Towards classification of K-matrices

arXiv: 1602.08471

1.1) Setup (Drinfeld-Jimbo QG)

$\mathfrak{g}_N = \mathfrak{sl}_N, \mathfrak{so}_N, \mathfrak{sp}_N$, $\hat{\mathfrak{g}}_N$ - loop alg. $\tilde{\mathfrak{g}}_N$ - affine

$U_q(\mathfrak{g}_N)$, $U_q(\hat{\mathfrak{g}})$, $U_q(\tilde{\mathfrak{g}}_N)$ / $\mathbb{K}$ = quad. closure of $\mathbb{C}(q)$

Generators: $x_i^{\pm}, k_i^{\pm}$ $\quad i \in I$

Simple roots: α_i

Dynkin diagram: (I, A) gen. Cartan mat.

Universal R-matrix $\mathcal{R} \in \overbrace{U_q(\tilde{\mathfrak{g}}_N) \otimes U_q(\tilde{\mathfrak{g}}_N)}$

- $\Delta(a)\mathcal{R} = \mathcal{R}\Delta^{op}(a) \quad \forall a \in U_q(\tilde{\mathfrak{g}}_N)$
- $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$

Drinfeld '85
Khoroshkin-Tolstoy '92

1.2) Vector representation

Set $\hat{\mathcal{U}}_q = \mathcal{U}_{q^{1/2}}(\widehat{so}_{2n+1})$ or $\mathcal{U}_q(\hat{\mathfrak{g}}_N)$ otherwise

$T_u : \hat{\mathcal{U}}_q \to \operatorname{End} \mathbb{K}^N[u, u^{-1}]$

e.g. sl_N : $\begin{cases} T_u(x_i^-) = E_{i+1,i} \quad T_u(x_i^+) = E_{i,i+1} \quad 1 \leq i < N \\ T_u(x_0^-) = u^{-1} E_{1N} \quad T_u(x_0^+) = u E_{N1} \end{cases}$

R-matrix $\quad R\left(\frac{u}{v}\right) = (T_u \otimes T_v)(\mathcal{R})$

$\rightsquigarrow R(u) = \tau(u) \left(R_q + (q - q^{-1}) \left(\frac{u}{1-u} P - \frac{u}{q^{2\kappa} - u} Q_q \right) \right)$

meromorphic function in u

- $(T_u \otimes T_v)(\Delta(a)) R(u/v) = R(u/v) (T_u \otimes T_v)(\Delta^{op}(a))$
- $R_{12}\left(\frac{u}{v}\right) R_{13}\left(\frac{u}{w}\right) R_{23}\left(\frac{v}{w}\right) = R_{23}\left(\frac{v}{w}\right) R_{23}\left(\frac{u}{w}\right) R_{12}\left(\frac{u}{v}\right)$

here R_q - constant R-matrix

P - permutation operator

$Q_q = 0$ for sl_N or $Q_q^2 = N Q_q$ otherwise

$\kappa = \frac{N}{2} \mp 1$ for so_N, sp_N

1.3) Reflection equation

untwisted RE

i) $R_{21}\left(\frac{u}{v}\right) K_1(u)\, R(uv)\, K_2(v) = K_2(v)\, R_{21}(uv)\, K_1(u)\, R\left(\frac{u}{v}\right)$

ii) $R\left(\frac{u}{v}\right) \tilde{K}_1(u)\, R\left(\frac{1}{uv}\right)^{t_1} \tilde{K}_2(v) = \tilde{K}_2(v)\, R\left(\frac{1}{uv}\right)^{t_1} \tilde{K}_1(u)\, R\left(\frac{u}{v}\right)$

twisted RE

<u>Rem</u> • for $sl_{N\geq 3}$ i) & ii) are independent

• for sl_2, so_N, sp_N they are equivalent

Let us make these statements precise:

