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MAS8091: MMath Project

Coxeter Groups

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May 1, 2015

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Abstract

We begin by discussing finite reflection groups. In particular, we see that all finite reflection groups can be realised as Coxeter Groups. Then we move on to the general theory of Coxeter groups, including the construction of a faithful geometric representation, the Exchange Condition, and parabolic subgroups. Next we consider the automorphisms of Coxeter groups. As an example we determine all automorphisms of Coxeter groups of type A_n and then all involutive automorphisms. Let (W, S) be a Coxeter system and Γ the group of graph automorphisms. We give a detailed combinatorial proof that the set of Γ fixed points of a certain subgroup $\mathcal{W} \leq W$ form a Coxeter group with canonically defined generators. Further, we construct an involutive automorphism for which the group of fixed points can be expressed as a semidirect product of Coxeter groups.

Introduction

Coxeter groups are abstract groups generated by involutions. They form a large class of groups and with the finite reflection groups amongst them. In fact, the finite Coxeter groups are precisely the finite reflection groups in Euclidean space. Coxeter groups were first classified in 1935 by their namesake, H. S. M Coxeter, in [1].

In the first chapter we discuss finite reflection groups and their associated root systems in some detail. Allowing us to see concrete motivational examples from one of the most important types of Coxeter groups. Finite reflection groups give a great deal of insight and are key to understanding other Coxeter groups.

In the second chapter we formally introduce Coxeter systems and some important properties that they possess. An emphasis is put on their geometric representation in order to highlight the parallels between finite Coxeter groups and finite reflection groups. As such, wel establish a standard geometric representation in detail. Further, using this representation we show that there exists a property, the Exchange Condition, which fully characterises a Coxeter System. We also state the classification of finite irreducible Coxeter groups.

The third chapter will focus on automorphisms of Coxeter groups. We begin by considering an example, determining all automorphisms and all involutive automorphisms of a certain type, A_n, of Coxeter group as classified in Chapter 2. Now let (W, S) be a Coxeter system and Γ the group of automorphisms such that $\gamma(S) = S$ for all $\gamma \in \Gamma$. We will follow Remark 8 of [2] to give a combinatorial proof of a known theorem, that $\mathcal{W}^\Gamma = \{w \in \mathcal{W} \mid \gamma(w) = w \text{ for all } \gamma \in \Gamma\}$ forms a Coxeter group with canonically defined generators. Here $\mathcal{W} \leq W$ subject to certain conditions that will be fully specified in Section 3.4. This theorem is proved using geometric arguments in [3]. Then we will construct an involutive automorphism as the product of an inner automorphism and a graph automorphism such that the group of Θ fixed points in W can be expressed as the semidirect product of the Coxeter group $\mathcal{W}^\Gamma$ and some parabolic subgroup, that is by definition also a Coxeter group.

Chapter 1

Finite Reflection Groups

We will begin by discussing finite reflection groups. Doing so will make the original motivation for defining Coxeter groups clear.

1.1 Preliminary Definitions

Before we begin the main content of this chapter we will introduce some necessary definitions as in [4].

Definition 1.1.1 (Bilinear form)**.** Let L be a space and $(\cdot, \cdot)$ be a function that takes two vectors $\lambda, \mu \in L$ to a scalar value. This function is called a bilinear form of L if it is linear in each of its arguments. That is, when λ (respectively μ) is fixed $(\cdot, \cdot)$ is a linear function of μ (respectively λ) on L.

Further, if $(\lambda, \mu) = (\mu, \lambda)$ for all $\lambda, \mu \in L$ then we say that $(\cdot, \cdot)$ is symmetric.

Definition 1.1.2 (Orthogonal Transformation)**.** Let $T : V \to V$ be a linear transformation. If T preserves a symmetric bilinear form then we say it is an orthogonal transformation.

1.2 Reflection Groups

The remainder of this chapter primarily follows [5].

Definition 1.2.1 (Reflection)**.** Let V be a real Euclidean space and $s : V \to V$ be a linear transformation. We say s is a reflection if there exists some non zero vector α such that $s(\alpha) = -\alpha$ and s fixes pointwise the hyperplane H_α orthogonal to α.
We write $s = s_\alpha$.

There is a formula for reflections.

$$s_\alpha \lambda = \lambda - 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha,$$

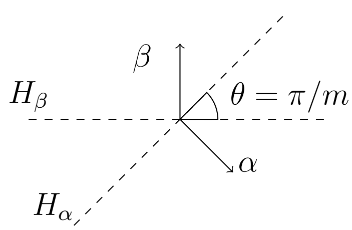
where (μ, λ) is a positive definite symmetric bilinear form. We can check that this formula is correct. Suppose $\lambda = \alpha$, substituting into the formula yields $s_\alpha(\lambda) = -\alpha$ as required. Further if we instead take $\lambda \in H_\alpha$ then $(\lambda, \alpha) = 0$ and so $s_\alpha \lambda = \lambda$. As $V = \mathbb{R}\alpha \oplus H_\alpha$ it follows that the formula holds for all $\lambda \in V$. Now it is clear that $s_\alpha^2 = 1$ and so all reflections s_α have order 2. In addition the calculation,

$$\begin{aligned}
(s_\alpha\lambda, s_\alpha\mu) &= \left(\lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \mu - 2\frac{(\mu,\alpha)}{(\alpha,\alpha)}\alpha\right) \\
&= \left(\lambda, \mu - 2\frac{(\mu,\alpha)}{(\alpha,\alpha)}\alpha\right) - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\left(\alpha, \mu - 2\frac{(\mu,\alpha)}{(\alpha,\alpha)}\right) \\
&= (\lambda,\mu) - 2\frac{(\mu,\alpha)}{(\alpha,\alpha)}(\lambda,\alpha) - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}(\alpha,\mu) + 4\frac{(\lambda,\alpha)(\mu,\alpha)}{(\alpha,\alpha)^2}(\alpha,\alpha) \\
&= (\lambda,\mu)
\end{aligned}$$

shows that s_α is an orthogonal transformation. We will denote the group of orthogonal transformations of V as $\mathrm{O}(V)$. Given a finite set of reflections we can define a subgroup of $\mathrm{O}(V)$.

Definition 1.2.2 (Finite Reflection Group)**.** Let S be a finite set of across hyperplanes that pass through the origin. The group $W = \langle S \rangle$ is called a finite reflection group.

Example 1.2.3 (The Dihedral Group)**.** Choose two straight lines H_α and H_β passing through the origin of Euclidean plane $\mathbb{R}^2$ with an angle of $\theta := \pi/m$ between them. Let s_α denote reflection across H_α and define s_β similarly.



Rotation through $2\pi/m$ can be achieved as a product of s_α and s_β when α and β are chosen to have the obtuse angle $\pi - \theta$ between them. It follows that the dihedral group of order $2m$ is generated by the two orthogonal reflections s_α and s_β. That is, D_{2m} is a finite reflection group.

1.3 Root Systems

For the remainder of this chapter take to be W be a finite reflection group acting on V. We continue to follow [5]. By definition each reflection $s_\alpha \in W$ fixes a hyperplane H_α. Let $L_\alpha = \mathbb{R}$ denote the line orthogonal to H_α. Now we will consider how each reflection acts on V.

Proposition 1.3.1. *Let $t \in O(V)$ and α be some non zero vector in V, then $ts_\alpha t^{-1} = s_{t\alpha}$.*

Proof. Clearly $ts_\alpha t^{-1}$ sends $t\alpha$ to its negative. Now we need to show that $ts_\alpha t^{-1}$ fixes $H_{t\alpha}$ pointwise. Note that λ lies in H_α if and only if $t\lambda$ lies in $H_{t\alpha}$, since $(\lambda, \alpha) = (t\lambda, t\alpha)$. Further, $(ts_\alpha t^{-1})(t\lambda) = ts_\alpha\lambda = t\lambda$ whenever λ lies in H_α. □

Corollary 1.3.2. *Let $w \in W$ and $s_\alpha \in W$, then $s_{w\alpha} \in W$.*

Proof. This follows immediately from the Proposition 1.3.1. □

We see that for each $w \in W$ we have $w(L_\alpha) = L_{w\alpha}$ and so W permutes the lines L_α. These lines are completely determined by W however the vectors α are not. However if we consider the set of all unit vectors lying on the lines L_α we see that it is stabilised. Such a choice gives a root system. We will now formally define root systems and see that every finite reflection group has an associated root system.

Definition 1.3.3 (Root System)**.** A root system Φ is a finite set of nonzero vectors in V such that:

- $\Phi \cap \mathbb{R}\alpha = \alpha, -\alpha$ for all $\alpha \in \Phi$;
- $s_\alpha\Phi = \Phi$ for all $\alpha \in \Phi$.

Each vector $\alpha \in \Phi$ is called a root.

We will refer to the two requirements of Definition 1.3.3. as the axioms of a root system. Often a third axiom, that Φ spans V is included but we will not require this here. Note that there is no requirement for the length of the roots $\alpha \in \Phi$. We may choose Φ to consist of unit vectors and it will often be convenient to do so but this is not a requirement. Similarly, all roots $\alpha \in \Phi$ may or may be of equal length.

The group W generated by all the reflections s_α, for all $\alpha \in \Phi$ is the finite reflection group associated with Φ. Thus W is completely determined by Φ and so it is possible to classify reflection groups by their root systems. This is not always as useful as it may first appear as $|\Phi|$ is often much larger than $\dim V$. As such we now wish to establish some minimal subset of Φ such that W can still be determined. We do this using simple roots however. Before we can define such roots we must introduce the notion of positive and negative roots.

Definition 1.3.4 (Total Ordering)**.** Let $\lambda, \mu, \nu \in V$ and $x \in \mathbb{R}$. A total ordering $<$ on V is a transitive relation such that

(i) Precisely one of $\lambda < \mu$, $\lambda = \mu$ or $\lambda > \mu$ holds.

(ii) If $\mu < \nu$ then $\mu + \lambda < \nu + \lambda$.

(iii) If $\mu < \lambda$ then $x\mu < x\lambda$ when $x > 0$ and $x\mu > x\lambda$ when $x < 0$.

Now we will impose a total ordering on V. To do so we must choose an arbitrary ordered basis $\lambda_1 \dots \lambda_n$ of V. We take the corresponding lexicographical order, where $\sum a_i\lambda_i < \sum b_i\lambda_i$ if $a_k < b_k$ when k is the smallest index such that $a_i \neq b_i$.Given this ordering we say that $\lambda \in V$ is positive when $0 < \lambda$.

Definition 1.3.5 (Positive System)**.** Let Π be a subset of some root system Φ. If Π consists of all the positive roots, corresponding to some total ordering of V, then Π is a positive system.

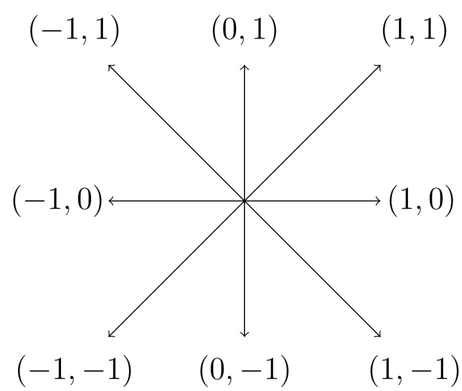
It is clear that positive systems exist. Further as roots occur in pairs $\{\alpha, -\alpha\}$ we can similarly define a negative system, denoted $-\Pi$, consisting only of the negative roots in Φ.

Definition 1.3.6 (Simple System)**.** Let Δ be a subset of some root system Φ. We say Δ is a simple system if:

- Δ is a vector space basis for the $\mathbb{R}$-span of Φ in V;
- each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign.

It can be shown that every $\Delta \in \Phi$ is contained in a unique positive system Π and that every positive system $\Pi \in \Phi$ contains a unique simple system Δ.

Example 1.3.7. We saw in Example 1.2.3. that all dihedral groups are finite reflection groups. Consider the dihedral group D_8. It's associated root system Φ is shown below.



We can choose $\Pi = \{(1,0),(1,1),(0,1),(-1,1)\}$ and $\Delta = \{(1,0),(-1,1)\}$.

Definition 1.3.8 (Height)**.** We can write any root $\beta \in \Phi$ uniquely as $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$. The value $\sum c_\alpha$ is called the height of β. We denote this as $\mathrm{ht}(\beta)$.

From the definition it is clear that the height of a root $\alpha \in \Phi$ depends directly on the choice of simple system Δ.

Fix a simple system Δ. It is clear from the definition that $w\Delta$ is also a simple system for any $w \in W$ and that it lies in the positive system $w\Pi$. Let $\alpha \in \Delta$, then we say s_α is a simple reflection. We will now see how the simple reflections act on Π.

Proposition 1.3.9. *Let $\alpha \in \Delta$ be a fixed simple system contained in the positive system Π. Then s_α permutes the roots in $\Pi \setminus \{\alpha\}$.*

Proof. Let $\beta \in \Pi \setminus \{\alpha\}$. Then, as β cannot be a multiple of α we may write

$$\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma,$$

where all $c_\gamma \geq 0$ and at least one $c_\gamma \neq c_\alpha > 0$ as the only multiple of $\alpha \in \Pi$ is $\alpha \neg \beta$. When we apply s_α to β we obtain $s_\alpha \beta = \beta - c\alpha$. This is a linear combination of Δ involving γ with the same coefficient c_γ. As all coefficients in such an expression must have the same sign we see that $s_\alpha \beta$ must be positive. That is $s_\alpha \beta \in \Pi$. Now, suppose $s_\alpha \beta = \alpha$. Then $\beta = s_\alpha s_\alpha \beta = -\alpha \notin \Pi$ which is a contradiction. So s_α is an injective map from $\Pi \setminus \{\alpha\}$ to itself and so must also be surjective. □

Theorem 1.3.10. *Let Φ be a root system then any two positive systems Π and Π' in Φ are conjugate under W.*

Proof. First note $|\Pi| = |\Pi'| = |\Phi|/2$. We will now proceed by induction on $r = |\Pi \cap -\Pi'|$. Suppose $r = 0$, then $\Pi = -\Pi'$ and so we are done. Suppose $r > 0$ it follows that the simple system $\Delta \in \Pi$ contains some root $\alpha \in -\Pi'$. By Proposition 1.3.9. $|s_\alpha \Pi \cap -\Pi'| = r - 1$. Then, as $s_\alpha \Pi$ is a positive system, by induction there exists some $w \in W$ such that $w(s_\alpha \Pi) = \Pi'$. □

We conclude this section by showing that for any fixed simple system Δ the corresponding set of simple reflections generate W.

Theorem 1.3.11. *Let W be a finite reflection group and Φ its associated roots system. Fix some simple system $\Delta \subseteq \Phi$. Then $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$.*

Proof. Let W' be the group generated by the s_α.

