 1$. (Note that physicists call it Heisenberg one.) Our NC Weyl algebra (denoted $\mathcal{W}(U(gl(m)_h))$ is generated by two subalgebras $U(gl(m)_h)$ and $\mathcal{D}$, also subject to some permutation relations.

Define permutation relations as follows

$$\partial_i^j \otimes n_k^l \to (\partial_i^j)_1(n_k^l) \otimes (\partial_i^j)_2, \text{ where } \Delta(\partial_i^j) = (\partial_i^j)_1 \otimes (\partial_i^j)_2$$

in Sweedler's notation.
Let us exhibit them explicitely

$$\partial_i^j \otimes n_k^l \to n_k^l \otimes \partial_i^j + \delta_i^l \delta_k^j + h(\partial_i^l \delta_k^j - \partial_k^j \delta_i^l).$$

Note that for $h = 0$ we get the usual Weyl algebra generated by the algebra $Sym(gl(m))$ and the usual partial derivatives.

Now, we define an analog of the de Rham complex on $U(gl(m)_h)$ as follows.
Introduce "pure differentials" dn_i^j by assuming that they anti-commute with each other.

Let

$$\omega = dn_{i_1}^{j_1} \bigwedge dn_{i_2}^{j_2} \bigwedge \ldots \bigwedge dn_{i_k}^{j_k} \otimes f, \;\; f \in U(gl(m)_h)$$

be a k-differential. Then by definition we put

$$d\,\omega = dn_{i_1}^{j_1} \bigwedge dn_{i_2}^{j_2} \bigwedge \ldots \bigwedge dn_{i_k}^{j_k} \bigwedge \sum_{i,j} dn_i^j \otimes \partial_j^i(f).$$

Theorem

$d^2 = 0$

Proof It is so since the pure differentials anticommute and the partial derivatives commute.

Now, consider the particular case $m = 2$ in more detail.
Denote a, b, c, d the standard generators of the algebra $U(gl(2)_h)$:

$$[a, b] = h\, b,\ [a, c] = -h\, c,\ [a, d] = 0, \ldots.., [d, c] = h\, c.$$

Now, pass to generators of the compact form, namely, $U(u(2)_h)$

$$t = \frac{1}{2}(a + d),\ \ x = \frac{i}{2}(b + c),\ \ y = \frac{1}{2}(c - b),\ \ z = \frac{i}{2}(a - d)$$

we get the standard $u(2)_h$ table of commutators

$$[x,\, y] = h\, z,\ \ [y,\, z] = h\, x,\ \ [z,\, x] = h\, y,\ \ t \text{ is central.}$$

The generator t is called time, x, y, z play the role of spacial variables.

On this algebra the coproduct mentioned above becomes

$$\Delta(\partial_t) = \partial_t \otimes 1 + 1 \otimes \partial_t + \frac{h}{2}(\partial_t \otimes \partial_t - \partial_x \otimes \partial_x - \partial_y \otimes \partial_y - \partial_z \otimes \partial_z),$$

$$\Delta(\partial_x) = \partial_x \otimes 1 + 1 \otimes \partial_x + \frac{h}{2}(\partial_t \otimes \partial_x + \partial_x \otimes \partial_t + \partial_y \otimes \partial_z - \partial_z \otimes \partial_y),$$

$$\Delta(\partial_y) = \partial_y \otimes 1 + 1 \otimes \partial_y + \frac{h}{2}(\partial_t \otimes \partial_y + \partial_y \otimes \partial_t + \partial_z \otimes \partial_x - \partial_x \otimes \partial_z),$$

$$\Delta(\partial_z) = \partial_z \otimes 1 + 1 \otimes \partial_z + \frac{h}{2}(\partial_t \otimes \partial_z + \partial_z \otimes \partial_t + \partial_x \otimes \partial_y - \partial_y \otimes \partial_x).$$

