Towards a classification of trigonometric reflection matrices

(Part 2)

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Algebra and Geometry seminar,
University of Newcastle, 27/04/2016

1. Affine quantum groups and coideal subalgebras
2. $\mathrm{Ad}(\widetilde{H}_q)$-equivalence and dressing
3. $\mathrm{Aut}(A)$-equivalence and rotation
4. Classification
5. Generalizations

V. Regelskis and B. Vlaar, "Classification of reflection matrices for quasistandard quantum affine Kac-Moody pairs of classical type". Preprint at arXiv:1602.08471.

Given: generalized Cartan matrix $A=(a_{ij})_{i,j\in I}$: $a_{ii}=2$, $a_{ij}\in\mathbb{Z}_{\leq 0}$ if $i\neq j$ such that $a_{ij}=0$ precisely if $a_{ji}=0$. Further assumptions:

- A is symmetrizable: there exist coprime $\varepsilon_i\in\mathbb{Z}_{>0}$ $(i\in I)$ such that $\varepsilon_i a_{ij}=\varepsilon_j a_{ji}$;
- A is indecomposable: there is no $\emptyset\subset I'\subset I$ such that $a_{ij}=0$ for all $i\in I'$, $j\in I\backslash I'$.

$\mathfrak{g}=\mathfrak{g}(A)$ is the corresponding Kac-Moody algebra, with Cartan subalgebra $\mathfrak{h}=\langle\{h_i\,|\,i\in I\},\{d_s\}\rangle$ of dimension $2|I|-\mathrm{rank}(A)$ and coweight lattice $P^\vee$.

- Derived subalgebra: $\mathfrak{g}'=[\mathfrak{g},\mathfrak{g}]=\langle h_i,e_i,f_i\,|\,i\in I\rangle$.
- Choose $\{\alpha_i\,|\,i\in I\}\subset\mathfrak{h}^*$ such that $\alpha_j(h_i)=a_{ij}$, $\alpha_j(d_s)=\{0,1\}$ for all $i,j\in I$.
- Nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}$ defined by $(h_i,h)=\varepsilon_i^{-1}\alpha_i(h)$ for $h\in\mathfrak{h}$, $i\in I$, induced form on $\mathfrak{h}^*$ given by $(\alpha_i,\alpha_j)=\varepsilon_i a_{ij}$ for $i,j\in I$.

For $r, s \in \mathbb{Z}_{\geq 0}$ with $r \leq s$ we denote

$$[r]_q = \frac{q^r - q^{-r}}{q - q^{-1}}, \qquad [r]_q! = \prod_{i=1}^{r} [i]_q, \qquad \binom{s}{r}_q = \frac{[s]_q!}{[r]_q! [s-r]_q!}.$$

Definition (Drinfeld-Jimbo quantum group)

Let A be a symmetrizable generalized Cartan matrix whose Kac-Moody algebra is $\mathfrak{g}$. Then $U_q(\mathfrak{g})$ is the unital associative algebra over $\mathbb{C}(q)$ generated by $\{x_i^\pm \,|\, i \in I\} \cup \{k_h \,|\, h \in P^\vee\}$ subject to

$$k_0 = 1, \qquad k_h k_{h'} = k_{h+h'}, \qquad k_h x_i^\pm = q^{\pm \alpha_i(h)} x_i^\pm k_h,$$

$$[x_i^+, x_j^-] = \delta_{ij} \frac{k_i^+ - k_i^-}{q_i - q_i^{-1}} \quad \text{for all } i, j \in I, \qquad \text{where } q_i = q^{\varepsilon_i},\ k_i^\pm = k_{\pm h_i}^{\varepsilon_i},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} (x_i^\pm)^{1-a_{ij}-r} x_j^\pm (x_i^\pm)^r = 0 \quad \text{for all} \quad i, j \in I.$$

Hopf algebra structure on $U_q(\mathfrak{g})$:

$$\begin{aligned}
\Delta(x_i^+) &= x_i^+ \otimes 1 + k_i^+ \otimes x_i^+, & \epsilon(x_i^+) &= 0, & S(x_i^+) &= -k_i^- x_i^+, \\
\Delta(x_i^-) &= x_i^- \otimes k_i^- + 1 \otimes x_i^-, & \epsilon(x_i^-) &= 0, & S(x_i^-) &= -x_i^- k_i^+, \\
\Delta(k_h) &= k_h \otimes k_h, & \epsilon(k_h) &= 1, & S(k_h) &= k_{-h},
\end{aligned}$$

Hopf subalgebra $U_q(\mathfrak{g}') = \langle x_i^\pm, k_i^\pm \,|\, i \in I \rangle$.

From now assume A to be of *affine* type: $\det(A)=0$ and proper principal minors are positive.

Untwisted affine types:

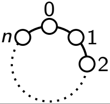
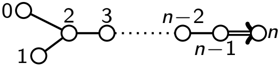
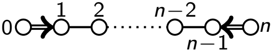
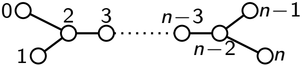
$$\mathrm{A}^{(1)}_{n\geq 1} = \widetilde{\mathrm{A}}_{n\geq 1}, \qquad \mathrm{B}^{(1)}_{n\geq 3} = \widetilde{\mathrm{B}}_{n\geq 3}, \qquad \mathrm{C}^{(1)}_{n\geq 2} = \widetilde{\mathrm{C}}_{n\geq 2}, \qquad \mathrm{D}^{(1)}_{n\geq 4} = \widetilde{\mathrm{D}}_{n\geq 4}$$
$$\mathrm{E}^{(1)}_{6,7,8} = \widetilde{\mathrm{E}}_{6,7,8}, \qquad \mathrm{F}^{(1)}_{4} = \widetilde{\mathrm{F}}_{4}, \qquad \mathrm{G}^{(1)}_{2} = \widetilde{\mathrm{G}}_{2}.$$