<u>Lem</u> The rep. T_u of $\hat{U}_q$ is σ_0-skew self-dual:

$$T_u(S^{-1}(\sigma_0(a))) = C^{-1}\, T^t_{\xi u}(a)\, C$$

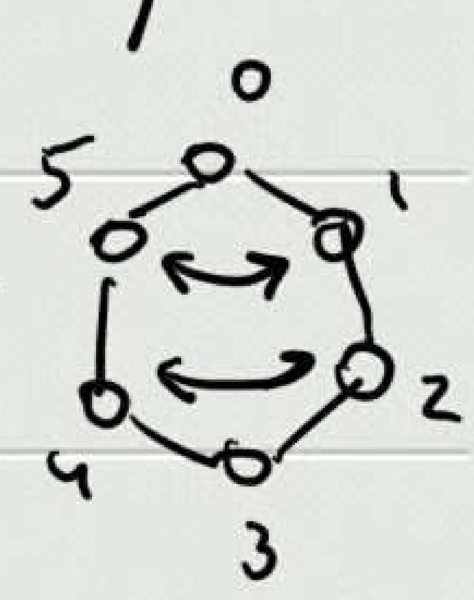
where: S - antipode, C - certain antidiag. mat.

$$\xi = \begin{cases} (-q)^N & \text{for } sl_N \\ q^{2K} & \text{for } so_N, sp_N \end{cases} \qquad \sigma_0 = \begin{cases} \prod_{i=1}^{\lfloor n/2 \rfloor} (i, n-i+1) & \text{for } sl_N \\ id & \text{for } so_N, sp_N \end{cases}$$

<u>Lem</u> Let $K(u)$ be a soln. of i) Then

$$\tilde{K}(u) = \begin{cases} C K(q^N u) & \text{for } so_N, sp_N \\ C K(qu) & \text{for } sl_2 \end{cases}$$

is a soln of ii)

$\sigma_0 \sim$ 

1.4) Goal and motivation

<u>Goal</u>: given $R(u)$ find all inequivalent $K(u)$ satisfying i) or $\tilde{K}(u)$ satisfying ii)

<u>Why</u>: - RTT presentation of QG's [FRT]
- Reps. of cyclotomic Hecke & BMW algebras [Chen-Guay-Ma, Jordan-Ma]
- Schur-Weyl duality
- Integrable models with open b.c.'s $\rightsquigarrow$ Bethe eqs, TQ-relations, etc.
- Boundary q-KZ eqs.

<u>What is known</u>: - classification of non-deg. constan K's [Noumi-Sugitani '95, Kulish-Sasaki-Schiebert '93]
- Lots of sols $K(u)$ but no "big picture" [Malara-Lima-Santos '05]

1.5) Boundary intertwining equation

- Let $B \subset \hat{\mathcal{U}}_q$ be a right coideal subalgebra i.e. $\Delta(b) \in B \otimes \hat{\mathcal{U}}_q \quad \forall b \in B$
- Let T_u and $T_u \otimes T_v$ be irreps of B

<u>Prop</u> Let $K(u) \in \operatorname{End} \mathbb{K}^N(u)$ & $\eta \in \mathbb{K}^\times$ be an invertible solution of:

i) $K(u)\, T_{\eta u}(b) = T_{\eta/u}(b)\, K(u)$ or

ii) $K(u)\, T_{\eta u}(b) = T^t_{\eta/u}(S(b))\, K(u)$ for all $b \in B$

Then $K(u)$ is also a soln. of RE i) or ii).

<u>Proof</u> Show that both sides of RE intertwine $(T_{\eta u} \otimes T_{\eta v})(\Delta^{op}(b))$ with i) $(T_{\eta/u} \otimes T_{\eta/v})(\Delta^{op}(b))$

ii) $(T^t_{\eta/u} \otimes T^t_{\eta/v})(S \otimes S)(\Delta^{op}(b))$

By Schur's lemma both sides are proportional up to a constant c. Take $u=v$ and compute determinant of both sides $\rightsquigarrow c = 1$