Also, the corresponding permutation relations are

$$[\partial_t, t] = \frac{h}{2}\partial_t + 1,\ [\partial_t, x] = -\frac{h}{2}\partial_x,\ [\partial_t, y] = -\frac{h}{2}\partial_y,\ [\partial_t, z] = -\frac{h}{2}\partial_z,$$

$$[\partial_x, t] = \frac{h}{2}\partial_x,\ [\partial_x, x] = \frac{h}{2}\partial_t + 1,\ [\partial_x, y] = \frac{h}{2}\partial_z,\ [\partial_x, z] = -\frac{h}{2}\partial_y,$$

$$[\partial_y, t] = \frac{h}{2}\partial_y,\ [\partial_y, x] = -\frac{h}{2}\partial_z,\ [\partial_y, y] = \frac{h}{2}\partial_t + 1,\ [\partial_y, z] = \frac{h}{2}\partial_x,$$

$$[\partial_z, t] = \frac{h}{2}\partial_z,\ [\partial_z, x] = \frac{h}{2}\partial_y,\ [\partial_z, y] = -\frac{h}{2}\partial_x,\ [\partial_z, z] = \frac{h}{2}\partial_t + 1.$$

Besides, the generators $\partial_t ..., \partial_z$ commute with each other and generate a commutative algebra $\mathcal{D}$. Thus, we get a Weyl algebra $\mathcal{W}(U(u(2)_h))$ generated by two subalgebras $U(u(2)_h)$ and $\mathcal{D}$ and the above permutation relations.

Given permutation relation it is possible to define the partial derivatives as operators. To this end we also need the counit on $\mathcal{D}$

$$\varepsilon(\partial_t) = ... = \varepsilon(\partial_z) = 0, \;\; \varepsilon(1) = 1$$

extended in the multiplicative way. Then we define $\partial(a)$ by permutating ∂ and a and by applying ε to the right factor from $\mathcal{D}$. For instance, in virtue of the permutation relations we have

$$\partial_x \, yz = (y\partial_x + \hbar\partial_z)\, z = y(z\partial_x - \hbar\partial_y) + \hbar(z\partial_z + \hbar\partial_t + 1).$$

Now, by applying the counit we conclude that $\partial_x(yz) = \hbar$. This result turns into the classical one as $h = 0$.

Problem: how is it possible to extend the quantum partial derivatives on fractions $a^{-1}\, b$?
In the classical case if V is a vector field and consequently, it is subject to the classical Leibniz rule, we have

$$0 = V(1) = V(aa^{-1}) = V(a)a^{-1} + aV(a^{-1}).$$

Thus, we get $V(a^{-1}) = -a^{-1}\, V(a)a^{-1}$.
We succeeded in extending their action to certain fractions.

Also, we succeeded in extension their actions on the so-called *quantum radius*

$$r_\hbar = \sqrt{x^2 + y^2 + z^2 + \hbar^2},\ \ h = 2i\hbar$$

and all analytical functions $f(r_\hbar)$.
This enables us to quantize in a new sense some dynamical models.

For instance, a "quantum version" of the Klein-Gordon operator is defined in the classical way

$$(\Box - m^2)\, f, \;\; \Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2.$$

For the Schrodinger type equation

$$\left(a\partial_t + b(\partial_x^2 + \partial_y^2 + \partial_z^2) + \frac{q}{r_\hbar} \right) f(t, r_\hbar) = 0$$

we have computed a first correction of the ground state energy.

Let us consider in more detail the Maxwell system and Dirac magnetic monopole.
The Maxwell system consists of 4 equations. The first couple of these equations is (we put $c = 1$)

$$\mathrm{div}\ \mathbb{H} = 0,\ \ \mathrm{curl}\ \mathbb{E} + \partial_{\mathrm{t}}\mathbb{H} = 0,$$

where $\mathbb{E} = (E_1, E_2, E_3)$ and $\mathbb{H} = (H_1, H_2, H_3)$ are vectors of electric and magnetic fields respectively. Also, div and curl stand for the divergence and curl respectively.
The second couple of the Maxwell system in vacuum is

$$\mathrm{div}\,\mathbb{E} = 0,\ \ \mathrm{curl}\,\mathbb{H} - \partial_{\mathrm{t}}\mathbb{E} = 0.$$

For the Dirac monopole $\mathbb{E}$ is trivial and $\mathbb{H}$ is stationary. Then, for $\mathbb{H}$ we get

$$\mathrm{div}\,\mathbb{H} = 0, \;\; \mathrm{curl}\,\mathbb{H} = (0, 0, 0).$$

In the classical setting one looks for a solution under the form $\mathbb{H} = f(r)(x, y, z)$. Then we get the following equation on f: $3f + r\,f' = 0$. This equation has the following general solution $f(r) = g\,r^{-3}$ where g is a real constant.