Suppose $\beta \in \Pi$ and consider $W'\beta \cap \Pi$. As $1 \in W'$ and so $\beta \in W'\beta$ we see that $W'\beta \cap \Pi$ is a nonempty set of positive roots. Now choose some $\gamma \in W'\beta \cap \Pi$ such that $\mathrm{ht}(\gamma) \leq \mathrm{ht}(\gamma')$ for all other $\gamma' \in W'\beta \cap \Pi$. Now write $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$, then $(\gamma, \alpha) > 0$ for some $\alpha \in \Delta$ as $0 < (\gamma, \gamma) = \sum c_\alpha (\gamma, \alpha)$. Clearly this holds when $\gamma = \alpha$. Now suppose $\gamma \neq \alpha$. Then $s_\alpha \gamma$ is a positive root by Proposition 1.3.8. This root is obtained from γ by subtracting some positive multiple of α and so $\mathrm{ht}(s_\alpha \gamma) < \mathrm{ht}(\gamma)$. But $s_\alpha \in W'$ implies $s_\alpha \gamma \in W'\beta$ and so we have a contradiction. Thus $\gamma = \alpha \in \Delta$.

We have just shown that the W'-orbit of any $\beta \in \Pi$ intersects Δ and so it follows that $\Pi \subset W'\Delta$. Now suppose β is negative. There exists some $w \in W'$ such that $w\beta w^{-1} = \alpha$ for some $\alpha \in \Delta$. The if we take $-\beta = w\alpha$ it follows that $\beta = (ws_\alpha)\alpha$ and $ws_\alpha \in W'$. Then $-\Pi \subset W'\Delta$ and we conclude $\Phi = W'\Delta$.

Now let s_β be some generator of W. Following the arguments of the previous paragraph we may write $\beta = w\alpha$ for some $w \in W'$ and $\alpha \in \Delta$. Then by Proposition 1.3.1. we have $s_\beta = ws_\alpha w^{-1}$. Thus we have shown $W = W'$. □

1.4 Deletion Condition

In order to prove our final result in this section we need to introduce an important property of finite reflection groups, the Deletion Condition. We will see in Section 2.6. that this condition can be used to characterise Coxeter groups. Before we do this we will define the length function.

Definition 1.4.1 (Length)**.** Let W be a finite reflection group. We define the length of an element $w \in W$ to be the minimal r such that $w = s_1 \dots s_r$ where $s_i = s_{\alpha_i}$ for some $\alpha_i \in \Delta$. That is, the length of w is the minimal number of simple reflections needed to write w in it's reduced form.

We write $\ell(w) = r$.

Proposition 1.4.2. *Let $w \in W$, then $det(w) = (-1)^r$.*

Proof. First recall that any reflection has det$= -1$. It immediately follows that any $w \in W$ that can be written as the product or r reflections has $\det(w) = (-1)^r$. Hence r has the same parity as $\ell(w)$. □

When Δ and Π are fixed we define $n(w)$ to be the number of positive roots sent to negative roots by w. That is, $n(w) := |\Pi \cap w^{-1}(-\Pi)|$.

Lemma 1.4.3. *Let $\alpha \in \Delta$ and $w \in W$. If $w\alpha > 0$ then $n(ws_\alpha) = n(w) + 1$ and if $w\alpha < 0$ then $n(ws_\alpha) = n(w) - 1$.*

Proof. Set $\Pi(w) := \Pi \cap w^{-1}(-\Pi)$, then $n(w) = |\Pi(w)|$. Suppose $w\alpha > 0$ then by Proposition 1.3.9. $\Pi(ws_\alpha)$ is the disjoint union of $s_\alpha\Pi(w)$ and $\{\alpha\}$. Now suppose $w\alpha < 0$. Proposition 1.3.9. implies that $s_\alpha\Pi(ws_\alpha) = \Pi(w) \setminus \alpha$, whereas α lies in $\Pi(w)$. □

Now we are able to prove an important result.

Theorem 1.4.4 (Deletion Condition)**.** *Given $w = s_1 \dots s_r$, where $s_i \in S$, that is not reduced, there exist indices $1 \leq i < j \leq r$ such that $w = s_1 ... \hat{s}_i ... \hat{s}_j ... s_r$. Here $\hat{s}$ denotes the omission of s.*

Proof. As $n(w) < r$ repeatedly applying Lemma 1.4.3. shows that we have $(s_1 \dots s_{j-1}\alpha_j < 0$ for some $j \leq r$. As $\alpha_j > 0$ there must exist some index $i < j$ such that $s_i(s_{i+1} \dots s_{j-1})\alpha_j < 0$ and $(s_{i+1} \dots s_{j-1})\alpha_j > 0$. Now apply Proposition 1.3.9. to the simple reflection s_i. This implies that $(s_{i+1} \dots s_{j-1})\alpha_j = \alpha_i$.

Now set $\alpha = \alpha_j$ and $w' = s_{i+1} \dots s_{j-1}$, then by the above $w'\alpha = \alpha_i$. Further Proposition 1.3.1. $w' s_\alpha w'^{-1} = s_{w'\alpha} = s_i$, meaning

$$(s_{i+1} \dots s_{j-1}) s_j (s_{j-1} \dots s_{i+1}) = s_i.$$

Now we multiply on the left by $s_1 \dots s_i$ and on the right by $s_{i+1} \dots \hat{s}_j \dots s_r$ to obtain

$$w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_r$$

as required. □

1.5 Every Finite Reflection Group is a Coxeter Group

Continuing to follow [5], we are now able to prove the main result of this chapter, that every finite reflection group is a Coxeter group.

Theorem 1.5.1. *Let Φ be a root system and W the associated finite reflection group. Fix a simple system $\Delta \in \Phi$ then W is generated by the set $S := \{s_\alpha \mid \alpha_\in \Delta\}$ subject only to relations of the form*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1,$$

where $\alpha, \beta \in \Delta$.

Proof. We have already seen in Theorem 1.3.9. that W is generated by the set $S = \{s_\alpha \mid \alpha_\in \Delta\}$. Hence we now now only need to show that each relation

$$s_1 ... s_r = 1 \tag{1}$$

is a consequence of the given relations. First we note that the number of simple reflections in (1) must be even as by Proposition 1.4.2. $textdet(s_1 \dots s_r) = 1$ and $\det(s_i)$ for all reflections. If $r = 2$ then (1) is $s_1 s_2 = 1$ which implies $s_2 = s_1^{-1} = s_1$. Hence (1) can be written as $s_1^2 = 1$, a given relation. Now we may proceed using induction on $r = 2p$, where $p > 1$. Note that we will henceforth use the relation $s_i^2 = 1$ freely to rewrite expressions. For example, we may rewrite (1) as

$$s_{i+1} \dots s_r s_1 \dots s_i = 1. \tag{2}$$

Now consider the element,

$$s_1 \dots s_{p+1} = s_r \dots s_{p+2}.$$

Clearly $\ell(s_r \dots s_{p+2}) \leq p - 1$ and so the LHS cannot be reduced. Hence we can apply the Deletion Condition to find indices $1 \leq i < j \leq p + 1$ such that

$$s_1 \dots s_{p+1} = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_{p+1}. \tag{3}$$

Rearranging (3) we obtain,

$$s_{i+1} \dots s_j = s_i \dots s_{j-1} \tag{4}$$

and

$$s_i \dots s_j \dots s_{i+1} = 1. \tag{5}$$

If (5) is a product of less than r simple reflections then by induction we conclude that it is a result of the given relations. Given this we may substitute (4) into (1) to obtain,

$$s_1 \dots s_i(s_i \dots s_{j-1}) s_{j+1} \dots s_r = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_r = 1.$$

Then, once again by induction, we see that this equation and so (1) is a result of the given equations.

Now suppose (5) is the product of r simple reflections. This is only possible if $i = 1$ and $j = p + 1$, that is when (4) reads

$$s_2 \dots s_{p+1} = s_1 \dots s_p. \tag{6}$$

In order to avoid this outcome we may choose to rewrite (1) as some alternate version of (3), for example

$$s_2 \dots s_r s_1.$$

In this case we repeat the same steps as above, yielding a relation that is a consequence of relations of the given form unless

$$s_3 \dots s_{p+2} = s_2 \dots s_{p+1}. \tag{7}$$

Now we can multiply (7) on the right by $s_3 s_2$ and on the left by $s_{p+1} \dots s_4$ to obtain

$$s_3(s_2 s_3 \dots s_{p+1}) s_{p+2} s_{p+1} \dots s_4 = 1.$$

The LHS is a product of r simple reflections so we may apply of original line of argument once again. Upon doing so we are successful in all cases but

$$s_2 \dots s_{p+1} = s_3 s_2 s_3 \dots s_p. \tag{8}$$

But then (6) and (8) must be equal forcing $s_1 = s_3$. Then by cyclically permuting factors we reach a successful conclusion unless $s_2 = s_4$. Continuing in the same way we see that the only case in which we don't reach a successful conclusion is when

$$s_1 = s_3 = ... = s_{r-1} \quad \text{and} \quad s_2 = s_4 = \cdots = s_r.$$

However in this case (1) can be rewritten as

$$s_\alpha s_\beta \dots s_\alpha s_\beta = (s_\alpha s_\beta)^m = 1.$$

Clearly this is a relation of the desired form.

□

By proving this theorem we have shown that all finite reflection groups are Coxeter groups. Notice we only used the deletion condition in order to do this. This is due to the fact that a group satisfying the deletion condition is equivalent to it being a Coxeter group.

Chapter 2

Coxeter Groups

In the previous chapter we saw many specific examples of Coxeter groups in the form of finite reflection groups. Now we will formally introduce Coxeter groups and some of their most important properties.

2.1 Coxeter Systems

This section is primarily based upon [5] and [6]. We begin by defining a Coxeter system. A Coxeter Group is an abstract group generated by a set of involutions subject to a given presentation.

Definition 2.1.1 (Coxeter System)**.** Let W be the group generated by the set S subject to relations of the form

$$(ss')^{m(s,s')} = 1,$$

for all $s, s' \in S$, where $m(s,s) = 1$ and $m(s,s') = m(s',s) \geq 2$ when $s \neq s'$. If $(ss')^m \neq 1$ for all $m \in \mathbb{Z}$ then we say $m(s,s') = \infty$.

- We say W is a Coxeter group and S a set of Coxeter generators.
- The pair (W, S) is called a Coxeter system.

In particular we notice that $m(s,s') = 2$ only when s and s' commute. We say that the Coxeter group (W, S) has rank $|S|$.

Examples 2.1.2. (i) Consider S_{n+1}, the symmetric group of order $n+1$. This is the group of permutations of a set containing $n+1$ elements. Let S be the set of n transpositions $S = \{(1,2),(2,3),\ldots,(n-1,n)\}$. Then (S_{n+1}, S) is a Coxeter system. Further S_{n+1} has rank n.

(ii) Now consider D_{2m}, the dihedral group of order m. This is the group of symmetries of a regular m-gon. D_{2m} is a Coxeter group with a set of 2 Coxeter generators $S = \{s_1,\ s_2\}$, where s_1 and s_2 are suitably chosen reflections. Further D_{2m} has rank 2.

Although often, when the set of Coxeter generators is understood, we will refer only to a Coxeter group it is important to always have the Coxeter system in mind. This is due to the fact that two different sets of Coxeter generators can generate Coxeter groups that are isomorphic as groups.

There are two further ways in which we can uniquely specify a Coxeter system. These are via a Coxeter matrix or a Coxeter graph, we will now define and give simple examples of each method.

Definition 2.1.3 (Coxeter Matrix)**.** The Coxeter system (W, S) specifies a Coxeter matrix $m : S \times S \to \{1, 2, \ldots, \infty\}$ such that $m(s, s') = m(s', s)$ and $m(s, s') \geq 2$ whenever $s \neq s'$. In addition $m(s, s) = 1$ for all $s \in S$.

Examples 2.1.4. (i) The symmetric group S_4 has Coxeter matrix:

$$m = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

(ii) D_{2m} has Coxeter matrix:

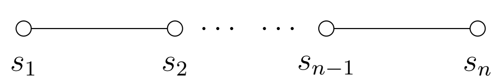
$$m = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}$$

Definition 2.1.5 (Coxeter Graph)**.** A Coxeter system can be completely specified by it's Coxeter graph, Γ:

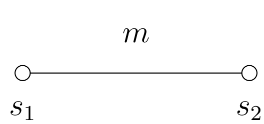
- Take the set S to be the vertices of the graph.
- Two vertices s, s' are connected by edge if $m(s, s') \geq 3$.
- Any edge of Γ connecting s and s' is labeled by $m(s, s')$ if $m(s, s') \geq 4$.

Note that two vertices $s, s' \in S$ are not joined by an edge only when $m(s, s') = 2$. That is, when they commute.

Examples 2.1.6. (i) The symmetric group, S_{n+1} on n generators has Coxeter graph Γ:



(ii) The Coxeter system (D_{2m}, S) has Coxeter graph Γ:



Now, in order to demonstrate our earlier claim that specifying a Coxeter group without a set of Coxeter generators is not sufficient to uniquely define a Coxeter system, we will consider an example.

Example 2.1.7. Let $W = D_{12}$, the dihedral group of order 12. Recall this is the group of symmetries of a regular hexagon. As such W has two Coxeter generators $S = \{s_1, s_2\}$ where s_1 and s_2 are suitably chosen reflections. Equivalently we can write $W = \langle x, y \mid x^6 = 1,\ y^2 = 1,\ (xy)^2 = 1\rangle$ where x is rotation through $\pi/3$ and y reflection through some fixed line. Construct $H = \{1, x^3\} \leq W$ and $K = \{1, y, x^2, x^4, yx^2, yx^4\} \leq W$. Now observe that $HK = W$ and $H \cap K = \{1\}$. Further $hk = kh$ for all $h \in H$ and $k \in K$. It follows that $W = H \times K$. Now we observe that $H \cong \mathbb{Z}_2$ and $K \cong S_3$. Thus $W \cong \mathbb{Z}_2 \times S_3$. Consider the respective Coxeter graphs of W and $\mathbb{Z}_2 \times S_3$ below.

Clearly these graphs are not isomorphic and so we conclude that two Coxeter groups being equivalent up to isomorphism does not imply that their corresponding Coxeter generators are.

2.2 The Length Function

We now follow [5]. We begin by recall the definition of the length function in the context of Coxeter groups,

Definition 2.2.1. Define the length of $w \in W$ to be the minimal r such that $w = s_1 \ldots s_r$ where $s_i \in S$. We write $\ell(w) = r$. When w is written as a product of $\ell(w)$ factors $s_i \in S$ we say w is reduced.

The following proposition gives some elementary properties of the length function.

Proposition 2.2.2. *Let $w, w' \in W$ then*

(i) $\ell(w) = \ell(w^{-1})$,

(ii) $\ell(w) = 0$ *if and only if* $w = 1$,

(iii) $\ell(w) - \ell(w') \leq \ell(ww') \leq \ell(w) + \ell(w')$.

Proof. We will prove (i) first. Suppose $\ell(w) = r$, then we may write $w = s_1 \ldots s_r$ where $s_i \in S$. Then $w^{-1} = s_r \ldots s_1$ and so $\ell(w^{-1}) \leq r = \ell(w)$. Similarly we can obtain $\ell(w) \leq \ell(w^{-1})$ and conclude $\ell(w) = \ell(w^{-1}$. We will now prove (iii). Let $w, w' \in W$ such that $\ell(w) = r$ and $\ell(w') = s$. It is clear that $\ell(ww') \leq r + s = \ell(w) + \ell(w')$. Now consider $\ell(ww'w'^{-1}) = \ell(w) \leq \ell(ww') + \ell(w')$ and so $\ell(w) - \ell(w') \leq \ell(ww')$. Finally the proof of (ii) is trivial. $\square$

Proposition 2.2.3. *There exists a unique homomorphism $\epsilon : W \to \{1, -1\}$ such that the image of each generator $s \in S$ is -1. This homomorphism is surjective.*

Proof. Define a homomorphism φ from the free group F onto the multiplicative group $\{1, -1\}$ sending all $s \in S$ to -1. Clearly $(ss')^{m(s,s')} \in \ker(\varphi)$ for all $s, s' \in S$. Hence there is an homomorphism $\epsilon : W \to \{1, -1\}$ sending all $s \in S$ to -1 and this homomorphism is surjective. $\square$

Using this homomorphism we can observe two additional properties of the length function.