Twisted affine types (note that $\mathrm{A}_3 \cong \mathrm{D}_3$):

$$\mathrm{A}^{(2)}_{2n\geq 2} = \widetilde{\mathrm{BC}}_{n\geq 1}, \qquad \mathrm{A}^{(2)}_{2n-1\geq 5} = \widetilde{\mathrm{B}}^{\vee}_{n\geq 3}, \qquad \mathrm{D}^{(2)}_{n+1\geq 3} = \widetilde{\mathrm{C}}^{\vee}_{n\geq 2},$$
$$\mathrm{E}^{(2)}_{6} = \widetilde{\mathrm{F}}^{\vee}_{4}, \qquad \mathrm{D}^{(3)}_{4} = \widetilde{\mathrm{G}}^{\vee}_{2}.$$

Generalized Cartan matrices (KM algebras, Drinfeld-Jimbo quantum groups, ...) can be graphically enumerated by Dynkin diagrams (I, A).

Standard labelling for affine diagrams: $I = \{0, \ldots, n\}$ so that $I\backslash\{0\}$ indexes corresponding objects of finite type.

Name	Diagram	Affine Lie algebra
$\mathrm{A}_1^{(1)}$		$\widehat{\mathfrak{sl}}_{N=n+1}$
$\mathrm{A}_{n\geq 2}^{(1)}$		
$\mathrm{B}_{n\geq 3}^{(1)}$		$\widehat{\mathfrak{so}}_{N=2n+1}$
$\mathrm{C}_{n\geq 2}^{(1)}$		$\widehat{\mathfrak{sp}}_{N=2n}$
$\mathrm{D}_{n\geq 4}^{(1)}$		$\widehat{\mathfrak{so}}_{N=2n}$

Group of *diagram automorphisms*:

$$\mathrm{Aut}(A) = \{\sigma : I \to I \text{ bijective} \,|\, a_{\sigma(i)\sigma(j)} = a_{ij} \quad \text{for all } i,j \in I\}.$$

Suppose $X \subset I$ is of finite type ($A_X := (a_{ij})_{i,j\in X}$ is of finite type). Then corresponding Weyl group W_X has longest element w_X (involution). $-w_X$ permutes $\Pi_X := \{\alpha_i \,|\, i \in X\}$, so induces involution on X.

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Definition (Admissible pair)

Let $X \subset I$ be of finite type and $\tau \in \mathrm{Aut}(A)$ be an involution. The pair (X,τ) is called *admissible* if

1. for all $i \in X$, $\alpha_i = \Theta(\alpha_i)$ where $\Theta := -w_X \circ \tau : \mathfrak{h}^* \to \mathfrak{h}^*$ (in particular, X is stable under τ);
2. for all $j \in I \backslash X$, $\tau(j) = j \implies \alpha_j(\rho_X^\vee) \in \mathbb{Z}$, where $\rho_X^\vee \in P_X^\vee$ is the half-sum of positive coroots.

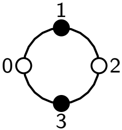
S. Kolb, Adv. Math. **267** (2014)

V. Back-Valente, N. Bardy-Panse, H. Ben Massaoud, G. Rousseau, J. Algebra **171** (1995)

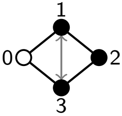
Satake diagrams enumerate admissible pairs.

Examples

1. A of type $\mathrm{A}_3^{(1)}$, $I = \{0,1,2,3\}$
   $X = \{1,3\}$, $\tau = \mathrm{id}$
   X is of type $\mathrm{A}_1 \times \mathrm{A}_1$ so
   - $w_X = r_1 r_3$: $w_X(\alpha_i) = -\alpha_i$ for $i \in X$
   - $\rho_X^\vee = \frac{1}{2}h_1 + \frac{1}{2}h_3$.



2. A of type $\mathrm{A}_3^{(1)}$, $I = \{0,1,2,3\}$.
   $X = \{1,2,3\}$, $\tau = (13)$.
   X is of type A_3 so
   - $w_X = r_1 r_2 r_3 r_2 r_1 r_2$: $w_X(\alpha_i) = -\alpha_{4-i}$ for $i \in X$
   - $\rho_X^\vee = \frac{3}{2}h_1 + 2h_2 + \frac{3}{2}h_3$.



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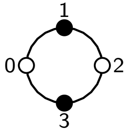
   (X,τ) is admissible since
   $\alpha_{\tau(i)} = -w_X(\alpha_i)$ for $i \in \{1,3\}$ and
   $\alpha_j(\rho_X^\vee) = \frac{1}{2}a_{1j} + \frac{1}{2}a_{3j} = -1$ ($j \in \{0,2\}$).



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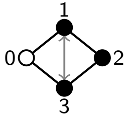

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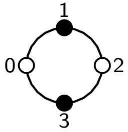

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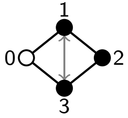
   $i \in \{1,2,3\}$ and $\alpha_0(\rho_X^\vee) = \frac{3}{2}a_{10} + \frac{3}{2}a_{30} = -3$.



More examples

3. A of type $\mathrm{D}_5^{(1)}$, $I = \{0,1,2,3,4,5\}$
   $X = \{1,3\}$, $\tau = (45)$
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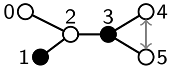

4. A of type $\mathrm{B}_4^{(1)}$, $I = \{0,1,2,3,4\}$
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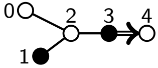

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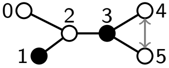

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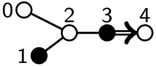

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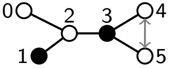
   (X,τ) is admissible since $\alpha_{\tau(i)} = -w_X(\alpha_i) = \alpha_i$ for $i \in \{1,3\}$ and $\alpha_2(\rho_X^\vee) = \frac{1}{2}a_{12} + \frac{1}{2}a_{32} = -1$.