Coideal subalgebras B

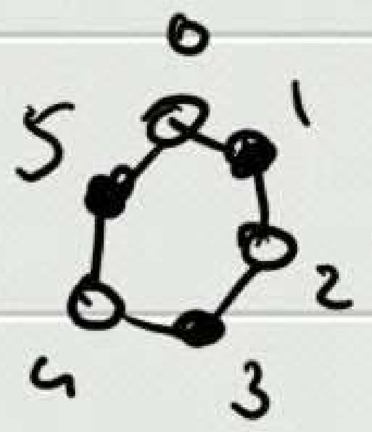
2.1) Admissible pairs & Satake diagrams

- Let (I, A) denote finite or affine Dynkin diag.
- Let $\mathfrak{g} = L(A)$ be the Lie alg generated by all $h \in \mathfrak{h}$ and e_i, f_i, $i \in I$. Set $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$
- Let (X, A_X) be a subdiagram of finite type
  - $W_X \subset W$ - Weyl group
  - $\omega_X \in W_X$ - longest element
  - $\rho_X^\vee$ - half sum of positive coroots

<u>Def</u> The pair (X, τ) where $\tau \in Aut(A)$ is called admissible if:

- For all $i \in X$, $\Theta(\alpha_i) = \alpha_i$, here $\Theta = -\omega_X \circ \tau : \mathfrak{h}^* \to \mathfrak{h}^*$
- For all $j \in I \setminus X$ s.t. $\tau(j) = j$ we have $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$

<u>Rem</u> Adm. pairs are in bijection with Satake diagrams.

e.g. A2 : 

$\Longleftrightarrow (\{1, 3, 5\}, id)$

2.2) Symmetric pairs

Define a map

$$\theta = \theta(X,\tau) = Ad(s) \circ Ad(w_X) \circ \tau \circ \omega \ : \mathfrak{g}' \to \mathfrak{g}'$$

where:

- ω – Chevalley involution $\quad \omega(e_i) = -f_i$, $\ \omega(h) = -h \ \forall h \in \mathfrak{h}'$
- $Ad(s)$ – mult. by a certain number $\quad s : I \to \mathbb{K}^\times$

i.e. $Ad(s)|_{\mathfrak{g}_{\alpha_i}}$ mult by $s(\alpha_i)$ s.t.

$$s(\alpha_i) = 1 \text{ for } i \in X \text{ or } \tau(i) = i$$

$$\frac{s(\alpha_i)}{s(\alpha_{\tau(i)})} = (-1)^{\alpha_i(2\rho_X^\vee)} \quad \tau(i) \neq i$$

- $Ad(w_X)$ – braid grp action

<u>Thm</u> (Kolb) θ is involution

<u>Lem</u> (Kolb) Let $\mathfrak{k}' = \{x \in \mathfrak{g}' \mid \theta(x) = x\}$ then:

$$\left.\begin{array}{l} \bullet\ e_i\ f_i \quad i \in X \\ \bullet\ h \in \mathfrak{h}' \text{ s.t. } \theta(h) = h \\ \bullet\ f_i + \theta(f_i) \end{array}\right\} \rightsquigarrow \mathfrak{k}'$$

<u>Rem</u> 1) The pair $(\mathfrak{g}', \mathfrak{k}')$ is symm pair

2) Replace $f_i + \theta(f_i)$ with $f_i + \theta(f_i) + s_i$ $\quad \in \mathbb{K}$

$\rightsquigarrow$ non-std. pair

2.3) Quantum symmetric pairs

Define "Quantum involution" by:

$\Theta_q = \Theta_q(X,\tau) = \mathrm{Ad}(s) \circ T_{w_X} \circ \tau \circ tw$

where:

- tw – a q-deformed Chevalley inv.
- T_{w_X} – Lusztig automorphism

<u>Def</u> $B_{c,s} \subset U_q(\mathfrak{g}')$, a "quantum analogue" of $U(\mathfrak{k}')$:

- $x_i^{\pm}, k_i^{\pm}$ for $i \in X$
- $k_j^{\pm} k^{\mp}_{\tau(j)}$ for $j \in I \setminus X$ & $\tau(j) \neq j$
- $b_j = x_j^- + c_j \, \Theta_q(x_j^- k_j^+) k_j^- + s_j k_j^-$

  where $c_j \in \mathbb{K}^\times$, $s_j \in \mathbb{K}$ ($\rightsquigarrow$ Bart's talk)

<u>Prop</u> (Kolb) $B_{c,s}$ is a right coideal subalg.