Proposition 2.2.4. *The homomorphism $\epsilon : W \to \{1, -1\}$ is given by $\epsilon(w) = (-1)^{\ell(w)}$. From this we conclude that $\ell(ws) = \ell(w) \pm 1$ for all $s \in S$ and similarly for $\ell(sw)$.*

Proof. Let $w = s_1 \dots s_r$ be a reduced expression for w. Then we see

$$\epsilon(w) = \epsilon(s_1) \dots \epsilon(s_r) = (-1)^r = (-1)^{\ell(w)}.$$

Now $\epsilon(ws) = -\epsilon(w)$, so $\ell(ws) \neq \ell(w)$ and so as $\ell(s) = 1$ we must have $\ell(ws) = \ell(w) \pm 1$. We can similarly show $\ell(sw) = \ell(w) \pm 1$. $\square$

All the properties of the length function given in this section will henceforth be used without further explanation.

2.3 The Standard Geometric Representation

We now establish a concrete geometric representation of a given Coxeter group (W, S) following [5]. It is not always possible to find a faithful representation of W as a group generated by orthogonal reflections in Euclidean space. However upon redefining a reflection to be a linear transformation that fixes pointwise some hyperplane and sends some non zero vector to its negative we can construct a satisfactory replacement.

Let (W, S) be some Coxeter group of rank n and V the n dimensional $\mathbb{R}$-vector space with fixed basis $\{\alpha_s \mid s \in S\}$. We define a symmetric bilinear form B on V as follows,

$$B(\alpha_s, \alpha_{s'}) = \begin{cases} -\cos \dfrac{\pi}{m(s,s')} & \text{if } m(s,s') < \infty, \\ -1 & \text{if } m(s,s') = \infty. \end{cases}$$

We can see that $B(\alpha_s, \alpha_s) = 1$ and $B(\alpha_s, \alpha_{s'}) \leq 0$ if $s \neq s'$. Let $H_s := \{v \in V \mid B(v, \alpha_s) = 0\}$, that is H_s is the subspace orthogonal to α_s with respect to B. Note that as $B(\alpha_s, \alpha_s) \neq 0$ for all $s \in S$ we have $\mathbb{R}\alpha_s \notin H_s$.

For each $s \in S$ we define a reflection $\sigma_s : V \to V$ as follows,

$$\sigma_s \lambda = \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

We see that $\sigma_s \alpha_s = -\alpha_s$ and σ_s fixes H_s pointwise. Recalling that $\mathrm{GL}(V)$ is the group of all bijective linear transformations from V to V, it is clear that σ_s has order 2 in $\mathrm{GL}(V)$.

Now, let $\lambda, \mu \in V$. We show that σ_s preserves the form B

$$\begin{aligned} B(\sigma_s\lambda, \sigma_s\mu) &= B(\lambda - 2B(\alpha_s, \lambda)\alpha_s, \mu - 2B(\alpha_s, \mu)\alpha_s) \\ &= B(\lambda, \mu) - 4B(\alpha_s, \lambda)B(\alpha_s, \mu) + 4B(\alpha_s, \lambda)B(\alpha_s, \mu)B(\alpha_s, \alpha_s) \\ &= B(\lambda, \mu). \end{aligned}$$

It follows that all $\gamma \in \langle \sigma_s \mid s \in S \rangle \subseteq \mathrm{GL}(V)$ preserve the form of B.

Proposition 2.3.1. *There exists a unique homomorphism $\sigma : W \to GL(V)$ that sends s to σ_s. The group $\sigma(W)$ preserves the form B on V and for each pair $s, s' \in S$ the order of ss' in W is $m(s, s')$. We will refer to this homomorphism as the standard geometric representation of W.*

Proof. That σ is unique follows immediately from $\sigma(s) = \sigma_s$. To show that such a homomorphism exists it suffices to check that

$$(\sigma_s\sigma_{s'})^{m(s,s')} = 1 \text{ for all } s \neq s'.$$

Let $m := m(s, s')$ and $V_{s,s'}$ be the two dimensional subspace of V generated by α_s and $\alpha_{s'}$. Take any $\lambda = a\alpha_s + b\alpha_{s'}$, where $a, b \in \mathbb{R}$. Then

$$B(\lambda, \lambda) = a^2 - 2ab\cos\left(\frac{\pi}{m}\right) + b^2 = \left(a - b\cos\left(\frac{\pi}{m}\right)\right)^2 + b^2\sin^2\left(\frac{\pi}{m}\right) \geq 0.$$

Hence the form B is positive semidefinite when restricted to $V_{s,s'}$. Further the form is non degenerate precisely when $\sin(\pi/m) \neq 0$. This is the case only when $m < \infty$.

Now we note that both σ_s and $\sigma_{s'}$ leave $V_{s,s'}$ stable. Thus we can calculate the order of $\sigma_s\sigma_{s'}$ restricted to $V_{s,s'}$.

Suppose $m < \infty$. Then the restriction of B to $V_{s,s'}$ is positive definite and so gives $V_{s,s'}$ a structure corresponding to the Euclidean plane. Thus both σ_s and $\sigma_{s'}$ act as orthogonal reflections. Observe that $B(\alpha_s, \alpha_{s'}) = -\cos(\pi/m) = \cos(\pi - (\pi/m))$. It follows that the angle between $\mathbb{R}^+\alpha_s$ and $\mathbb{R}^+\alpha_{s'}$ is $\pi - (\pi/m)$ and so the angle between the two orthogonal reflecting lines must be π/m. We saw in Example 1.2.3 that this implies $\sigma_s\sigma_{s'}$ is a rotation through $2\pi/m$. We conclude that $\sigma_s\sigma_{s'}$ has order m. Additionally, as B is non degenerate, we see that V is the orthogonal direct sum of $V_{s,s'}$ and its orthogonal complement , $V_{s,s'}^{\perp} := \{x \in V \mid B(x, y) = 0 \text{ for all } y \in V_{s,s'}\}$. Now we see that both σ_s and $\sigma_{s'}$ must fix the complement pointwise and so it follows that $\sigma_s\sigma_{s'}$ also has order m on V.

Now suppose we have $m = \infty$. Then $B(\alpha_s, \alpha_{s'}) = -1$. Consider $\lambda = \alpha_s + \alpha_{s'}$, it is clear that $B(\lambda, \alpha_s) = 0 = B(\lambda, \alpha_{s'})$ and so $\sigma_s(\lambda) = \sigma_{s'}(\lambda) = \lambda$. Further we have $\sigma_s\sigma_{s'}\alpha_s = \sigma_s(\alpha_s + 2\alpha_{s'}) = 3\alpha_s + 2\alpha_{s'} = 2\lambda + \alpha_s$. Then an induction shows $(\sigma_s\sigma_{s'})^k\alpha_s = 2k\lambda + \alpha_s$, for all $k \in \mathbb{Z}$. It follows that $\sigma_s\sigma_{s'}$ has infinite order on $V_{s,s'}$ and so must also have infinite order on V. $\square$

Now it remains to show that this geometric representation is faithful. To do this we first specify an associated root system of the standard geometric representation of W.

Definition 2.3.2. The root system associated with the standard geometric representation of W is $\Phi = \{w(\alpha_s) \mid w \in W \text{ and } s \in S\}$.

Note that in this definition we have used $w(\alpha_s)$ to denote $\sigma(w)(\alpha_s)$, we will continue to do so from here on. We see that any $\alpha \in \Phi$ is a unit vector as W preserves the form of B on V. Further we have $\Phi = -\Phi$ as $s(\alpha_s) = -\alpha_s$. Any root $\alpha \in \Phi$ can be written uniquely as

$$\alpha = \sum_{s \in S} c_s \alpha_s,$$

where $c_s \in \mathbb{R}$. We say α is positive, that is $\alpha > 0$ if all $c_s \geq 0$. We can similarly define the case where α is negative. We denote the sets of positive and negative roots as Φ^+ and Φ^- respectively.

To prove our next theorem we need to introduce parabolic subgroups.

Definition 2.3.3 (Parabolic Subgroup)**.** The subgroup W_I generated by $I \subseteq S$, is called a parabolic subgroup when (W, S) is a Coxeter Group.

In addition, let ℓ_I be the length function relative to the generating set I. Then it follows from $I \subseteq S$ that $\ell(w) \leq \ell_I(w)$. This will be sufficient for our current needs however we will further discuss the properties of parbolic subgroups in Section 2.7.

Theorem 2.3.4. *Let $w \in W$ and $s \in S$. Then $\ell(ws) > \ell(w)$ implies $w(\alpha_s) > 0$. Similarly $\ell(ws) < \ell(w)$ implies $w(\alpha_s) < 0$.*

Proof. Let $\ell(ws) > \ell(w)$. We will now use induction on $\ell(w)$. When $\ell(w) = 0$ there is nothing to prove. If $\ell(w) > 0$ then it follows that there exists some $s' \in S$ such that $\ell(ws') < \ell(w)$. Clearly $s \neq s'$. Thus if we set $I := \{s, s'\}$ it follows that W_I is a dihedral group.

Now consider the set

$$A := \{v \in W \mid v^{-1}w \in W_I \text{ and } \ell(v) + \ell_I(v^{-1}w) = \ell(w)\}.$$

Clearly $w \in A$ so A is nonempty. Fix $v \in A$ such that v has minimal length and set $v_I := v^{-1}w \in W_I$. Hence $w = vv_I$ and $\ell(w) = \ell(v) + \ell_I(v_I)$.

Observe $(s'w^{-1})w = s'$ and $\ell(ws') + \ell_I(s') = (\ell(w) - 1) + 1 = \ell(w)$, so $ws' \in A$. It follows that $\ell(v) \leq (\ell(ws') = \ell(w) - 1$.

Now we will compare the lengths of v and vs. Suppose $\ell(vs) < \ell(v)$. Then

$$\begin{aligned} \ell(w) &\leq \ell(vs) + \ell((sv^{-1})w) \leq \ell(vs) + \ell_I(sv^{-1}w) = \ell(v) - 1 + \ell_I(sv^{-1}w) \\ &\leq \ell(v) + \ell_I(v^{-1}w) = \ell(w). \end{aligned}$$

Forcing $\ell(w) = \ell(vs) + \ell((sv^{-1})w)$ and thus $vs \in A$. This contradicts $\ell(vs) < \ell(v)$ and so we conclude that $\ell(v) > \ell(vs)$. Thus by induction $v(\alpha_s) > 0$. We can similarly show $v(\alpha_{s'}) > 0$.

Now, as $w = vv_I$, if we show that $v_I(\alpha_s) = a\alpha_s + b\alpha_{s'}$ where $a, b \geq 0$ we will be done. Suppose we have $\ell_I(v_I s) < \ell(v_I)$. Then

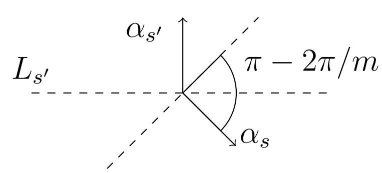
$$\begin{aligned}\ell(ws) &= \ell(vv^{-1}ws) \leq (v) + \ell(v^{-1}ws) = \ell(v) + \ell(v_I s)\\ &\leq \ell(v) + \ell_I(v_I s) < \ell(v) + \ell_I(v_I) = \ell(w),\end{aligned}$$

contradicting $\ell(ws) > \ell(w)$. Thus $\ell_I(v_I s) \geq \ell_I(v_I)$. Then, as W_I is dihedral, any reduced expression for $v_I \in W_I$ must end in s'. That is, either $v_I = (ss')^k$ or $v_I = s'(ss')^l$ where $k \leq m/2$ and $l < m/2$.

Suppose $m(s, s') = \infty$. In the proof of Proposition 2.3.1. we calculated $(ss')^k(\alpha_s) = 2k\lambda + \alpha_s$, where $\lambda = \alpha_s + \alpha_{s'}$ so when $v_I = (ss')^k$ we have $v_I(\alpha_s) = 2k\alpha_{s'} + (2k+1)\alpha_{s'}$. Now suppose we have $v_I = s'(ss')^l = s'(2l\lambda + \alpha_s) = \alpha_s + (2l+1)\lambda = (2l+1)\alpha_{s'} + 2(l+1)\alpha_s$. In both cases $v_I(\alpha_s)$ has been written as a non negative linear combination of α_s and $\alpha_{s'}$ so we are done.

Now suppose $m := m(s, s') < \infty$. Then the element $w_I \in W_I$ such that $\ell_I(w_I) = m$ has a reduced expression ending with s and so $\ell_I(v_I) < m$. It follows that either $v_I = (ss')^k$ or $v_I = s'(ss')^l$ where $k, l < m/2$. Recall from the proof of Proposition 2.3.1. that we are in the Euclidean plane and α_s and $\alpha_{s'}$ are unit vectors with an angle of $\pi - \pi/m$ between them. Define the positive cone $C := \{a\alpha_s + b\alpha_{s'} \mid a, b \in \mathbb{Z}^+\}$. We have seen that ss' rotates α_s through an angle of $2\pi/m$ towards $\alpha_{s'}$.

It follows that if v_I begins with s it moves α_s through an angle of at most $\pi - 2\pi/m$ towards $\alpha_{s'}$ and so $v_I(\alpha_s) \in C$. If v_I begins with an s' then $v_I(\alpha_s) \in C$ as the angle between α_s and the reflecting line $L_{s'}$ is $\pi/2 - \pi/m$.



It is clear from the diagram that $v_I(\alpha_s) \in C$ for all possible v_I, that is $v_I(\alpha_s) = a\alpha_s + b\alpha_{s'}$ where $a, b \geq 0$ completing the proof of the first statement in the theorem.

In order to prove the second statement of the theorem we simply observe that it is the result of applying the first statement to ws instead of w. $\square$

by the above theorem we see that $\Phi = \Phi^+ \cup \Phi^-$. Hence we have in effect specified the set of simple roots to be $\Delta = \{\alpha_s \mid s \in S\}$.

Corollary 2.3.5. *The standard geometric representation $\sigma : W \to GL(V)$ is faithful.*

Proof. Suppose there exists $w \neq 1 \in W$ such that $w \in ker(\sigma)$, that is $w(\alpha_s) = \alpha_s$. As $w \neq 1$ we can find some $s \in S$ for which $\ell(ws) < \ell(w)$. Then by Theorem 2.4.3. $w(\alpha_s) < 0$ contradicting $w(\alpha_s) = \alpha_s > 0$. $\square$

As a result of this corollary we see that we can identify W with $\langle \sigma_s \mid s \in S\rangle \leq \mathrm{GL}(V)$.

2.4 The Geometric Interpretation of the Length Function

Continuing to follow [5], we can now describe a geometric interpretation of the length function and consequently obtain a description of how W permutes Φ. First recall the definition of $n(w) := |\Pi \cap w^{-1}(-\Pi)|$.