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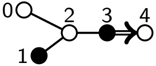
   As required, $\alpha_{\tau(i)} = -w_X(\alpha_i) = \alpha_i$ for $i \in \{1,3\}$ and $\alpha_2(\rho_X^\vee) = \frac{1}{2}a_{12} + \frac{1}{2}a_{32} = -1$, but $\alpha_4(\rho_X^\vee) = \frac{1}{2}a_{34} = -\frac{1}{2}$. So (X,τ) is *not* admissible.



Let $\mathfrak{d} \in \{1, \frac{1}{2}, \frac{1}{4}\}$ as needed. Given $\boldsymbol{c} \in (\mathbb{C}(q^{\mathfrak{d}})^{\times})^{I\backslash X}$, $\boldsymbol{s} \in \mathbb{C}(q^{\mathfrak{d}})^{I\backslash X}$, the coideal subalgebra $B^{\boldsymbol{c},\boldsymbol{s}} = B^{\boldsymbol{c},\boldsymbol{s}}(X,\tau) \subset U_q(\mathfrak{g}')$ is generated by the elements

- $x_i^{\pm}, k_i^{\pm}$ for $i \in X$;
- $k_\mu = \prod_{i \in I} k_i^{m_i}$ for those $\mu = \sum_{i \in I} m_i \alpha_i \in Q$ such that $\mu = \Theta(\mu)$; these are the elements $k_i^+ k_{\tau(i)}^-$ for $i \in I$ such that $i \neq \tau(i)$;
- $b_j = x_j^- + c_j \theta_q(x_j^- k_j^+) k_j^- + s_j k_j^-$ for $j \in I\backslash X$, with the "quantum involution"

$$\theta_q = \theta_q(X,\tau) := \mathrm{Ad}(s_{X,\tau}) \circ T_{w_X} \circ \mathrm{tw} \circ \tau$$

Call $B^{\boldsymbol{c},\boldsymbol{s}}$ *standard* if $s_j = 0$ for all $j \in I\backslash X$.

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Call $B^{\boldsymbol{c},\boldsymbol{s}}$ *standard* if $s_j = 0$ for all $j \in I\backslash X$.

$B^{\boldsymbol{c},\boldsymbol{s}}$ is quantum analogon (of enveloping algebra) of certain fixed-point subalgebra $\mathfrak{k}' \subset \mathfrak{g}'$ if $\boldsymbol{c} \in \mathcal{C}$, $\boldsymbol{s} \in \mathcal{S}$ for suitable $\mathcal{C} \subset (\mathbb{C}(q^{\mathfrak{d}})^\times)^{I\backslash X}$, $\mathcal{S} \subset \mathbb{C}(q^{\mathfrak{d}})^{I\backslash X}$. $B^{\boldsymbol{c},\boldsymbol{s}}$ is called a *QSP (quantum symmetric pair) algebra*.

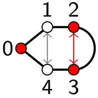
Given admissible pair (X, τ), define

$$
\begin{aligned}
I_{\mathrm{pair}} &= \{(j, \tau(j)) \in (I \backslash X)^2 \,|\, j < \tau(j), (\alpha_j, \Theta(\alpha_j)) \neq 0\} \\
&= \{(j, \tau(j)) \in (I \backslash X)^2 \,|\, j < \tau(j) \text{ and } (a_{j\tau(j)} \neq 0 \text{ or } \exists i \in X : a_{ij} \neq 0)\}, \\
I_{\mathrm{ns}} &= \{j \in I \backslash X \,|\, j = \tau(j) \text{ and } \forall i \in X : a_{ij} = 0\}, \\
I_{\mathrm{nse}} &= \{j \in I_{\mathrm{ns}} \,|\, \forall i \in I_{\mathrm{ns}} : a_{ij} \in 2\mathbb{Z}\}
\end{aligned}
$$

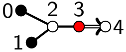
and families of tuples

$$
\begin{aligned}
\mathcal{C} &= \{\boldsymbol{c} \in (\mathbb{C}(q)^\times)^{I \backslash X} \,|\, c_j \neq c_{\tau(j)} \text{ and } j < \tau(j) \implies (j, \tau(j)) \in I_{\mathrm{pair}}\}, \\
\mathcal{S} &= \{\boldsymbol{s} \in \mathbb{C}(q)^{I \backslash X} \,|\, s_j \neq 0 \implies j \in I_{\mathrm{nse}}\},
\end{aligned}
$$

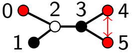
Examples (orbits corresponding to I_{pair} and I_{nse} in red):



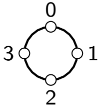
$I_{\mathrm{ns}} = I_{\mathrm{nse}} = \{0\}$
$I_{\mathrm{pair}} = \{(2,3)\}$



$I_{\mathrm{ns}} = \{3,4\}$
$I_{\mathrm{nse}} = \{3\}$, $I_{\mathrm{pair}} = \emptyset$



$I_{\mathrm{ns}} = I_{\mathrm{nse}} = \{0\}$
$I_{\mathrm{pair}} = \{(4,5)\}$



$I_{\mathrm{ns}} = \{0,1,2,3\}$
$I_{\mathrm{nse}} = I_{\mathrm{pair}} = \emptyset$

Hopf algebra automorphisms

For convenience, write

$$\widehat{U}_q := \begin{cases} U_{q^{1/2}}(\mathfrak{g}') & \text{if } \mathfrak{g} = \widehat{\mathfrak{so}}_{2n+1}, \\ U_q(\mathfrak{g}') & \text{if } \mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}, \widehat{\mathfrak{so}}_{2n}, \widehat{\mathfrak{sp}}_{2n}. \end{cases}$$

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Note that $\mathrm{Aut}(A) < \mathrm{Aut}_{\mathrm{Hopf}}(\widehat{U}_q)$ by means of $\sigma(x_i^\pm) = x^\pm_{\sigma(i)}, \sigma(k_i^\pm) = k^\pm_{\sigma(i)}$ for $i \in I$.