<u>Rem</u> 1) For suitable $c \in (\mathbb{K}^\times)^{I\setminus X}$ and $s \in (\mathbb{K})^{I\setminus X}$

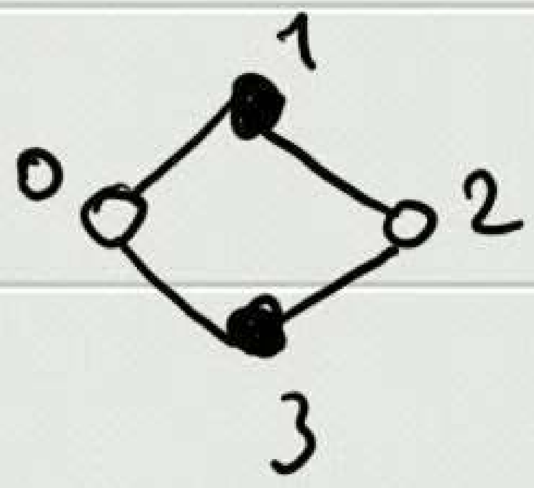
$B_{c,s}$ is q-analogue of $U(\mathfrak{k}')$

2) Depending on properties of c & s

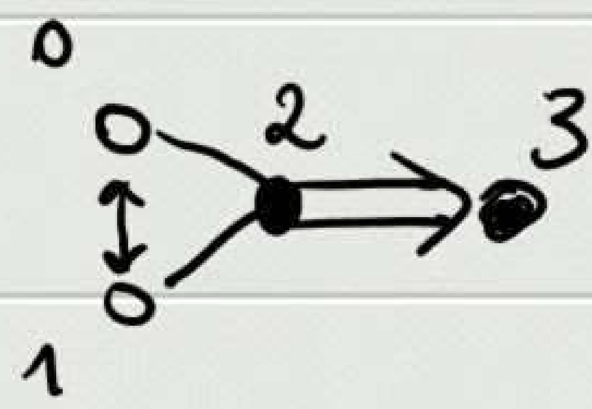
$B_{c,s}$ can be standard, quasistandard, prop. non-standard or q-Onsager

$\rightsquigarrow$ Bart's talk.

2.4) Examples

A.2 $\quad (X,\tau) = (\{1,3\}, \mathrm{id}) \quad \leftrightsquigarrow$ 

$$B_{c,s} = \begin{cases} x_i^{\pm},\ k_i^{\pm} \quad i = 1,3 \qquad w_X = w_1 w_3 \\ b_0 = x_0^- - c_0\, T_{1,2}(x_0^+)\, k_0^- \\ b_2 = x_2^- - c_2\, T_{1,2}(x_2^+)\, k_2^- \end{cases}$$

B.1(b) $\quad (X,\tau) = (\{2,3\}, (01)) \quad \leftrightsquigarrow$ 

$$B_{c,s} = \begin{cases} x_i^{\pm},\ k_i^{\pm} \quad i = 2,3 \qquad w_X = w_1 w_2 w_1 w_2 \\ k_0^{\pm} k_1^{\mp} \\ b_0 = x_0^- - c_0\, T_{1,2,1,2}(x_1^+)\, k_0^- \\ b_1 = x_1^- + c_1\, T_{1,2,1,2}(x_0^+)\, k_1^- \end{cases}$$

C.1 $\quad (X,\tau) = (\emptyset, \mathrm{id}) \quad \leftrightsquigarrow$ 

$$B_{c,s} = \{\, b_j = x_j^- - c_j\, x_j^+ k_j^- - s_j k_j^-$$

where $s_1 = s_2 = \ldots = s_{n-1} = 0$ & $s_0, s_n \in \mathbb{K}$