Proposition 2.4.1. (i) *Let $s \in S$. Then s permutes the roots in $\Pi \setminus \{\alpha_s\}$ and sends α_s to its negative.*

(ii) *Let $w \in W$. Then $\ell(w) = n(w)$.*

Proof. We begin by proving (i). By definition s sends α_s to its negative.

Now let $\alpha \in \Pi \setminus \{\alpha_s\}$. Then, as α cannot be a multiple of α_s we may write

$$\alpha = \sum_{t \in S \setminus s} c_t \alpha_t,$$

where all $c_t \geq 0$ and at least one $c_t \neq 0$. When we apply s to α it affects the sum only by adding some non-negative multiple of α_s. This can only result in a non-negative sum of positive roots so gives some $\alpha' \in \Pi$ that is not equal to α_s. Hence we have $s(\Pi \setminus \{\alpha_s\}) \subseteq \Pi \setminus \{\alpha_s\}$. By applying s to both sides of the inclusion we see that the reverse holds and so $s(\Pi \setminus \{\alpha_s\}) = \Pi \setminus \{\alpha_s\}$.

Now we will prove (ii). Let $w \in W$. We proceed by induction on $\ell(w)$. The case where $\ell(w) = 0$ is trivial and the case where $\ell(w) = 1$ is precisely (i). Now suppose $\ell(w) > 1$. We consider two cases. If $\ell(ws) = \ell(w) + 1$ then by Theorem 2.3.4. we have $w(\alpha_s) > 0$. Then by (i) $\Pi(ws) = s(\Pi(w)) \cup \{\alpha_s\}$ and $s(\Pi(w)) \cap \{\alpha_s\} = \emptyset$. Thus $n(ws) = n(w) + 1$ and so by induction $\ell(w) = n(w)$. Similarly for $\ell(ws) = \ell(w) - 1$ Theorem 2.3.4. gives $w(\alpha_s) < 0$. Then we obtain $\Pi(ws) = s(\Pi(w) \setminus \{\alpha_s\})$ where $\alpha_s \in \Pi(w)$, implying $n(ws) = n(w) - 1$ and so by induction we are done.

□

2.5 Reflections

Still continuing to follow [5] we now consider the reflections in W. Let $\alpha = w(\alpha_s)$ where $w \in W$ and $s \in S$. We will now consider the action of wsw^{-1} on V,

$$\begin{aligned} wsw^{-1}(\lambda) &= w\{w^{-1}(\lambda) - 2B(w^{-1}(\lambda), \alpha_s)\alpha_s\} \\ &= \lambda - 2B(w^{-1}(\lambda), \alpha_s)w(\alpha_s) \\ &= \lambda - 2B(\lambda, w(\alpha_s))w(\alpha_s) \\ &= \lambda - 2B(\lambda, \alpha)\alpha. \end{aligned}$$

Clearly wsw^{-1} does not depend on the choice of w or s, only upon α, and so we write $wsw^{-1} = s_\alpha$. Note that $s_\alpha(\alpha) = -\alpha$ and each s_α fixes pointwise the hyperplane orthogonal to it and so each acts on V as a reflection. We denote the set of all such reflections as follows,

$$T := \bigcup_{w \in W} wSw^{-1}.$$

Hence for each root $\alpha \in \Phi$ we have specified an associated reflection $s_\alpha \in \mathrm{GL}(V)$.

Now suppose $s_\alpha = s_\beta$, where $\alpha, \beta \in \Phi$ then $s_\alpha(\beta) = \beta - 2B(\beta,\alpha)\alpha$, that is, $\beta = B(\beta,\alpha)\alpha$. Both α and β are unit vectors so we conclude $\alpha = \beta$. Hence we see that the correspondence between roots and reflections is one-to-one enabling us to move freely between the two.

Lemma 2.5.1. *Let $\alpha, \beta \in \Phi$ and $\beta = w(\alpha)$ for some $w \in W$, then $ws_\alpha w^{-1} = s_\beta$.*

Proof. The bilinear form B is W invariant and so the lemma follows immediately from the formula for s_β. □

Proposition 2.5.2. *Let $w \in W$ and $\alpha \in \Pi$. Then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) > 0$.*

Proof. Let $\ell(ws_\alpha) > \ell(w)$. We will proceed by induction on $\ell(w)$. When $\ell(w) = 0$ the result is trivial. Now suppose $\ell(w) > 0$. There must exist some $s \in s$ such that $\ell(sw) < \ell(w)$. Then $\ell((sw)s_\alpha) = \ell(s(ws_\alpha) \geq \ell(ws_\alpha) - 1 => \ell(w) - 1 = \ell(sw)$ and so by induction $sw(\alpha) > 0$. Now suppose $w(\alpha) < 0$, then we must have $w(\alpha) = -\alpha_s$ by Proposition 2.4.1. Then $sw(\alpha) = \alpha_s$, and so by Lemma $(sw)s_\alpha(sw)^{-1} = s$. We can rearrange to get $ws_\alpha = sw$ but then $\ell(ws_\alpha) = \ell(sw)$ contradicting $\ell(ws_\alpha) > \ell(w) > \ell(sw)$. We conclude that $w(\alpha) > 0$.

If we apply the statement we have just proved to ws_α instead of w then $\ell(ws_\alpha) < \ell(w)$ implies $ws_\alpha(\alpha) < 0$, that is $w(\alpha) > 0$. Thus we conclude that $w(\alpha) > 0$ only if $\ell(ws_\alpha) > \ell(w)$. □

2.6 The Strong Exchange Condition

We are now able to state and prove the Strong Exchange Condition, a key combinatorial property of a Coxeter group W concerning the reduced expressions of its elements. This section is based off [5] and [6].

Theorem 2.6.1 (Strong Exchange Condition)**.** *Let $w = s_1 ... s_r$, where $s_i \in S$. If $\ell(wt) < \ell(w)$ for some $t \in T$, then we have $wt = s_1 ... \hat{s}_i ... s_k$, for some index i. Further, i is unique if the expression for w was reduced.*

Proof. Write $t = s_\alpha$ where $\alpha > 0$. Then as we have $\ell(wt) < \ell(w)$ Proposition 2.5.2. implies $w(\alpha) < 0$. Further as $\alpha > 0$ there must exist some index $i \leq r$ such that $s_{i+1} \dots s_r(\alpha) > 0$ but $s_i s_{i+1} \dots s_r(\alpha) < 0$. Now by Proposition 2.4.1. there is only one positive root that is sent to its negative by s_i, that is α_{s_i}. Then we have $s_{i+1} \dots s_r(\alpha) =$

α_{s_i}. Then by Lemma 2.5.1. we may obtain $(s_{i+1} \dots s_r)t(s_r \dots s_{i+1}) = s_i$, that is $wt = s_1 ... \hat{s}_i ... s_k$ as required.

Now suppose w is reduced, that is $\ell(w) = r$, and suppose there exist two distinct indices $i < j$ such that $wt = s_1 \dots \hat{s}_i \dots s_j \dots s_k = s_1 \dots s_i \dots \hat{s}_j \dots s_k$. We can cancel to obtain $s_{i+1} \dots s_j = s_i \dots s_{j-1}$. Then we may may write $w = s_1 ... \hat{s}_i ... \hat{s}_j ... s_r$. This contradicts $\ell(w) = r$ and so $i = j$. $\square$

If we were to require $t \in S$ for the above theorem we would obtain a weaker statement called the Exchange Condition.

Corollary 2.6.2 (Deletion Condition)**.** *Given $w = s_1 \dots s_r$, where each $s_i \in S$, if w is not reduced there exist indices $1 \leq i < j \leq r$ such that $w = s_1 ... \hat{s}_i ... \hat{s}_j ... s_r$. Here $\hat{s}$ denotes the omission of s.*

Proof. Choose the largest i such that $s_i s_{i+1} \dots s_r$ is not reduced. Then $\ell(s_i s_{i+1} \dots s_r) < \ell(s_{i+1} \dots s_r)$. Hence by the Exchange condition $s_i s_{i+1} \dots s_r = s_{i+1} \dots \hat{s}_j \dots s_r$. We multiply on the left by $s_1 \dots s_{i-1}$ to obtain $s_1 \dots s_r = s_1 ... \hat{s}_i ... \hat{s}_j ... s_r$. $\square$

Theorem 2.6.3. *The three following statements are equivalent.*

(i) *(W, S) is a Coxeter system.*

(ii) *The Exchange condition holds for (W, S).*

(iii) *The Deletion condition holds for (W, S).*

Proof. It is clear from the proof of the Exchange Condition that (i) $\implies$ (ii). Further the proof of the Deletion Condition uses only the Exchange Condition so (ii) $\implies$ (iii). Finally recall the proof of Theorem 1.4.1., that any group generated by a set of involutions is a Coxeter group, relied solely on the Deletion Condition. Hence we conclude the proof by noting that (iii) $\implies$ (i). $\square$

2.7 Parabolic Subgroups

This section is also based off [6] and to a lesser extent [5]. Recall the definition of a parabolic subgroup. For any Coxeter group with at least two generators we can construct a non trivial subgroup that is also a Coxeter group by taking a parabolic subgroup on some $I \subset S$.

Proposition 2.7.1. (i) *(W_I, I) is a Coxeter group.*

(ii) *$\ell_I(w) = \ell(w)$, for all $w \in W_I$.*

(iii) *$W_I \cap W_J = W_{I \cap J}$.*

Proof. First we will prove (ii). Let $w \in W_I$. Then we may write $w = s_1 s_2 \dots s_n$ where $s_i \in I$ for $1 \leq i \leq n$. By the Deletion Condition we may assume that this expression is reduced in W_I. As $I \subseteq S$ it follows that the expression is reduced in W so $\ell_I(w) = \ell(w)$.

Now we will prove (i). The Exchange Property holds in (W, S), as it is a Coxeter group. In addition, $\ell_I(w) = \ell(w)$ by (i) so $\ell_I(sw) = \ell(sw)$, for all $s \in I$ and so the Exchange Property holds for (W_I, I). Equivalently we have that (W_I, I) is a Coxeter group.

No $s \in I$ can be written as the product $s_1 \dots s_n$ where $s_i \in s$, similarly for $t \in J$ so it follows that I and J are minimal generating sets for W_I and W_J respectively. Then we see $W_{I \cap J} = \langle I \cap J \rangle = \langle I \rangle \cap \langle J \rangle = W_{I \cap J}$. □

Any finite parabolic subgroup, W_I, has a unique longest element that we shall denote as w_I. Further, this element is an involution. It is always true that $\ell(w_0) = \ell(w_0^{-1})$ and as w_0 is unique we must therefore have $w_0 = w_0^{-1}$.

Define $X_I := \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in I\}$ and $X_I' := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}$ where $I \subseteq S$.

Proposition 2.7.2. *Any element $w \in W$ is in X_I if and only if all reduced expressions for w begin with $s \notin I$.*

Proof. Clearly if all reduced expressions for w begin with $s \notin I_0$ then $\ell(w) < \ell(tw)$ for all $t \in I$. If $w \in X_I$, then $\ell(tw) = \ell(w) + 1$, for all $t \in I_0$. Hence in any reduced expression of $w = s_1 ... s_n$ we have $s_1 \neq t$. □

Following the same steps as before we can see that an element $w \in W$ is in X_I' if and only if all reduced expressions for w end with $s \notin I$.

Proposition 2.7.3. *Any $w \in W$ can be written uniquely as $w = uv$, where $u \in W_I$ and $v \in X_I$. The lengths satisfy $\ell(w) = \ell(u) + \ell(v)$. In addition, v is the minimal coset representative of $W_I w$. That is v is the unique smallest element in the coset $W_I w$.*

Proof. Let $w \in W$ then choose some $s_1 \in I$ such that $\ell(s_1 w) < \ell(w)$, if such an s_1 exists. Keep repeating the process of choosing $s_i \in I$ such that $\ell(s_i \dots s_1 w) < \ell(s_{i-1} \dots s_1 w)$ until no such s_i exists. We will write this as $v = s_k \dots s_1$. Clearly, $\ell(v) = k \leq \ell(w^{-1}) = \ell(w)$. By construction we have $\ell(sv) > \ell(v)$ for all $s \in I$, so $v \in X_I$. In addition we have $u = s_k \dots s_1 \in W_I$, hence $w = uv$, where $u \in W_I$ and $v \in X_I$. We also have by construction that $\ell(w) = \ell(u) + \ell(v)$.

Now suppose we have some $xy = uv$, where $x \in W_I$ and $y \in X_I$. Then $y = x^{-1}uv$. If $x \neq u$ then we may write $y = zv$, where zv is reduced, for some $z \in W_I$. But then we have a reduced expression for y that begins with $s \in I$. This is a contradiction so $x = u$. This implies $y = v$ and thus the factorisation is unique. □

For X_I' a "mirrored" version of the proposition holds.

Proposition 2.7.4. *Any $w \in W$ can be written uniquely as $w = uv$, where $u \in X_I'$ and $v \in W_I$. The lengths satisfy $\ell(w) = \ell(u) + \ell(v)$. In addition, v is the minimal coset representative of wW_I. That is, u is the unique smallest element in the coset wW_I.*

To prove this we follow the same steps as for the corresponding proposition for X_I.

2.8 Irreducible Components

The following is again based off [5].

Definition 2.8.1 (Irreducible)**.** A Coxeter system (W, S) is irreducible if the Coxeter graph Γ is connected and non empty. Otherwise we say that (W, S) is reducible.

Equivalently (W, S) is reducible if and only if there exist some non trivial $I, J \subseteq S$ such that $I \cup J = S$, $I \cap J = \emptyset$ and every $s \in I$ commutes with every $s' \in J$.

Proposition 2.8.2. *Let (W, S) be a Coxeter system with Coxeter graph Γ. Suppose Γ has connected components $\Gamma_1, \dots \Gamma_n$ each corresponding to the respective subset $S_1, \dots S_n$ of S. Then $W = W_{S_1} \times \cdots \times W_{S_n}$ and each Coxeter system (W, S_i) is irreducible.*

Proof. We will use induction on n. If $n = 0$ or 1 then the proposition is clearly true. Note that the product of all the parabolic subgroups W_{S_i} where $1 \leq i \leq n$ contains S and so must be equal to the whole of W. Suppose $n \geq 2$ then by induction $W_{S \setminus S_n}$ is the direct product of the W_{S_i}'s where $i \neq n$. The elements of S_n commute with the elements of S_i where $i \neq n$ and so the elements of $S \setminus S_n$ commute with the elements of S_n. Further by part (iii) of Theorem 2.7.1. $W_{S \setminus S_n} \cap W_{S_n} = W_{S \setminus S_n \cap S_n}$ and so the intersection is trivial. Hence we have a direct product of $W_{S \setminus S_n}$ and W_{S_n}. $\square$

It follows from this proposition that much of the theory concerning finite Coxeter groups can be reduced to the case when Γ is connected.

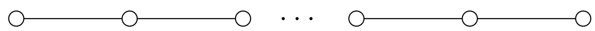
2.9 Classification

The irreducible finite Coxeter groups are classified by their graphs, the full details of this classification are given in [7]. On the following page we list all possible finite irreducible Coxeter graphs.