Define $(\widehat{U}_q)_\beta := \{u \in \widehat{U}_q \,|\, k_i^+ u k_i^- = q^{(\alpha_i,\beta)} u\}$ for all $\beta \in Q$. For $s \in \widetilde{H}_q := \mathrm{Hom}(Q, \mathbb{C}(q^{\mathfrak{d}})^\times)$ define $\mathrm{Ad}(s) \in \mathrm{Aut}_{\mathrm{Hopf}}(\widehat{U}_q)$ by

$$\mathrm{Ad}(s)(a) = s(\beta)a \qquad \text{for all } a \in (\widehat{U}_q)_\beta \text{ and } \beta \in Q.$$

In fact, $\mathrm{Aut}_{\mathrm{Hopf}}(\widehat{U}_q) = \mathrm{Ad}(\widetilde{H}_q) \rtimes \mathrm{Aut}(A)$.

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Let $\Sigma \leq \mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{A})$ for a Hopf algebra $\mathcal{A}$. Two coideal subalgebras $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{A}$ are called Σ-*equivalent* if there exists $\sigma \in \Sigma$ such that $\sigma(\mathcal{B}) = \mathcal{B}'$; if $\Sigma = \mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{A})$ we simply call them *equivalent*.

Lemma

Suppose $Z(u) \in \mathrm{End}(\mathbb{C}^N)(u)$ *satisfies* $[R(u/v), Z(u) \otimes Z(v)] = 0$. *If* $K(u)$ *satisfies*

$$R_{21}(u/v)K_1(u)R(uv)K_2(v) = K_2(v)R_{21}(uv)K_1(u)R(u/v)$$

then for any $\eta \in \mathbb{C}^\times$, $K^Z(u) := Z(\eta/u)^{-1}K(u)Z(\eta u)$ *satisfies*

$$R_{21}(u/v)K_1^Z(u)R(uv)K_2^Z(v) = K_2^Z(v)R_{21}(uv)K_1^Z(u)R(u/v).$$

Similarly for twisted RE, provided we take $\widetilde{K}(u) := Z(\eta/u)^{\mathrm{t}}K(u)Z(\eta u)$.

Fix adm. pair (X, τ). Choose $I^* \subset I\backslash X$ so that it intersects all τ-orbits in singletons, say

$$I^* := \{i \in I\backslash X \,|\, i \leq \tau(i)\}.$$

Let $\overline{\mathcal{C}}, \overline{\mathcal{S}}$ be algebraic closures of $\mathcal{C}$, $\mathcal{S}$.

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The following generalizes S. Kolb, Adv. Math. **267** (2014), Prop. 9.2 (1) :

Proposition

Let $\boldsymbol{c} \in \overline{\mathcal{C}}$, $\boldsymbol{s} \in \overline{\mathcal{S}}$ and $\boldsymbol{\gamma} \in (\overline{\mathbb{C}(q^{\mathfrak{d}})}^{\times})^{I^*}$. There exist $x_{\boldsymbol{\gamma},\boldsymbol{c}} \in \widetilde{H}_q$, $\boldsymbol{c}' \in \overline{\mathcal{C}}$ and $\boldsymbol{s}' \in \overline{\mathcal{S}}$ such that $B_{\boldsymbol{c}',\boldsymbol{s}'} = \mathrm{Ad}(x_{\boldsymbol{\gamma},\boldsymbol{c}})(B_{\boldsymbol{c},\boldsymbol{s}})$ and $c'_j = \gamma_j$ for all $j \in I^*$.

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Remark

Imposing $c'_j = \gamma_j$ for $j \in I^*$ fixes all entries of $\boldsymbol{c}'$ and $\boldsymbol{s}'$ except those $c'_{\tau(j)}$ where $(j, \tau(j)) \in I_{\mathrm{pair}}$ and those s'_j where $j \in I_{\mathrm{nse}}$.

Recall representation T of $U_q(= U_q(\mathfrak{sl}_N), U_q(\mathfrak{so}_N), U_q(\mathfrak{sp}_N))$ and its extension $T_u : \widehat{U}_q \to \mathrm{End}(\mathbb{C}_q^N)[u, u^{-1}]$, where $\mathbb{C}_q^N = \mathbb{C}^N \otimes \mathbb{C}(q^{\mathfrak{d}})$.

For example, for $\widehat{U}_q = U_q(\widehat{\mathfrak{sl}}_{N=n+1})$, write E_{ij} for the matrix $(\delta_{ik}\delta_{jl})_{1\le k,l\le N}$ and set

$$T_u(x_i^+) = E_{i,i+1}, \quad T_u(x_i^-) = E_{i+1,i}, \quad T_u(k_i^\pm) = \sum_{j=1}^N q^{\pm(\delta_{ij}-\delta_{i+1,j})}E_{jj}$$

for $1 \le i < N$ and

$$T_u(x_0^+) = uE_{N1}, \quad T_u(x_0^-) = u^{-1}E_{1N}, \quad T_u(k_0^\pm) = \sum_{j=1}^N q^{\pm(\delta_{jN}-\delta_{j0})}E_{jj}.$$

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Let $\omega \in (\mathbb{C}(q^{\mathfrak{d}})^\times)^N$ if $\mathfrak{g}_N = \mathfrak{sl}_N$ and $\omega \in (\mathbb{C}(q^{\mathfrak{d}})^\times)^n$ otherwise.