More precisely, the field $\mathbb{H} = \frac{g}{r^3}(x, y, z)$ is a solution of the equation $\mathrm{div}\,\mathbb{H} = 0$ only on the set $\mathbb{R}^3 \setminus (0, 0, 0)$, whereas, on the whole space $\mathbb{R}^3$ this field meets the equation

$$\mathrm{div}\,\mathbb{H} = 4\,\mathrm{g}\pi\delta(\mathrm{r}),$$

where $\delta(r)$ is the delta-function on the space $\mathbb{R}^3$ located at the point $(0, 0, 0)$.

The second equation of the above system is automatically met by $\mathbb{H} = f(r)(x, y, z)$ with any rational function $f(r)$.

Now, let us apply our NC quantization to this model. By looking for a solution the above system under the form $\mathbb{H} = (x, y, z) f(r_\hbar)$, where div and curl have the same meaning as above, we have found the following NC field

$$\mathbb{H} = \frac{g}{r_\hbar(r_\hbar^2 - \hbar^2)}(x, y, z).$$

We call this solution *NC Dirac monopole*. Emphasize that for $\hbar \to 0$ we retrieve the classical Dirac monopole.

Let $R(u,v)$ be the Yang current braiding (R-matrix)

$$R(u,v) = P - \frac{I}{u-v}.$$

It meets the QYBE in the following form

$$R_{12}(u,v)R_{23}(u,w)R_{12}(v,w) = R_{23}(v,w)R_{12}(u,w)R_{23}(u,v).$$

By definition [Drinfeld] the Yangian $\mathbf{Y}(gl(m))$ is defined via coefficients of the matrix

$$L(u) = \sum_{k\geq 0} L(k)u^{-k}, \text{ or equivalently } l_i^j(u) = \sum_{k\geq 0} l_i^j(k)u^{-k},$$

(here $L(k) = (l_i^j(k))$ and $L(0) = I$), subject to

$$R(u,v)L_1(u)L_2(v) - L_1(v)L_2(u)R(u,v) = 0.$$

It possesses the following properties.

1. It has the locality property.
2. It has a bi-algebra structure.
3. It has an analog of determinant defined by

$$det_{\mathbf{Y}}(L) = Tr_1 ... Tr_m\, A_m\, L_1(u) L_2(u-1) ... L_m(u-(m-1)),$$

where A_m is the projector of skew-symmetrization.
This element is central.

4. In a similar manner quantum minors can be defined. They form a commutative family.

5. The map (called evaluation morphism)

$$L(u) \mapsto I + \frac{M}{u}$$

where M is generating matrix of the Lie algebra $gl(m)$

$$M_1 M_2 - M_2 M_1 = M_2 - M_1$$

defines a surjective morphism

$$\mathbf{Y}(gl(m)) \to U(gl(m)).$$

Consequently, any $U(gl(m))$-module becomes $\mathbf{Y}(gl(m))$-module.

How is it possible to define analogs of Yangians, associated to other R-matrices $R(u, v)$?

First, describe Yang-Baxterization procedure.

Proposition.

1. If R is an involutive symmetry,

$$R(u, v) = R - \frac{a}{u - v}$$

is an R-matrix.
2. If R is a Hecke symmetry,

$$R(u, v) = R - \frac{(q - q^{-1})u}{u - v}$$

is an R-matrix.

Definition

Braided Yangian $\mathbf{Y}(R)$ is the unital algebra defined by

$$R(u,v)L_1(u)RL_1(v) = L_1(v)RL_1(u)R(u,v)$$

As usual, $L(u) = \sum_{k\geq 0} L(k)u^{-k}$ and $L(0) = I$.

Definition

RTT-type Yangian $\mathbf{Y}_{RTT}(R)$ is the unital algebra defined by

$$R(u,v)T_1(u)T_2(v) = T_1(v)T_2(u)R(u,v)$$

Also, $T(u) = \sum_{k\geq 0} T(k)u^{-k}$ and $T(0) = I$.

For the braided Yangians there are evaluation morphisms. They have different properties in the cases corresponding to involutive R and Hecke ones.
Also, in a braided Yangian there is a determinant and it is central. This enables us to construct a Bethe families by considering a chain of braided Yangians

$$\mathbf{Y}_1 \hookrightarrow \mathbf{Y}_2 \hookrightarrow \ldots \hookrightarrow \mathbf{Y}_m$$

and the corresponding determinants in the standard case.

Many thanks