From Proposition 2.8.2. it follows that all finite Coxeter groups have Coxeter graphs consisting of only of disconnected copies of these graphs. Equivalently, all finite Coxeter groups correspond to arbitrary direct products of groups of the given types.

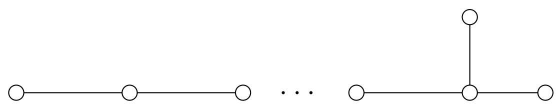
Recall the Coxeter graphs of S_{n+1} and D_{2m} seen in Example 2.2.7. From these we see that the symmetric group is of type A_n and D_{2m} is of type $I_2(m)$.

$A_n (n \geq 1)$

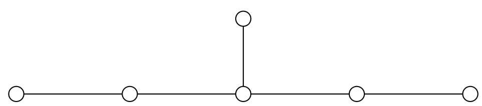


$B_n (n \geq 2)$

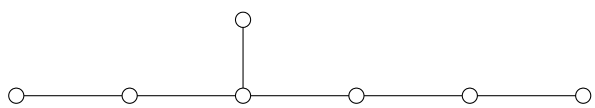
$D_n (n \geq 4)$



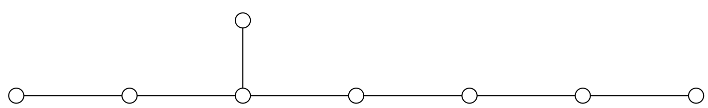
E_6



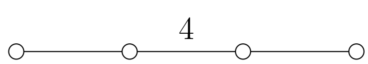
E_7



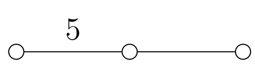
E_8



F_4

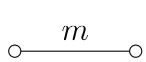


H_3



H_4

$I_2(m)$



Chapter 3

Automorphisms of Coxeter Groups

In this section we will consider automorphisms of Coxeter Groups. Before we do this we give some preliminary definitions.

Definition 3.0.1 (Automorphism)**.** Let G be a group and $\varphi : G \to G$ be an isomorphism. We call φ an automorphism. The set obtained by taking all the automorphisms of some group G is a group denoted as $\mathrm{Aut}(G)$.

Definition 3.0.2 (Inner Automorphism)**.** Let G be a group and $w \in G$. We say the automorphism $\mathrm{ad}(w) : G \to G$ such that $\mathrm{ad}(w)(x) = w^{-1}xw$ for all $x \in G$ is an inner. We denote the group of all inner automorphisms $\mathrm{Inn}(G)$.
Additionally, $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

Definition 3.0.3 (Outer Automorphism)**.** An outer automorphism is an automorphism that is not inner. Further $\mathrm{Out}(G) \cong \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Definition 3.0.4 (Internal Semidirect Product)**.** Let G be a group, $H \leq G$ and $N \lhd G$. Suppose $G = HN$ and $H \cap N = \{1\}$ then we say G is the internal semidirect product of H and N,

$$G = N \rtimes H.$$

3.1 Automorphisms of Coxeter Groups of Type A_n

We will now determine the automorphism group of all Coxeter groups of type A_n. This section will be based upon [8] and [9]. A Coxeter group of type A_n corresponds to the symmetric group S_{n+1}. Elements of the symmetric group have the same cycle structure if and only if they are in the same conjugacy class. Let w_k, where $1 \leq k \leq \frac{n+1}{2}$, be some element of S_{n+1} that can be written as the product of k disjoint 2-cycles. Then the set of all such elements form a conjugacy class, $cl(w_k)$. Each automorphism of S_{n+1} maps the set of conjugacy classes onto itself bijectively. Also note that automorphisms preserve order and all elements in S_{n+1} of order 2 can be written as the product of k disjoint transpositions.

Lemma 3.1.1. *Let α be an automorphism of S_{n+1} such that $\alpha(cl(w_1)) = cl(w_1)$. Then α is an inner automorphism.*

Proof. The set $\{(1,2),(1,3),\dots,(1,n)\}$ generates S_{n+1}. Then for all $a \neq 1$ we have $\alpha(1,a) = (1_a, a')$ where $1_a \neq a'$. Now consider $\alpha((1,a)(1,b)) = (1_a,a')(1_b,b')$. As $(1,a)(1,b) = (1,b,a)$ has order 3 $(1_a,a')(1_b,b')$ must also. That is, either $1_b \in \{1_a, a'\}$ or $b' \in \{1_a, a'\}$. We can assume without loss of generality that $1_b \in \{1_a, a'\}$.

Suppose we have $1_c = 1_a$ for some c and $1_d = a'$ for some $d \neq c$. Then we have

$$\alpha((1,a)(1,c)) = \alpha((1,c,a)) = (1_a,a')(1_c = 1_a, c') = (1_a, c', a')$$

and

$$\alpha((1,a)(1,d)) = \alpha((1,d,a)) = (1_a,a')(1_d = a', d') = (1_a, a', d').$$

Then

$$\alpha((1,c,a)(1,d,a)) = (1_a,c',a')(1_a,a',d') = (a',d',c')$$

which has order 3 but $(1,c,a)(1,d,a) = (1,d)(a,c)$ has order 2 so we have a contradiction. Thus either $1_b = 1_a$ for all b or $1_b = a'$ for all b.

Now assume $1_a = 1_b$ for all b. Then $\alpha(1,b) = (1_a, b')$ for all b as α is injective it follows that $b' \neq c'$ for all $b \neq c$. Now we can take x to be the permutation that takes 1 to 1_a and all other b to b'. Then $x(1,b)x^{-1}$ takes $1_a \mapsto 1 \mapsto b \mapsto b'$ and $b' \mapsto b \mapsto 1 \mapsto 1_a$. Every other $c \neq b, 1$ is fixed. So $x(1,b)x^{-1} = \alpha((1,b))$. Recall that $(1,b)$ forms a generating set for A_n, it follows $x(a,b)x^{-1} = \alpha((a,b))$. That is, α is inner. We can use the same argument to show that α is inner when $a' = 1_b$ for all b. ∎

So an outer automorphism of S_{n+1} can only exist if there is some $cl(w_k)$ that $cl(w_1)$ is interchanged with. Clearly this is only possible when $|cl(w_k)| = |cl(w_1)|$. We can determine the size of this conjugacy class by considering the number of possible combinations of k disjoint transpositions. First we choose a pair of distinct elements from the set of size $n+1$ permuted by S_{n+1}, then 2 more distinct elements from the remaining $n-1$ and so on until we choose the kth pair from the remaining $n+1-2(k-1) = n-2k+3$ possibilities. As the transpositions are disjoint their order doesn't matter. That gives the following formula,

$$\begin{aligned}
|cl(w_k)| &= \binom{n+1}{2} \times \binom{n-1}{2} \times \cdots \times \binom{n-2k+3}{2} \times \frac{1}{k!} \\
&= \frac{(n+1)n}{2} \times \frac{(n-1)(n-2)}{2} \times \cdots \times \frac{(n-2k+3)(n-2k+1)}{2} \times \frac{1}{k!} \\
&= \frac{(n+1)!}{(n-2k+1)!} \times \frac{1}{2^k \times k!} \\
&= \binom{n+1}{2k} \times (2k-1) \times \cdots \times 3 \times 1.
\end{aligned}$$

In particular the conjugacy class of transpositions has size

$$|cl(w_1)| = \binom{n+1}{2}.$$

Now suppose $2 \leq k < \frac{n-1}{2}$ we have

$$\binom{n+1}{2} < \binom{n+1}{2k}$$

and so

$$\binom{n+1}{2} < \binom{n+1}{2k} \times (2k-1) \times \cdots \times 3 \times 1.$$

That is, $|cl(w_1)| < |cl(w_k)|$.

If $k = (n-1)/2$, then

$$\binom{n+1}{2} = \binom{n+1}{n-1} < \binom{n+1}{n-1} \times (n-2) \times \cdots \times 3 \times 1$$

That is, $|cl(w_1)| < |cl(w_k)|$.

If n is even and $k = n/2$, then

$$|cl(w_k)| = \binom{n+1}{n} \times (n-1) \times \cdots \times 3 \times 1 = (n+1)(n-1) \times \cdots \times 3 \times 1.$$

When $n = 2$ then $k = 1$ so we are only interested in $n > 2$. In this case we have $n/2 < (n-1) \times \cdots \times 3 \times 1$, so we can conclude that

$$|cl(w_1)| = \frac{(n+1)n}{2} < (n+1)(n-1) \times \cdots \times 3 \times 1 = |cl(w_k)|$$

If n is odd and $k = (n+1)/2$,

$$|cl(w_k)| = \binom{n+1}{n+1} \times n \times \cdots \times 3 \times 1 = n \times \cdots \times 3 \times 1.$$

When $n = 3$ we can easily see that

$$|cl(w_1)| = \frac{(n+1)n}{2} = 6 < n \times \cdots \times 3 \times 1 = 3 = |cl(w_k)|.$$

Now suppose $n > 5$. Then we see that

$$n + 12 < (n-2) \times \cdots \times 3 \times 1$$

and so

$$|cl(w_1)| = \frac{(n+1)n}{2} < n \times \cdots \times 3 \times 1 = 3 = |cl(w_k)|.$$

Finally if $n = 5$ we have

$$\frac{n+1}{2} = 3 = (n-2) \times \cdots \times 3 \times 1$$

and so we conclude that $|cl(w_1)| = |cl(w_k)|$ if and only if $n = 5$ and $k = 3$.

We have now seen that all automorphisms of S_{n+1} must be inner when $n \neq 5$. That is, every automorphism acts on S_{n+1} by conjugation of some element $w \in S_{n+1}$. The inner automorphism $\mathrm{ad}(w) = \mathrm{id}$ if and only if $w \in Z(A_n)$. Now recall that the center of S_{n+1} is trivial. It follows that

$$\mathrm{Aut}(S_{n+1}) \cong S_{n+1}.$$

Now suppose $n = 5$. Define the map $\xi : S_6 \to S_6$ as

$$(1,2) \to (13)(24)(56)$$

$$(2,3) \to (16)(25)(34)$$

$$(3,4) \to (14)(23)(56)$$

$$(4,5) \to (16)(24)(35)$$

$$(5,6) \to (12)(34)(56)$$

This is an automorphism. Further ξ maps $cl(w_1)$ to $cl(w_3)$ so it is an outer automorphism. By checking that $\xi^2(s) = s$ for all $s \in S$ we can show that ξ is involutive.

Suppose we have some outer automorphism $\alpha \neq \xi$, by Lemma 3.1.1. and our counting argument α must interchange $cl(w_1)$ and $cl(w_3)$. Then $\alpha^{-1}\xi(cl(w_1)) = cl(w_1)$ and so $\alpha^{-1}\xi \in \mathrm{Inn}(S_6)$ again by Lemma 3.1.1. It follows that $\mathrm{Aut}(S_6)/\mathrm{Inn}(S_6)$ has order 2 and so is isomorphic to $\mathbb{Z}_2 \cong \langle\xi\rangle$. Now as $\mathrm{Inn}(s_6) \cong S_6 \lhd \mathrm{Aut}(S_6)$ we see that

$$\mathrm{Aut}(S_6) \cong S_6 \rtimes \langle\xi\rangle.$$

3.2 Involutive Automorphisms of Coxeter Groups of Type A_n

We will now specify all the involutive automorphisms of Coxeter group of type A_n. These precisely the involutive automorphisms of S_{n+1}.

2.a $n \neq 5$

First if $n \neq 5$ then the involutive automorphisms of S_{n+1} are inner and so correspond to the action by conjugation of some element $x \in S_{n+1}$. Such an automorphism $\mathrm{ad}(x)$ is involutive if and only if $\mathrm{ad}(x) \circ \mathrm{ad}(x)(w) = w$ for all $w \in S_{n+1}$. That is, $x^2wx^{-2} = w$, this is the case precisely when x has order 2 as $Z(S_{n+1})$ is trivial when $n \neq 5$. That is, the set of involutive automorphisms of S_{n+1} is

$$\{\mathrm{ad}(x) \mid x \in S_{n+1} \text{ and } x^2 = 1\}.$$

2.b $n = 5$

Now if $n = 5$, clearly the set of inner automorphisms

$$\{\mathrm{ad}(x) \mid x \in S_6 \text{ and } x^2 = 1\}$$

is a subset of the involutive automorphisms of S_6. Further the outer automorphism ξ is involutive. Hence the only remaining question is whether there are any involutive automorphisms that can be expressed as a product of ξ and some inner automorphism α. That is, does there exist some $x \in S_6$ such that $(\mathrm{ad}(x) \circ \xi)^2 = \mathrm{id}$. We can find a simplified version of this question.

Lemma 3.2.1. *The automorphism ad$(x) \circ \xi$ is involutive if and only if $x\xi(x) = 1$.*

Proof. Let $w \in S_6$ and observe

$$\xi \circ \mathrm{ad}(x) \circ \xi(w) = \xi(x\xi(w)x^{-1}) = \xi(x)w\xi(x)^{-1}.$$

Then by substitution we see

$$\mathrm{ad}(x) \circ \xi \circ \mathrm{ad}(x) \circ \xi = \mathrm{ad}(x)\mathrm{ad}(\xi(x)) = \mathrm{ad}(x\xi(x)).$$

Then

$$\mathrm{ad}((x) \circ \xi)^2 = \mathrm{ad}(x\xi(x)) = \mathrm{id}$$

and this holds if and only if

$$x\xi(x) = 1.$$

□

By using magma we can find an answer to this question, either directly or from simplified form. The code is included in Appendix A.

Theorem 3.2.2. *There are 36 automorphisms ad$(x) \in$ Inn(S_6) such that $(ad(x) \circ \xi)^2 =$ id. These automorphisms correspond precisely to elements $x \in S_6$ listed below.*

Id(G)	(1, 5, 6, 3, 4)	(1, 2, 4, 6, 3)
(1, 2)(3, 4)	(1, 2, 6, 5, 3)	(2, 4, 5, 3, 6)
(1, 5)(2, 4)	(1, 6, 4, 5, 3)	(1, 3, 4, 5)(2, 6)
(2, 3)(4, 5)	(2, 3, 4, 6, 5)	(1, 5, 3, 2)(4, 6)
(1, 3)(2, 5)	(1, 6, 3, 2, 5)	(1, 4, 5, 2)(3, 6)
(1, 4)(3, 5)	(1, 4, 3, 2, 6)	(1, 5, 4, 3)(2, 6)
(1, 2, 4, 5, 6)	(2, 5, 6, 4, 3)	(1, 2, 3, 5)(4, 6)
(1, 6, 5, 4, 2)	(1, 6, 2, 3, 4)	(1, 6)(2, 5, 3, 4)
(1, 3, 5, 6, 2)	(2, 6, 3, 5, 4)	(1, 2, 5, 4)(3, 6)
(1, 5, 2, 6, 4)	(1, 5, 2, 3, 6)	(1, 4, 2, 3)(5, 6)
(1, 4, 6, 2, 5)	(1, 4, 3, 6, 5)	(1, 3, 2, 4)(5, 6)
(1, 3, 5, 4, 6)	(1, 3, 6, 4, 2)	(1, 6)(2, 4, 3, 5)

We have now completely specified the involutive automorphisms for Coxeter groups of type A_n.

3.3 Pairs

In [8] it is shown that a large class of finite Coxeter groups only have automorphisms that are inner by graph, where an automorphism is said to be inner by graph if it is in the set generated by the inner and the graph automorphisms.