Proposition

Let ω be as above and let $\eta \in \mathbb{C}(q^{\mathfrak{d}})^\times$. There exists $y_{\omega,\eta} \in \widetilde{H}_q$ and a diagonal matrix $G(\omega) \in \mathrm{End}(\mathbb{C}_q^N)$ such that

$$G(\omega) T_{\eta u}(a) = T_u(\mathrm{Ad}(y_{\omega,\eta})(a)) G(\omega) \qquad \text{for all } a \in U_q(\widehat{\mathfrak{g}}_N).$$

Corollary

Given ω *as above,* $K_\omega(u) := G(\omega)^{-1}K(u)G(\omega)$ *satisfies*

$$K_\omega(u)T_{\eta u}(b) = T_{\eta/u}(b)K_\omega(u) \qquad \text{for all } b \in B_{\mathbf{c},\mathbf{s}}$$

precisely if

$$K(u)T_u(b) = T_{1/u}(b)K(u) \qquad \text{for all } b \in \mathrm{Ad}(y_{\omega,\eta})(B_{\mathbf{c},\mathbf{s}}).$$

It is always possible to choose ω such that $x_{\gamma,\mathbf{c}} = y_{\omega,\eta}$ (recall $c'_j = \gamma_j$ for $j \in I^*$). Then the second intertwining equation above simplifies:

$$K(u)T_u(b) = T_{1/u}(b)K(u) \qquad \text{for } b \in B_{\mathbf{c}',\mathbf{s}'}.$$

$K(u)$ is called the *bare* K-matrix and $G(\omega)^{-1}K(u)G(\omega)$ the *dressed* K-matrix. $K(u)$ only depends on $|I_{\mathrm{pair}}| + |I_{\mathrm{nse}}|$ free parameters (namely, those $c'_{\tau(j)}$ where $(j,\tau(j)) \in I_{\mathrm{pair}}$ and those s'_j where $j \in I_{\mathrm{nse}}$).

1. Affine quantum groups and coideal subalgebras
2. $\mathrm{Ad}(\widetilde{H}_q)$-equivalence and dressing
3. $\mathrm{Aut}(A)$-equivalence and rotation
4. Classification
5. Generalizations

If (X, τ) is an admissible pair and if $\sigma \in \mathrm{Aut}(A)$, then it can be verified (X^σ, τ^σ) is an admissible pair, where

$$X^\sigma = \sigma(X), \qquad \tau^\sigma = \sigma \circ \tau \circ \sigma^{-1}.$$

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$$X^\sigma = \sigma(X), \qquad \tau^\sigma = \sigma \circ \tau \circ \sigma^{-1}.$$

Let (X,τ) be an admissible pair and let $\sigma \in \mathrm{Aut}(A)$. Given $\boldsymbol{c} \in \mathcal{C}$ and $\boldsymbol{s} \in \mathcal{S}$, we have

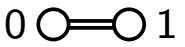
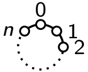
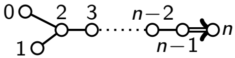
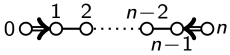
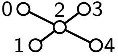
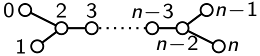
$$\sigma(B_{\boldsymbol{c},\boldsymbol{s}}(X,\tau)) = B_{\sigma(\boldsymbol{c}),\sigma(\boldsymbol{s})}(X^\sigma,\tau^\sigma),$$

where $\sigma(\boldsymbol{c}) \in \sigma(\mathcal{C})$ is determined by $(\sigma(\boldsymbol{c}))_{\sigma(i)} = c_i$ for $i \in I \backslash X$ (and likewise for $\boldsymbol{s}$).

$\sigma \in \mathrm{Aut}(A)$ is called a *symmetry of* T_u if $\exists Z^\sigma(u) \in \mathrm{End}(\mathbb{C}_q^N)(u)$ such that

$$Z^\sigma(u)\, T_u(\sigma(a)) = T_u(a) Z^\sigma(u) \qquad \text{for all } a \in \widehat{U}_q$$

Symmetries of T_u form subgroup $\Sigma(A) < \mathrm{Aut}(A)$.

Name	Diagram	$\mathrm{Aut}(A)$	$\Sigma(A)$	
$\mathrm{A}_1^{(1)}$		C_2	C_2	
$\mathrm{A}_{n\geq 2}^{(1)}$		D_N	C_N	$(N = n+1)$
$\mathrm{B}_{n\geq 3}^{(1)}$		C_2	C_2	
$\mathrm{C}_{n\geq 2}^{(1)}$		C_2	C_2	
$\mathrm{D}_4^{(1)}$		S_4	D_4	$\Sigma(A) = \langle (34), (04)(13) \rangle$
$\mathrm{D}_{n\geq 5}^{(1)}$		D_4	D_4	

Type $\mathrm{A}^{(1)}_{n\geq 2}$: "reflections" in the dihedral group $\mathrm{Aut}(A) \cong D_N$ ($N = n+1$) are not symmetries of T_u: instead we have "σ-skewed self-duality" of T_u: $\exists C \in \mathrm{End}(\mathbb{C}_q^N)$ such that

$$CT_u(\sigma(a)) = \Big(T_{(-q)^N u}(S(a))\Big)^{\mathrm{t}} C \qquad \text{for all } a \in U_q(\widehat{\mathfrak{g}}_N),$$

where $\sigma = \prod_{i=1}^{\lfloor \frac{N}{2} \rfloor} (i, N-i) = (1n)(2, n-1)\cdots$.

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Lemma

Suppose A is of type $\mathrm{A}^{(1)}_{n\geq 2}$. If $(X', \tau') = (X^\sigma, \tau^\sigma)$ for some $\sigma \in \mathrm{Aut}(A)$ then there exists $\tilde{\sigma} \in \Sigma(A)$ such that $(X', \tau') = (X^{\tilde{\sigma}}, \tau^{\tilde{\sigma}})$.