Suppose we have a Coxeter system (W, S). Let $I_0 \subset S$ and γ be an automorphism of W such that $\gamma(S) = S$, that is, γ is an automorphism of the Coxeter graph Γ. In the next section of this chapter we will follow [2] to show that we can construct a Coxeter group using pairs $\{I_0, \gamma\}$ such that

(i) $\gamma(I_0) = I_0$;

(ii) $w_{I_0 \cup J} \in N_W(W_{I_0})$ where J is a Γ-orbit in W.

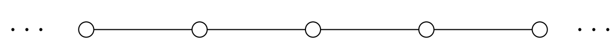
In the final section of this chapter we will construct an involutive automorphism that is inner by graph and is dependent on a pair $\{I_0, \gamma\}$ as above with one additional condition

(iii) $\gamma|_{W_{I_0}} = \mathrm{ad}(w_{I_0})$.

As such, we will now determine all such pairs, given conditions (i) and (ii), when we have a Coxeter group of type A_n. In the following we will not allow the choice $I_0 = S$ although this choice would fulfill our criteria. There are only two involutive automorphisms that preserve the whole generating set S, the identity automorphism and the graph automorphism induced by $\mathrm{ad}(w_0)$. We will consider each of these choices for γ in turn.

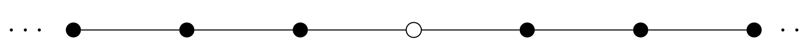
First we take $\gamma = \mathrm{id}$. The γ-orbits are $J_i = \{s_i\}$. Note that as $\gamma = \mathrm{id}$ condition (i) is satisfied for all possible choices of I_0.

Suppose we choose $I_0 = \emptyset$. We can represent this choice on the Coxeter diagram for type A_n using white vertices to denote $s \notin I_0$ and black vertices to denote $s \in I_0$,



Then $w_{I_0 \cup J} = w_J \in N_W(\emptyset)$ and so condition (ii) is satisfied. Hence we may have $I_0 = \emptyset$.

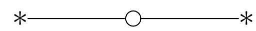
Suppose n is odd and consider $I_0 = S \setminus \{s_{(n+1)/2}\}$,



This gives $w_{I_0 \cup J} = w_0$. Then $w_0 W_{I_0} w_0 = W_{I_0}$ and so condition (ii) is satisfied. So we may choose $I_0 = S \setminus \{s_{(n+1)/2}\}$ when n is odd.

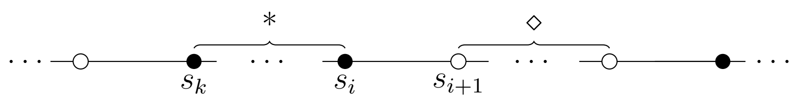
Suppose we choose I_0 to be,

where $*$ denotes a block of k consecutive black vertices for some fixed $k \in \mathbb{Z}$. Any choice of orbit $J = \{s_i\}$ will commute with all but two blocks of black vertices $*_1$ and $*_2$. Further each block of black vertices $*_i$ commutes with every other block $*_j$ where $i \neq j$. Thus we may write $w_{I_0 \cup J} = w_{K_0 \cup J} w_{L_0}$, where K_0 consists of the vertices in $*_1$ and $*_2$ and $L_0 = I_0 \setminus K_0$. Now if we show that $w_{K_0 \cup J}$ and w_{L_0} it follows that $w_{I_0 \cup J}$. As $w_{L_0} \in W_{I_0}$ it satisfies condition (ii). Now consider the subgraph consisting of vertices $K_0 \cup J$,



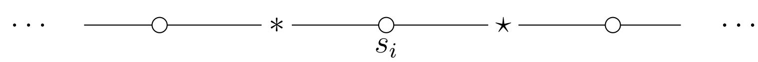
From the above diagram we see that this reduces to the case $n = 2k+1$, that is n is odd and $I_0 = S \setminus \{s_{(n+1)/2}\} = S \setminus \{s_{k+1}\}$. We have seen above that this satisfies condition (ii).

We will now rule out all other possibilities. Suppose there exists some block $\diamond$ consisting of $l \geq 2$ consecutive white vertices. Let s_i be the black vertex immediately to the left of $\diamond$ lying in some block $*$ of $k \geq 1$ black vertices. The block $*$ commutes with all other $s \in I_0$, further s_{i+1} commutes with all $s \in I_0$ such that $s \neq s_i$. Thus we can consider the subgraph consisting of the vertices in $*$ and s_{i+1}.



Now upon considering the orbit $J = \{s_{i+1}\}$ we see that $w_{I_0 \cup J}$ corresponds to the longest element of this subgraph. Then $\mathrm{ad}(w_{I_0 \cup J})$ interchanges s_{i+1} and the leftmost vertex $s_k \in *$. Thus condition (ii) is not satisfied. We conclude that when $I_0 \neq \emptyset$ at least one of the two adjacent vertices s_i, s_{i+1} must be in I_0.

We have just seen that when $I_0 \neq \emptyset$ every white vertex lies between two blocks, $*$ and $\star$, of black vertices. Now suppose $*$ contains $k \geq 1$ black vertices and $\star$ contains $l \geq 1$ black vertices where $k \neq l$.

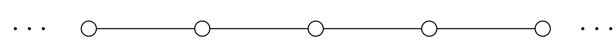


Consider the orbit $J_i = \{s_i\}$ consisting of the white vertex between $*$ and $\star$. As all $s \in *$ commute with all $t \in I_0 \setminus *$ and respectively for $\star$ we can consider the subgraph consisting only of the vertices in $J_i \cup * \cup \star$. Then in this subgraph $w_{I_0 \cup J_i}$ is equivalent to the longest element. We have seen that in this case condition (ii) does not hold. Hence we cannot have blocks $*$ and $\star$ of unequal size on either side of a white vertex. Further it follows from this argument we that both $s_1, s_n \in I_0$ whenever I_0 is not empty.

We have now exhausted all possible choices of I_0 when $\gamma = \mathrm{id}$.

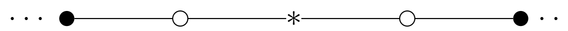
Now we will determine all permissible subsets $I_0 \subseteq S$ given $\gamma = \mathrm{ad}(w_0)$. If n is even then the γ-orbits are $J_i = \{s_i, s_{n-i+1}\}$, if n is odd then the γ-orbits are $J_i = \{s_i, s_{n-i+1}\}$ for $i \neq (n+1)/2$ and $J_{(n+1)/2} = \{s_{(n+1)/2}\}$. Condition (i) is satisfied if and only if I_0 is the union of of some arbitrary set Λ of γ-orbits, that is $I_0 = \bigcup_{i \in \Lambda} J_i$. Hence we only need to consider possible I_0 of this form.

First we consider $I_0 = \emptyset$,

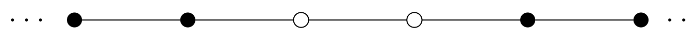


Following the same reasoning as for $\gamma = \mathrm{id}$ we can see that this choice is permissible.

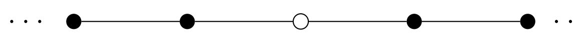
Now suppose we have $I_0 = S \setminus J_i$. This can be represented graphically in one of three ways. Either,



Or if n is even I_0 could be,

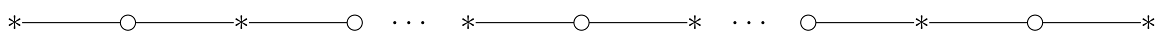


Finally if n is odd we may have,



In all three cases we have $w_{I_0 \cup J_i} = w_0$. Hence $w_{I_0 \cup J_i} W_{I_0} w_0 W_{I_0} w_0 = \gamma(W_{I_0}) = W_{I_0}$. Thus condition (ii) is satisfied and so we may choose $I_0 = S \setminus J_i$ for any γ-orbit J_i.

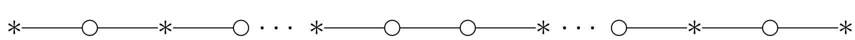
Let n be odd and consider I_0 as given below,



Each block $*_j$ commutes with every other block $*_k$ when $j \neq k$. Consider the orbit $J_{(n+1)/2}$ and label the blocks of black vertices on either side $*_1$ and $*_2$. The orbit $J_{(n+1)/2}$ commutes with all vertices $s \in I_0 \setminus (*_1 \cup *_2)$ so we can consider only the subgraph consisting of the vertices in $*_1 \cup *_2 \cup J_{(n+1)/2}$. Then conjugation by $w_{I_0 \cup J_{(n+1)/2}}$ fixes $s_{(n+1)/2}$ and interchanges each vertex in $*_1$ with one in $*_2$. Thus condition (ii) is satisfied. Now consider the arbitrary orbit $J_i = \{s_i, s_{n-i+1}\}$ orbit J_i. By commutativity we can similarly reduce this case to the disconnected subgraph,

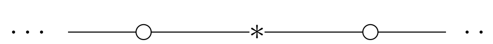
As the graph is disconnected we see that it reduces to the case we considered above and so condition (ii) is satisfied.

If n is even then we may have I_0,



This follows via the same argument as for the previous case where n was odd.

Suppose we take I_0 to be a block $*$ of k vertices with a block of l white vertices on either side. That is,



Any orbit $J \notin I_0$ will either commute with all elements of I_0 or it will be next to leftmost and rightmost vertices in I_0. In either case condition (ii) is satisfied.

Now we will rule out all other remaining possibilities. Suppose we have a block, $\diamond$, of white vertices from two or more different orbits, that is . Further suppose $\diamond$ lies between two blocks of black vertices, $*_L$ and $*_R$. Then if we take J_L to be the orbit of the leftmost vertex $s_L \in \diamond$ we obtain $w_{I_0 \cup J_L} t w_{I_0} = s_L \notin I_0$ for some $t \in *_L$ unless J_L consists of the two vertices on either side of $*_L$. This implies that the center of the graph is to the left of $\diamond$. In this case we can take J_R to be the orbit of the rightmost vertex $s_R \in \diamond$ and obtain $w_{I_0 \cup J_R} t' w_{I_0} = s_R \notin I_0$ for some $t' \in *_R$. Here J_R cannot consist of the two vertices on either side of $*_R$ as the center of the graph is to the left of s_R. We have shown that we cannot have a block of $k \geq 2$ white vertices between two blocks of black vertices unless n is even in which case we may take $J_{n/2}$ to be such a block with $k = 2$.

Suppose $J_i = \{s_i, s_{n-i+1}\} \subseteq I_0$ and there is some s_j where $i < j < n+i-1$ that is not in I_0. Then if $J_{i-1} \nsubseteq I_0$ and either $J_{i-1} = J_1$ or $J_{i-2} \nsubseteq I_0$ we have $w_{I_0 \cup J_{i-1}} s w_{I_0 \cup J_{i-1}} = s_{i-1}$ where s is in the same block of black vertices as s_i. Thus condition (ii) is not satisfied.

We have now exhausted all possible choices of I_0 when $\gamma = \mathrm{ad}(w_0)$ and so determined all possible pairs $\{\gamma, I_0\}$.

Proposition 3.3.1. *The pairs $\{\gamma, I_0\}$ such that $\gamma(I_0) = I_0$ and $w_{I_0 \cup J} \in N_W(W_{I_0})$ for all $J \in \mathcal{J}$ are precisely those given in Table 3.1.*

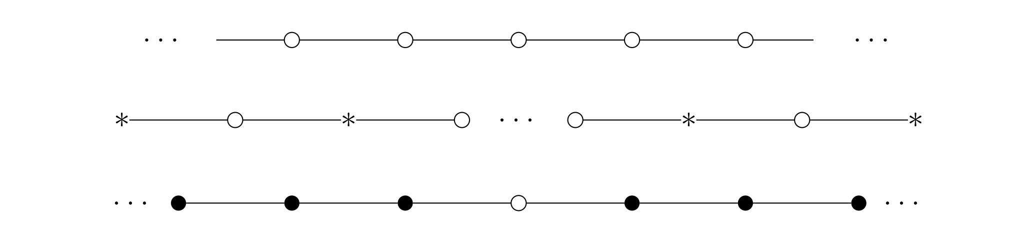
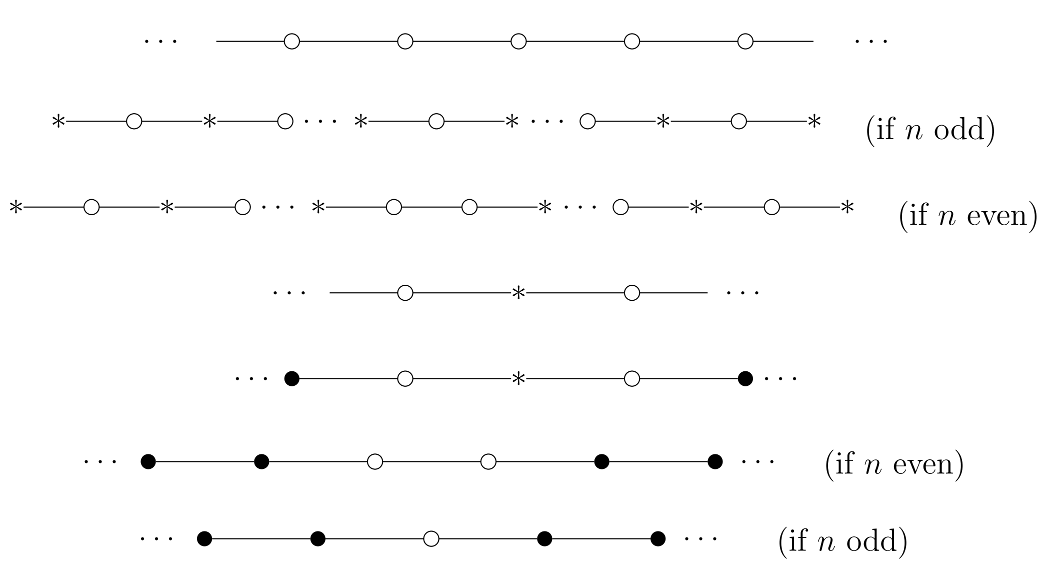
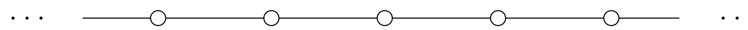
$\gamma = \mathrm{id}$	
$\gamma = \mathrm{ad}(w_0)$	

Table 3.1: Pairs $\{\gamma, I_0\}$ in A_n subject to conditions (i) and (ii).

We will now determine the subset of pairs $\{\gamma, I_0\}$ obtained when we add condition (iii)

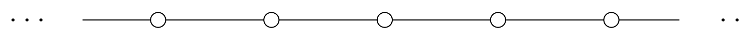
It is clear that the pairs we can choose will be a subset of all the pairs $\{\gamma, I_0\}$.