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For A of type $A^{(1)}_{n\geq 1}$ and $\sigma = (012\ldots n)$ we can take

$$Z(u)^\sigma = \sum_{1\leq i<N} E_{i,i+1} + uE_{N1}.$$

Proposition

Let (X, τ) be an admissible pair and let $\sigma \in \Sigma(A)$. If

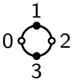
$$K(u) T_{\eta u}(b) = T_{\eta/u}(b) K(u) \qquad \textit{for all } b \in B_{\boldsymbol{c},\boldsymbol{s}}$$

then $K^\sigma(u) := Z^\sigma(\frac{\eta}{u})^{-1} K(u) Z^\sigma(\eta u)$ *satisfies*

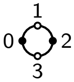
$$K^\sigma(u) T_{\eta u}(b) = T_{\eta/u}(b) K^\sigma(u) \qquad \textit{for all } b \in \sigma(B_{\boldsymbol{c},\boldsymbol{s}}).$$

Example

The bare K-matrix $K(u) = q^{-\frac{1}{2}} E_{21} - q^{\frac{1}{2}} E_{12} + q^{-\frac{1}{2}} E_{43} - q^{\frac{1}{2}} E_{34}$ solves the boundary intertwining equation for the coideal subalgebra given by



. The bare K-matrix associated to

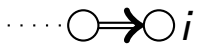


is given by

$$K^\sigma(u) = Z^\sigma(1/u)^{\mathrm{t}} K(u) Z^\sigma(u) = q^{-\frac{1}{2}} E_{32} - q^{\frac{1}{2}} E_{23} + q^{-\frac{1}{2}} u^{-1} E_{14} - q^{\frac{1}{2}} u E_{41}.$$

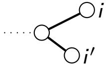
Auxiliary terminology

Let $\tau : I \to I$ be a diagram involution. Let $Y \subset I$ be stable under τ. Y is called *lateral w.r.t.* τ if it is one of the following types:

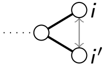


$Y = \{i\}$ is of type B_1

$Y = \{i\}$ is of type C_1

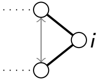


or

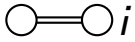


$|i - i'| = 1$

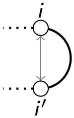
$Y = \{i, i'\}$ is of type D_2



or



$Y = \{i\}$ is a *hinge*



$Y = \{i, i'\}$ is a *hinge*

- There are ≤ 2 subsets of I lateral w.r.t. τ, denoted Y_1 and Y_2.

Types of admissible pairs:

- (X, τ) is said to be of *identity type* if τ fixes I minus any subsets of type D_2. Two subtypes:
  - (X, τ) is said to be of *plain type* if $I \backslash X$ is connected.
  - (X, τ) is said to be of *alternating type* if all $j \in I \backslash X$ with ≥ 2 neighbours in I have ≥ 2 neighbours in X.
- (X, τ) is said to be of *parallel type* if τ has at least one hinge. This is possible in types $\mathrm{A}^{(1)}_{n \geq 1}$, $\mathrm{C}^{(1)}_{n \geq 2}$, $\mathrm{D}^{(1)}_{n \geq 4}$ (also $\mathrm{D}^{(2)}_{n+1 \geq 3}$).

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A *component* of X is a subset $X' \subseteq X$ such that $a_{ij} = 0$ for all $i \in X'$, $j \in X \backslash X'$. General decomposition of X into components:

$$X = X_1 \cup X_{\mathrm{alt}} \cup X_2,$$

where

- either $X_i = \emptyset$ or $Y_i \subset X_i$ $(i = 1, 2)$ and
- X_{alt} is of type $\mathrm{A}_1^{\times t}$ (with $t = 0$ unless (X, τ) is of alternating type).

Recall: the pair (X, τ) with $X \subset I$ of finite type and $\tau \in \mathrm{Aut}(A)$ an involution is admissible if:

1. for all $i \in X$, $\alpha_{\tau(i)} = -w_X(\alpha_i)$;
2. for all $j \in I \backslash X$, $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$ if $\tau(j) = j$.

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From condition 1. it follows that

- if a component of X is of type $\mathrm{D}_{t \geq 2}$ then it contains an even number of τ-orbits;
- if (X, τ) is of parallel type, each connected component of X is of type $\mathrm{A}_{t \geq 1}$, symmetrically arranged around a hinge.

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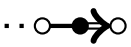
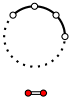
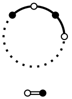
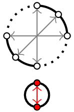
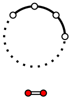
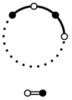
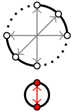
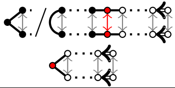
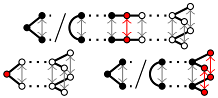
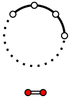
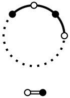
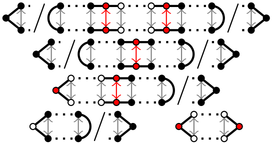
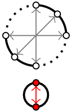
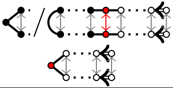
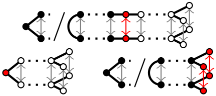
If (X, τ) is of identity type, then condition 2. implies the following:

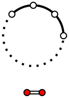
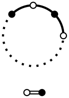
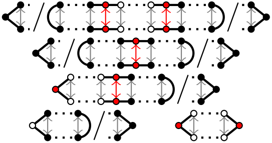
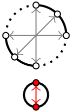
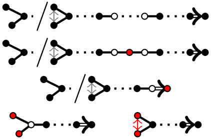
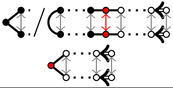
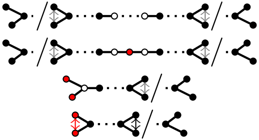
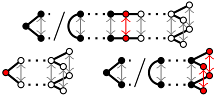
- The components of X are *either* of type B_t and/or D_t, *or* of type A_1 and/or C_t. In the latter case each node outside X neighbouring ≥ 2 nodes in I must neighbour 2 nodes in X.
- If I has a subset of type B_1 then (X, τ) must be of plain type:  and  are not admissible.

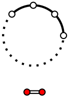
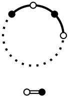
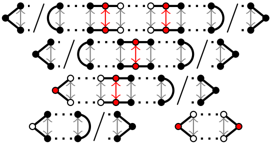
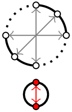
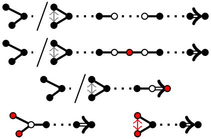
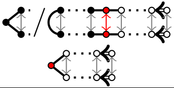
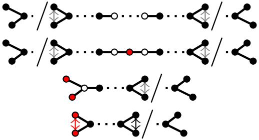
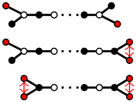
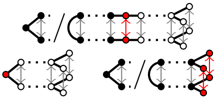
	Plain (1)	Alternating (2)	Parallel (3)	Exceptional (4)
A				
B				
C				
D				

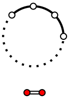
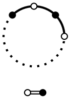
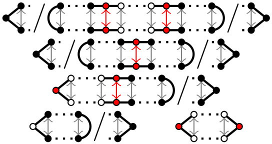
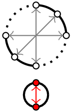
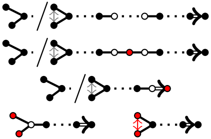
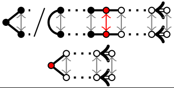
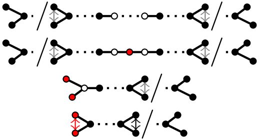
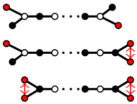
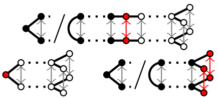
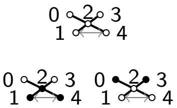
	Plain (1)	Alternating (2)	Parallel (3)	Exceptional (4)
A				
B				
C				
D				

	Plain (1)	Alternating (2)	Parallel (3)	Exceptional (4)
A				
B				
C				
D				

	Plain (1)	Alternating (2)	Parallel (3)	Exceptional (4)
A				
B				
C				
D				

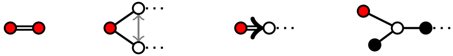
	Plain (1)	Alternating (2)	Parallel (3)	Exceptional (4)
A				
B				
C				
D				

	Plain (1)	Alternating (2)	Parallel (3)	Exceptional (4)
A				
B		Nothing!		
C				
D				

	Plain (1)	Alternating (2)	Parallel (3)	Exceptional (4)
A				
B				
C				
D				

Quasistandard QSP algebras

Define I_{qs} as consisting of those elements of I_{nse} of the following type:



Typical elements of $I_{\mathrm{nse}} \backslash I_{\mathrm{qs}}$ are:

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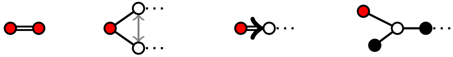

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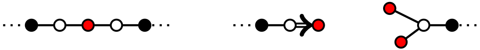

Definition

A QSP algebra $B^{\boldsymbol{c},\boldsymbol{s}}$ is called *quasistandard* if $s_j \neq 0 \implies j \in I_{\mathrm{qs}}$.

Three types of quasistandard QSP algebras:

1. QSP algebras which are "standard by nature", i.e. $I_{\mathrm{nse}} = \emptyset$;
2. Nonstandard QSP algebras which have been "standardized" ($s_j = 0$ forced for all $j \in I \backslash X$);
3. QSP algebras for which $I_{\mathrm{qs}} \neq \emptyset$.

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Nice property of quasistandard QSP algebras: corresponding K-matrices are of "generalized cross-form", i.e. given standard basis $\{v_1, \ldots, v_N\}$ of $\mathbb{C}^N$, there exists involution ψ on $\{1, \ldots, N\}$ such that

$$K(u) v_i \in \mathbb{C} v_i \oplus \mathbb{C} v_{\psi(i)} \qquad \text{for all } i \in \{1, \ldots, N\}.$$

- More precisely, nondiagonal nonzero entries of such K-matrices are on at most two antidiagonals.
- Forcing $s_j = 0$ for such K-matrices does not cause off-diagonal entries to vanish.

General formula

For the untwisted cases, the bare K-matrices are of the form

$$K(u) = \mathrm{Id} + \frac{u - u^{-1}}{k_1(u)}\Big(D_1(u) + \frac{D_2(u)}{k_2(u)}\Big)$$

where $k_1(u)$ and $k_2(u)$ are given by

$$\begin{aligned} &\mathrm{A.3, BCD.1, CD.2:} \quad && k_1(u) = \lambda\mu - u, \quad && k_2(u) = \lambda^{-1} + (\mu u)^{-1}, \\ &\mathrm{CD.3:} && k_1(u) = \mu^{-1} - \mu u, && k_2(u) = \lambda + (\lambda u)^{-1}, \end{aligned}$$

and the matrices $D_1(u)$ and $D_2(u)$ are defined as follows...