Suppose $\gamma = \mathrm{id}$ then $\gamma|_{W_{I_0}} = \mathrm{ad}(w_{I_0})$ is equivalent to $s = w_{I_0} s w_{I_0}$ for all $s \in I_0$. That is, the restriction holds if and only if every element $s \in I_0$ commutes with every other element in I_0. If we look at Table 3.1. we see that this condition is satisfied in only two cases. We can take $I_0 = \emptyset$,

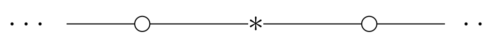


or we can take I_0 to be

Now consider $\gamma = \mathrm{ad}(w_0)$. Clearly we may have $I_0 = \emptyset$,



Now suppose we have a block of black vertices, B with either a white vertex or nothing on either side of it. Then all elements of B commute with all elements of $I_0 \setminus B$, meaning $w_{I_0} B w_{I_0} \in B$. If B is not central then it can not contain the γ-orbits of the vertices it contains and so $w_0 B w_0 \notin B$. If B is central then $w_0 B w_0 = w_{I_0} B w_{I_0}$. Hence the only other possible choice $I_0 = A_{n-2i}$ for all $1 < i < \lfloor (n+1)/2 \rfloor$, where I_0 is centered,



Proposition 3.3.2. *The pairs $\{\gamma, I_0\}$ such that $\gamma(I_0) = I_0$, $w_{I_0 \cup J} \in N_W(W_{I_0})$ for all $J \in \mathcal{J}$ and $\gamma|_{W_{I_0}} = ad(w_{I_0})$ are precisely those given in Table 3.2.*

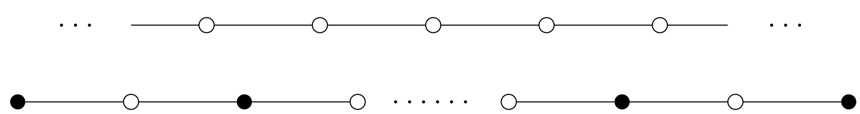
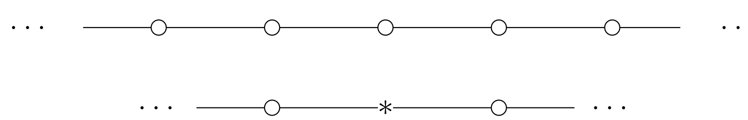
$\gamma = \mathrm{id}$	
$\gamma = \mathrm{ad}(w_0)$	

Table 3.2: Pairs $\{\gamma, I_0\}$ in A_n subject to conditions (i), (ii) and (iii).

3.4 Fixed Point Subgroups

In this section we take I_0 to be fixed a subset of S such that $\gamma(I_0) = I_0$ and $w_{I_0 \cup J} \in N_W(W_{I_0})$ for all $\gamma \in \Gamma$. We define the Γ-fixed points of W to be $W^\Gamma := \{w \in W \mid \gamma(w) = w \text{ for all } \gamma \in \Gamma\}$ and $\mathcal{W} := \{w \in W \mid w \in X_{I_0} \text{ and } wW_{I_0} = W_{I_0} w\}$.

Define $s_J := w_{I_0 \cup J} w_{I_0} = w_{I_0} w_{I_0 \cup J}$ and $\mathcal{J}$ to be the set of all Γ-orbits in $(W,\ S \setminus I_0)$. In [2] it is shown that $(W^\Gamma, \{w_J \mid J \in \mathcal{J}\})$ is a Coxeter system. Further the paper states that a similar method can be used to show that the Γ fixed points of $\mathcal{W}$ form a Coxeter group with generators of the form s_J. In fact this is the general case, $(W^\Gamma, \{w_J \mid J \in \mathcal{J}\})$ is the special case where $I_0 = \emptyset$. We will now work through this proof in full detail following the same structure as in [2].

Proposition 3.4.1. *$s_J \in \mathcal{W}^\Gamma$ and $s_J^2 = 1$ for all $J \in \mathcal{J}$.*

Proof. By definition we have $s_J = w_{I_0 \cup J} w_{I_0} = w_{I_0} w_{I_0 \cup J}$. Thus

$$s_J^2 = w_{I_0 \cup J} w_{I_0} w_{I_0 \cup J} w_{I_0} = w_{I_0 \cup J} w_{I_0} w_{I_0} w_{I_0 \cup J} = 1.$$

Now we will show $s_J \in \mathcal{W}^\Gamma$. This is equivalent to showing the following three conditions hold,

(i) $\gamma(s_J) = s_J$,

(ii) $s_J W_{I_0} = W_{I_0} s_J$,

(iii) $s_J \in X_{I_0}$.

For condition (i), note that for all $\gamma \in \Gamma$ we have $\gamma(I_0) = I_0$ by definition and, as J is an orbit, $\gamma(J) = J$. Then $\gamma(w_{I_0}) = w_{I_0}$ and $\gamma(w_{I_0 \cup J}) = w_{I_0 \cup J}$ as γ preserves length. Thus $\gamma(s_J) = s_J$.

Now let $w \in W_{I_0}$. Consider $s_J w s_J = w_{I_0} w_{I_0 \cup J} w w_{I_0 \cup J} w_{I_0}$. As $w_{I_0 \cup J} \in N_W(W_{I_0})$ we have $w_{I_0 \cup J} w w_{I_0 \cup J} = w' \in W_{I_0}$. Then clearly $w_{I_0} w' w_{I_0} \in W_{I_0}$ and so condition (ii) is satisfied.

Finally consider

$$\ell(s_J) = \ell(w_{I_0 \cup J} w_{I_0}) = \ell(w_{I_0 \cup J}) - \ell(w_{I_0}).$$

As $\ell(t w_{I_0}) < \ell(w_{I_0})$ for all $t \in I_0$ we have

$$\ell(t s_J) = \ell(t w_{I_0} w_{I_0 \cup J}) > \ell(w_{I_0 \cup J}) - \ell(w_{I_0}) = \ell(s_J).$$

That is, $s_J \in X_{I_0}$. □

Lemma 3.4.2. *Any $w \in \mathcal{W}^\Gamma$ may be written as $w = s_{J_1} \dots s_{J_r}$ such that $\ell(w) = \ell(s_{J_1}) + \dots + \ell(s_{J_r})$. That is, $\mathcal{W}^\Gamma = \langle s_J \mid J \in \mathcal{J} \rangle$.*

Proof. Let $w \in \mathcal{W}^\Gamma$. We use induction on $\ell(w)$. First suppose $\ell(w) = 0$. Then $w = 1$ and so there is nothing to prove.

Now suppose $\ell(w) > 0$. By definition $w \in \mathcal{W}$ and so we may let $u = w_{I_0} w = w w_{I_0}$ where $\ell(u) = \ell(w_{I_0}) + \ell(w)$. As $\ell(w) > 0$ and $w \in X_{I_0}$ by Proposition 2.7.2. there must exist some $s \in S \setminus I_0$ such that $\ell(su) < \ell(u)$. Choose the Γ-orbit of s to be J_1. Then $\ell(\gamma(su)) = \ell(\gamma(s)u) = \ell(tu) < \ell(u)$, for all $t \in J_1$. Similarly, as $\gamma(I_0) = I_0$, we have $\ell(s'u) < \ell(u)$ for all $s' \in I_0$. Write $u = vx$ where $v \in W_{I_0 \cup J_1}$, $x \in X_{I_0 \cup J_1}$ and $\ell(u) = \ell(v) + \ell(x)$. For all $t' \in I_0 \cup J_1$ we have $t'v \in W_{I_0 \cup J_1}$ and so $\ell(t'vx) = \ell(t'v) + \ell(x)$. This implies $\ell(t'v) < \ell(v)$ for all $t' \in I_0 \cup J_1$, that is, $v = w_{I_0 \cup J_1}$. Then $w = w_{I_0} w_{I_0 \cup J_1} x = s_{J_1} x$ and $\ell(w) = \ell(u) - \ell(w_{I_0}) = \ell(w_{I_0 \cup J_1}) - \ell(w_{I_0}) + \ell(x) = \ell(s_{J_1}) + \ell(x)$.

We can proceed by induction on x. □

From here on we will use $x \bullet y$ to denote xy with $\ell(xy) = \ell(x) + \ell(y)$.

Lemma 3.4.3. *Let $w \in \mathcal{W}^\Gamma$. If we have two expressions*

$$w = s_{J_1} \bullet \cdots \bullet s_{J_r} = s_{I_1} \bullet \cdots \bullet s_{I_p}$$

where $J_i, I_i \in \mathcal{J}$, then $r = p$.

Proof. The proof uses induction on $\ell(w)$. If $\ell(w) = 0$ then $w = 1$ and so there is nothing to prove.

Assume $\ell(w) > 0$, it follows that $r > 0$ and $p > 0$. If we have $I_1 = J_1$ then

$$w' = s_{I_1} w = s_{J_2} \bullet \cdots \bullet s_{J_r} = s_{I_2} \bullet \cdots \bullet s_{I_p}.$$

By induction we may assume $r - 1 = p - 1$ and so we have $r = p$.

Now suppose $I_1 \neq J_1$. Let $K := I_0 \cup I_1 \cup J_1$ and $X_K = \{x \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in K\}$ be the set of distinguished coset representatives of W_K in W. Let $u = w_{I_0} w = w_{I_0 \cup J_1} \bullet s_{J_2} \bullet \cdots \bullet s_{J_r} = w_{I_0 \cup I_1} \bullet s_{I_2} \bullet \cdots \bullet s_{I_p}$. We may write $u = v \bullet x$, where $v \in W_K$ and $x \in X_K$. Then we have $\ell(su) < \ell(u)$ for all $s \in K$. Hence we must have $\ell(sv) < \ell(v)$ for all $s \in K$, that is, $v = w_K$ and so W_K is finite. Now consider $\mathcal{W}_K^\Gamma := \{w \in W_K \mid wW_{I_0} = W_{I_0} w \text{ and } w \in X_{I_0}\}$.

We will show that $w_{I_0} w_K \in \mathcal{W}_K^\Gamma$. First $\gamma(w_{I_0} w_K) = \gamma(w_{I_0})\gamma(w_K) = w_{I_0} w_K$. Observe that $\ell(w_{I_0} w_K) = \ell(w_K) - \ell(w_{I_0})$ and $\ell(tw_{I_0} w_K) \geq \ell(w_K) - \ell(tw_{I_0}) = \ell(w_K) - \ell(w_{I_0}) + 1$ for all $t \in I_0$. It follows that $\ell(tw_{I_0} w_K) > \ell(w_{I_0} w_K)$ for all $t \in I_0$, that is, $w_{I_0} w_K \in X_{I_0}$. Now let $s \in I_0$ and v' be a reduced expression for $w_{I_0} w_K = w_K w_{I_0}$. Consider $v' s v'$, clearly $sv' \in W_K$ and $\ell(v' s v') = \ell(s)$. Then $v' s v'$ is not reduced so we can apply the Deletion condition $\ell(v')$ times to obtain a reduced expression. As sv' and $v's$ are reduced expressions applying the Deletion Condition forces us to delete one factor from the left v' and one factor from the right v'. Upon doing so $\ell(v')$ times we are left with only s, that is $v' s v' = s$. Hence $w_{I_0} w_K W_{I_0} = W_{I_0} w_{I_0} w_K$ and so $w_{I_0} w_K \in \mathcal{W}_K^\Gamma$.

It follows that $x \in \mathcal{W}^\Gamma$. So by Lemma 3.4.2. we may write $x = s_{L_1} \bullet \cdots \bullet s_{L_q}$ where $L_i \in \mathcal{J}$. Now we will consider

$$s_{J_1} \bullet \cdots \bullet s_{J_r} = w = w_{I_0} w_K \bullet x. \tag{1}$$

By Lemma 3.4.2. $\mathcal{W}_K^\Gamma = \langle s_{I_1}, s_{J_1} \rangle$. As s_{I_1} and s_{J_1} are involutions we have that $\mathcal{W}_K^\Gamma$ is the dihedral group of order $2m$ for some $m \in \mathbb{Z}$. As $w_{I_0} w_K$ is an involution and $w_{I_0} w_K \in \mathcal{W}_K^\Gamma$ we must have,

$$w_{I_0} w_K = \underbrace{s_{I_1} \bullet s_{J_1} \bullet s_{I_1} \bullet \ldots}_{m} = \underbrace{s_{J_1} \bullet s_{I_1} \bullet s_{J_1} \bullet \ldots}_{m}.$$

Now by substituting into (1) we obtain

$$s_{J_1} \bullet \cdots \bullet s_{J_r} = w = \underbrace{s_{J_1} \bullet s_{I_1} \bullet s_{J_1} \bullet \ldots}_{m} \bullet s_{L_1} \bullet \cdots \bullet s_{L_q}.$$

Canceling s_{J_1} gives

$$s_{J_2} \bullet \cdots \bullet s_{J_r} = \underbrace{s_{I_1} \bullet s_{J_1} \bullet \ldots}_{m-1} \bullet s_{L_1} \bullet \cdots \bullet s_{L_q}.$$

We conclude $r-1 = (m-1)+q$. Applying the same strategy to

$$s_{I_1} \bullet \cdots \bullet s_{I_p} = w = \underbrace{s_{I_1} \bullet s_{J_1} \bullet s_{I_1} \bullet \ldots}_{m} \bullet s_{L_1} \bullet \cdots \bullet s_{L_q}.$$

gives $p-1 = (m-1)+q$ and so we have $r = p$. □

Define $\lambda : \mathcal{W}^\Gamma \to \mathbb{N}_0$ to be the length function on $\mathcal{W}^\Gamma$ with respect to the generators $\{s_J \mid J \in \mathcal{J}\}$.

Lemma 3.4.4. *For any $w \in \mathcal{W}^\Gamma$ that can be written as $s_{J_1} \ldots s_{J_p}$ and $\lambda(w) = p$ we have $w = s_{J_1} \bullet \cdots \bullet s_{J_p}$.*

Proof. We will use induction on p. If $p = 0$ or $p = 1$ then clearly the lemma is true. Now consider w such that $p \geq 2$. We may set $w' := s_{J_2} \ldots s_{J_p}$ where $\lambda(w') = p-1$ so by assumption $w' = s_{J_2} \bullet \cdots \bullet s_{J_p} \in \mathcal{W}^\Gamma$.

Now suppose $\ell(sw') > \ell(w')$ for all $s \in J_1$. Then $w' \in X_{J_1}$, further $w' \in X_{I_0}$ by definition of $\mathcal{W}^\Gamma$ and so $\ell(s_{J_1}w') = \ell(s_{J_1}) + \ell(w')$. Hence we have shown that $w = s_{J_1} \bullet \cdots \bullet s_{J_p}$.

Otherwise, $\ell(sw') < \ell(w')$ for some $s \in J_1$. Then we have that $\ell(\gamma(sw')) = \ell(tw')$ for all $t \in J_1$ and so as in the proof of Lemma 3.4.2. we may write $w' = s_{L_1} \bullet \cdots \bullet s_{L_q}$ where $L_i \in \mathcal{J}$ and $L_1 = J_1$. By Lemma 3.4.3. we have that $p-1 = q$. Thus $w = s_{J_1}w' = s_{L_2} \bullet \cdots \bullet s_{L_q}$ and $\ell(w) < \ell(w')$. This is impossible so we are done. □

Corollary 3.4.5. *Let $w, w' \in \mathcal{W}^\Gamma$. We have $\ell(ww') = \ell(w) + \ell(w')$ if and only if $\lambda(ww') = \lambda(w) + \lambda(w')$.*

Proof. Let $\lambda(w) = p$ and $\lambda(w') = q$. Then by Lemma 3.4.2. we have $w = s_{J_1} \bullet \cdots \bullet s_{J_p}$ and $w' = s_{I_1} \bullet \cdots \bullet s_{I_q}$ where $J_i, I_i \in \mathcal{J}$.

Suppose $\ell(ww') = \ell(w) + \ell(w')$. Then $ww' = s_{J_1} \bullet \cdots \bullet s_{J_p} \bullet s_{I_1} \bullet \cdots \bullet s_{I_q}$. Let $r := \lambda(ww') \leq p+q$. Again by Lemma 3.4.2. we have $ww' = s_{L_1} \bullet \cdots \bullet s_{L_r}$ where $L_i \in \mathcal{J}$. Then Lemma 3.4.3. implies $r = q+p$.

Now if $\lambda(ww') = \lambda(w) + \lambda(w') = p+q$. Then we have $ww' = s_{J_1} \ldots s_{J_p} s_{I_1} \ldots s_{I_q}$ and by Lemma 3.4.4. $ww' = s_{J_1} \bullet \cdots \bullet s_{J_p} \bullet s_{I_1} \bullet \cdots \bullet s_{I_q}$. Hence we see that $\ell(ww') = \ell(w) + \ell(w')$. □

Theorem 3.4.6. *Suppose we have $w \in \mathcal{W}^\Gamma$ and $J \in \mathcal{J}$. Let $\lambda(w) = p$ and $w = s_{J_1} \ldots s_{J_p}$ where $J_i \in \mathcal{J}$. If $\lambda(s_J w) \leq \lambda(w)$, then there exists some $i \in \{1, \ldots p\}$ such that $s_J w = s_{J_1} \ldots s_{J_{i-1}} s_{J_{i+1}} \ldots s_{J_p}$. That is, the Exchange Condition is satisfied for the pair $(\mathcal{W}^\Gamma,\ \{s_J \mid J \in \mathcal{J}\})$.*

Proof. Suppose we have $\ell(sw) > \ell(w)$ for all $s \in I_0 \cup J$, then $w \in X_{I_0 \cup J}$ and so $\ell(s_J w) = \ell(s_J) + \ell(w)$. Corollary 3.4.5. implies that $\lambda(s_J w) = \lambda(s_J) + \lambda(w) > \lambda(w)$. This contradicts our assumption that $\lambda(s_J w) \leq \lambda(w)$. Hence there must exist some $s \in I_0 \cup J$ such that $\ell(sw) < \ell(w)$. In fact as $w \in X_{I_0}$ we can further see that $s \in J$. In addition, by Lemma 5.4.4 we have that $w = s_{J_1} \bullet \cdots \bullet s_{J_p}$. If we take reduced expressions for all s_{J_i} then we have a reduced expression for w. The Exchange Condition holds for $(W,\ S)$ so there exists some index $i \in \{1 \dots p\}$ such that

$$sw = s_{J_1} \dots s_{J_{i-1}} x s_{J_{i+1}} \dots s_{J_p}$$

where $x \in W_{J_i \cup I}$ is equivalent to the reduced expression of s_{J_i} less one factor. Now take $z := s_{J_1} \dots s_{J_{i-1}}$, then we have

$$z^{-1} s z = x s_{J_i}$$

Note that $z \in \mathcal{W}^\Gamma$ so we have $z^{-1}\gamma(s)z = \gamma(x s_{J_i}) \in W_{I_0 \cup J_i}$ for all $\gamma \in \Gamma$. In addition, $zs = sz$ for all $s \in I_0$ so $z^{-1}sz = z^{-1}zs = s \in W_{I_0 \cup J_i}$. We conclude $u = z^{-1} s_J z \in W_{I_0 \cup J_i}$. Further as $u = s_{J_1} \dots s_{J_i} \dots s_{J_1}$ we have $u \in \mathcal{W}^\Gamma$. Thus, $u \in \mathcal{W}^\Gamma_{I_0 \cup J_i}$. By Lemma 3.4.2 $u \in \mathcal{W}^\Gamma_{I_0 \cup J_i} = \langle s_{J_i} \rangle = \{1, s_{J_i}\}$. Then we must have $u = s_{J_i}$ giving $s_J = z s_{J_i} z^{-1}$ and so $s_J w = s_{J_1} \dots s_{J_{i-1}} s_{J_{i+1}} \dots s_{J_p}$ as required. □

Corollary 3.4.7. *$(\mathcal{W}^\Gamma,\ \{s_J \mid J \in \mathcal{J}\})$ is a Coxeter system.*

Proof. By Theorem 3.4.6 the Exchange Condition is satisfied for the pair $(\mathcal{W}^\Gamma,\ \{s_J \mid J \in \mathcal{J}\})$. This is equivalent to the pair being a Coxeter System. □

3.5 An Involutive Automorphism

Define $\Theta := \mathrm{ad}(w_{I_0}) \circ \Gamma$, where Γ is defined as in the previous section but with the additional restriction

$$\gamma|_{W_{I_0}} = \mathrm{ad}(w_{I_0}).$$

This map is the product of two automorphisms and so must itself be an automorphism. Upon observing that γ preserves length and $\gamma(I_0) = I_0$ by definition we see that w_{I_0} is γ invariant for all $\gamma \in \Gamma$. It follows that

$$\begin{aligned}\Theta^2 &= \mathrm{ad}(w_{I_0}) \circ \Gamma \circ \mathrm{ad}(w_{I_0}) \circ \Gamma \\ &= \Gamma \circ \mathrm{ad}(w_{I_0}) \circ \mathrm{ad}(w_{I_0}) \circ \Gamma \\ &= \mathrm{id}.\end{aligned}$$

Hence we have that Θ is an involutive automorphism. We have constructed Θ to be inner by graph in the sense of [8]. The choice of Γ is natural as it is the group of graph automorphisms. The choice of $\mathrm{ad}(w_0)$ is less obvious but still not unreasonable as the longest element is always involutive.

Proposition 3.5.1. $W_{I_0} \subseteq W^\Theta$.

Proof. Let $w \in W_{I_0}$, then $w = s_1 \dots s_n$ where $s_i \in I_0$. We defined Θ to be the product of two group homomorphisms so it follows that Θ must also be a group homomorphism. That is,

$$\Theta(w) = \Theta(s_1) \dots \Theta(s_n).$$

As a result to complete the proof it is sufficient to prove that all $s \in I_0$ are contained in W^Θ. That is, $\Theta(s) = s$ for all $s \in I_0$. Consider

$$\Theta(s) = w_{I_0}\gamma(s)w_{I_0} = w_{I_0}w_{I_0}sw_{I_0}w_{I_0} = s.$$

□

Proposition 3.5.2. $\mathcal{W}^\Gamma \subseteq W^\Theta$.

Proof. Let $w \in \mathcal{W}^\Gamma$. First we will note that $\gamma(w) = w$ by definition so we have

$$\Theta(w) = w_{I_0}\gamma(w)w_{I_0} = w_{I_0}ww_{I_0}.$$

Now by definition we have $wW_{I_0} = W_{I_0}w$ and so $ww_{I_0} = uw$, where $u \in W_{I_0}$. Also by definition we have $w \in X_{I_0}$ meaning $\ell(ww_{I_0}) = \ell(w) + \ell(w_{I_0}) = \ell(uw)$. Then as $\ell(uw) \leq \ell(u) + \ell(w)$ we see that $u = w_{I_0}$. That is, w_{I_0} and w commute. Thus

$$\Theta(w) = w.$$

□

We know $\mathcal{W}^\Gamma$ is a Coxeter group and that W_{I_0} is a parabolic subgroup of W. That is, they are both groups. Hence having shown that each of them is contained by W^Θ is sufficient to conclude that $\mathcal{W}^\Gamma \leq W^\Theta$ and $W_{I_0} \leq W^\Theta$.

Proposition 3.5.3. $W^\Theta = W_{I_0}\mathcal{W}^\Gamma$.

Proof. To prove this we will show containment in both directions.

First we will show that $W_{I_0}\mathcal{W}^\Gamma \subseteq W^\Theta$. Let $w \in W_{I_0}\mathcal{W}^\Gamma$. We may write $w = uv$, where $u \in W_{I_0}$ and $v \in \mathcal{W}^\Gamma$. Upon noting that W_{I_0} and $\mathcal{W}^\Gamma$ are both subgroups of W^Θ we see that $w = uv \in W^\Theta$ as it is the product of two elements in W^Θ.

Now we will show that $W^\Theta \subseteq W_{I_0}\mathcal{W}^\Gamma$. Let $w \in W^\Theta$. Then we may write $w = uv$, where $u \in W_{I_0}$ and $v \in X_{I_0}$. We have $w_{I_0}\gamma(v)w_{I_0} = v$ and $v \in X_{I_0}$ hence $\gamma(v) \in X'_{I_0}$. Further γ preserves length so for all $s \in I_0$ we have

$$\ell(vs) = \ell(\gamma(vs)) = \ell(\gamma(v)t),$$

where $t \in I_0$. Thus $v \in X'_{I_0}$.

Now consider

$$w_{I_0}vw_{I_0} = \gamma(v).$$

Clearly the LHS is not reduced so by the Deletion Condition we may delete two factors from the LHS to obtain an equivalent expression. Both of these factors may not lie in

v as v is reduced. If one of the factors lies in v and the other in w_{I_0} on either side it contradicts the fact that vw_{I_0} and $w_{I_0}v$ are both reduced. Hence one factor must come from each of the w_{I_0}'s. It follows that any reduced expression of $\gamma(v)$ must in fact be v. Thus

$$\gamma(v) = v.$$

Further if we let x be some reduced expression for $w_{I_0}s$, where $s \in I_0$ we obtain

$$w_{I_0}vx = vs$$

and following the argument above an equivalent expression is obtained by deleting one factor from w_{I_0} and one factor from x. If we do this the maximum amount of times our equation becomes

$$tv = vs,$$

where $t \in I_0$. That is for all $s \in I_0$, there exists some $t \in I_0$ such that $svt = v$. Hence we have

$$vW_{I_0} = W_{I_0}v.$$

It follows that

$$v \in \mathcal{W}^\Gamma.$$

Hence we have shown that

$$w = uv \in W_{I_0}\mathcal{W}^\Gamma.$$

□

Corollary 3.5.4. $W^\Theta \cong W_{I_0} \rtimes \mathcal{W}^\Gamma$

Proof. We have seen that $W^\Theta = W_{I_0}\mathcal{W}^\Gamma$. Let $w \in \mathcal{W}^\Gamma$, then $w \in X_{I_0}$ and so any reduced expression of w must not begin with $s \in I_0$. This means $w \notin W_{I_0}$ unless $w = 1$ and so we conclude $W_{I_0} \cap \mathcal{W}^\Gamma = \{1\}$. Now let $w' \in W^\Theta$ can be written as $w = uv$ where $u \in W_{I_0}$ and $v \in \mathcal{W}^\Gamma$. Clearly u commutes with W_{I_0} and by the definition of $\mathcal{W}^\Gamma$ we also have $vW_{I_0} = W_{I_0}v$. We conclude that $wW_{I_0} = W_{I_0}w$ and so $W_{I_0} \lhd W^\Theta$. It follows that $W^\Theta \cong W_{I_0} \rtimes \mathcal{W}^\Gamma$. □

We have shown that $W^\Theta \cong W_{I_0} \rtimes \mathcal{W}^\Gamma$. That is, W^Θ is isomorphic to the semidirect product of two Coxeter groups. It is also true that $\mathcal{W}^\Gamma \lhd W^\Theta$ and $W^\Theta = \mathcal{W}^\Gamma W_{I_0}$, so we could further say $W^\Theta \cong \mathcal{W}^\Gamma \rtimes W_{I_0}$.

Conclusion

We have given a brief overview of the theory of Coxeter groups and it's motivation from finite reflection groups. In the first two chapters of the report the approach we took was primarily geometric. Alternatively we could have approached the subject from a more combinatorial point of view. Such an approach would have been less natural in terms of motivation, however, as evidenced in Chapter 3, the combinatorial properties of Coxeter groups are very useful. One of the most fundamental combinatorial properties of Coxeter groups that has been ommitted in this report is the Bruhat order.

We have proved that $(\mathcal{W}^\Gamma, \{s_J \mid J \in \mathcal{J})$ is a Coxeter system and we have constructed a group of involutive automorphisms $\Theta := \mathrm{ad}(w_{I_0}) \circ \Gamma$. Further we have shown, $W^\Theta \cong W_{I_0} \rtimes \mathcal{W}^\Gamma$ where W_{I_0} and $(\mathcal{W}^\Gamma$ are Coxeter groups with canonically defined Coxeter generators, I_0 and $\{s_J \mid J \in \mathcal{J}\}$ respectively.

The motivation for doing this was to make progress towards a classification of the involutive automorphisms of Coxeter groups. As such two further questions arise.

Let $\Theta := \mathrm{ad}(w_{I_0}) \circ \Gamma$, where $\gamma(I_0) = I_0$, $w_{I_0 \cup J} \in N_W(W_{I_0})$ and $\gamma|_{W_{I_0}} = \mathrm{ad}(w_{I_0})$ for all $\gamma \in \Gamma$. Further define $\Theta' := \mathrm{ad}(w_{I_0'}) \circ \Gamma'$, where $\{\gamma', I_0'\}$ is a pair such that $\gamma'(I_0') = I_0'$ and $w_{I_0' \cup J} \in N_W(W_{I_0'})$ for all $\gamma' \in \Gamma'$. Now the questions are as follows.

(i) Fix Θ'. Does there exist some $\beta \in \mathrm{Aut}(W)$ such that $\beta\Theta'\beta^{-1} = \Theta$, for some Θ as above?

(ii) Given (i) is true, then is every involutive automorphism of W conjugate to some $\Theta := \mathrm{ad}(w_{I_0'}) \circ \Gamma'$?

If the answer to both of these questions were positive then we would obtain a classification for the involutive automorphisms of finite Coxeter groups.

Appendix A

Magma Code

Given in this appendix is the code used to determine the involutive automorphisms of S_6.

Define S_6: G := Sym(6);

Define the generators:

```
g1 := G! (1,2);
g2 := G! (2,3);
g3 := G! (3,4);
g4 := G! (4,5);
g5 := G! (5,6);
```

Define the automorphism ξ:

```
xi:= hom< G -> G | g1 ->
g1*g2*g1*g2*g3*g2*g5, g2 ->
g1*g2*g3*g4*g5*g4*g3
*g2*g1*g2*g3*g4*g3*g2*g3,
g3 -> g1*g2*g3*g2*g1*g2*g5, g4->g1*g2*g3*g4*g5*g4*g3
*g2*g1*g2*g3*g2*g3*g4*g3, g5 -> g1*g3*g5>;
```

The following code counts the number of $w \in S_6$ for which $(\mathrm{ad}(x) \circ \xi)^2(w) = w$ for each $x \in S_6$ and returns x precisely when this number is $|S_6| = 720$. That is, when $\mathrm{ad}(x) \circ \xi$ is involutive.

```
for i in G do
n:=0;
for j in G do
l1:=i*j/i;
l2:=xi(l1);
l3:=i*l2/i;
if xi(l3) eq j then
n:=n+1;
if n eq 720 then
print i;
end if;
end if;
end for;
end for;
```

Alternatively the code given below checks directly whether $x \in S_6$ satisfies $x\xi(x) = 1$

```
for i in G do
for j in G do
if alpha(i)*j eq Id(G) then
print i;
end if;
end for;
```

Both sets of code give identical output.

Appendix B

Bibliography

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