$$
\begin{aligned}
\mathrm{A.3}: \quad & D_1(u) = \sum_{s<i\leq N} E_{ii}, \\
& D_2(u) = \sum_{1\leq i\leq r} \left(\lambda E_{ii} + \lambda^{-1} E_{s-i+1,s-i+1} + E_{i,s-i+1} + E_{s-i+1,i}\right), \\
\mathrm{C.1}: \quad & D_1(u) = 0, \\
& D_2(u) = \sum_{1\leq i\leq n} \left(\lambda E_{-i,-i} + \lambda^{-1} E_{ii} + E_{-i,i} + E_{i,-i}\right), \\
\mathrm{D.2}: \quad & D_1(u) = \delta_{o_1,1} E_{nn} \\
& D_2(u) = \delta_{o_1,1}(\lambda - \mu^{-1}u) E_{-n,-n} + \delta_{o_2,1}(\lambda + \lambda^{-1}) E_{-1,-1} + \\
& \qquad + \sum_{o_2<i<\overline{o_1}} \left(\lambda E_{-i,-i} + \lambda^{-1} E_{ii} + \epsilon_i (E_{i,-i+\epsilon_i} + E_{-i+\epsilon_i,i})\right),
\end{aligned}
$$

where $\lambda, \mu \in \mathbb{C}^\times$ are free parameters, $\epsilon_i = (-1)^{i+o_2}$, $r = (N + o_1 + o_2)/2 - p_1 - p_2 + 1$ and $s = N - o_1 - 2p_1$.

$$\text{BD.1}: \quad D_1(u) = \sum_{\bar r \le i \le n} (\lambda\mu u E_{-i,-i} + E_{ii}),$$

$$D_2(u) = \sum_{\bar s \le i < \bar r} (\lambda E_{-i,-i} + \lambda^{-1} E_{ii} + E_{-i,i} + E_{i,-i}),$$

$$\text{C.2}: \quad D_1(u) = \sum_{\bar r \le i \le n} (\lambda\mu u E_{-i,-i} + E_{ii}),$$

$$D_2(u) = \sum_{\bar s \le i < \bar r} \big(\lambda E_{-i,-i} + \lambda^{-1} E_{ii} + \epsilon_i (E_{-i-\epsilon_i,i} + E_{i,-i-\epsilon_i})\big),$$

where $\lambda = q^{N/2-s}$, $\mu = q^{-r}$, $\epsilon_i = (-1)^{\bar\imath - r}$ and $(r, s) = (o_1 + p_1, n - o_2 - p_2)$.

$$\mathrm{CD.3}: \quad D_1(u) = \sum_{1\le i\le n} \mu E_{ii},$$

$$D_2(u) = \sum_{\bar{r}\le i\le n} \Big((\lambda\mu)^{-1} E_{-i,-i} + \lambda\mu E_{-\bar{\imath},-\bar{\imath}} - E_{-i,-\bar{\imath}} - E_{-\bar{\imath},i} +$$

$$- u^{-1}\big(\lambda\mu^{-1} E_{ii} + \lambda^{-1}\mu E_{\bar{\imath}\bar{\imath}} - E_{i\bar{\imath}} - E_{\bar{\imath}i}\big)\Big),$$

where $\lambda = q^{-n/2+r}$, $\mu \in \mathbb{C}^\times$ is a free parameter and $r = (n-o)/2 - p$.

Additional properties of K-matrices

- Assuming irreducibility of $T_u|_{B_{c,s}}$, one can derive "unitarity" $K(u)K(u^{-1}) = (\mathrm{scalar})\mathrm{Id}$.

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Additional properties of K-matrices

- Assuming irreducibility of $T_u|_{B_{c,s}}$, one can derive "unitarity" $K(u)K(u^{-1}) = (\mathrm{scalar})\mathrm{Id}$.
- Similarly, if $T_{\pm 1}|_{B_{c,s}}$ is irreducible, one can derive "regularity" $K(\pm 1) = (\mathrm{scalar})\mathrm{Id}$.
- $\hat{R}(u) = PR(u)$ satisfies Hecke-type identity:

$$\left(\hat{R}(u) - f_1(u)\mathrm{Id}\right)\left(\hat{R}(u) - f_2(u)\mathrm{Id}\right)\left(\hat{R}(u) - f_3(u)\mathrm{Id}\right) = 0$$

for some $f_i(u) \in \mathbb{C}(q^{\mathfrak{d}})^\times(u)$. For all quasistandard K-matrices we obtain similar identities, of degree ≤ 4.

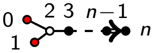
- Call (X, τ) *restrictable* if $\tau(0) = 0 \notin X$. Then $(X, \tau|_{\{1,\ldots,n\}})$ is an admissible pair w.r.t. the Cartan matrix $A_{\{1,\ldots,n\}}$ of finite type. In this case $K^{\mathrm{fin}} := \lim_{u \to 0} K(u)$ solves the finite refl. eqn.

$$R^{\mathrm{fin}}_{21} K^{\mathrm{fin}}_1 R^{\mathrm{fin}} K^{\mathrm{fin}}_2 = K^{\mathrm{fin}}_2 R^{\mathrm{fin}}_{21} K^{\mathrm{fin}}_1 R^{\mathrm{fin}}$$

where $R^{\mathrm{fin}} = \lim_{u \to 0} R(u)$. Moreover we always get an *affinization identity*, i.e. there exists $d^{\pm}(u) \in \mathbb{C}(q^{\mathfrak{d}})(u)$:

$$K(u) = \frac{d_+(u) K^{\mathrm{fin}} + d_-(u) (K^{\mathrm{fin}})^{-1}}{d_+(u) + d_-(u)}$$

and the Hecke-type identity is of degree ≤ 2.

- Non-quasistandard cases. E.g. $\left(\mathrm{B}_n^{(1)}\right)^{\mathrm{id}}_{0;n-2}$ = 

gives

$$K(u) = \mathrm{Id} + \frac{u - u^{-1}}{k(u)} \sum_{-1 \le i \le 1} k_i(u) D_i(u)$$

  with a *third-order* Hecke relation.
- q-Onsager cases
- "quartic" admissible pairs