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MMath Mathematics Report

Spherical Trigonometry Through the Ages

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Contents

Abstract

The art of applying trigonometry to triangles constructed on the surface of the sphere is the branch of mathematics known as spherical trigonometry. It allows new theorems relating the sides and angles of spherical triangles to be derived.

The history and development of spherical trigonometry throughout the ages is discussed. The geometry on the surface of the sphere is introduced to aid the understanding of fundamental proofs of spherical results. Questions such as "how big can we make a spherical triangle?" are answered so the strange geometry the sphere yields can be visualised. How spherical trigonometry was simplified from ancient to medieval times is investigated. Crucial applications to astronomy and cartography are identified, such as determining the coordinates of the Sun, finding the qibla, and calculating distances on the surface of the Earth. The use of hyper-complex numbers, known as quaternions, and their properties are introduced to determine spherical results, as well as briefly looking at the application of quaternions to 3-dimensional rotations.

Chapter 1

Introduction

The face of mathematics has been continuously growing and changing since the dawn of humanity. Things that we think of as simple, such as the concept of a number, have in fact been under development since the Old Stone Age. Cave men did not have the same advanced set of symbols and words to represent numbers as we do, nevertheless they still needed to quantify how much food to gather. As we have become more knowledgeable of the mathematics around us, we have accepted what those before us have discovered, and developed it further. It is therefore not surprising that as time has gone by, the topics we are interested in and what is taught in schools have changed dramatically.

It is widely believed that for many centuries people around the world thought that the Earth was flat. The belief of this myth is in itself a myth. As far back as the famous Greek philosopher Pythagoras in the 6th century BC, and Aristotle c. 340 BC, (Heath, 1932) scholars had evidence and observations to confirm that the world is spherical. Although there were

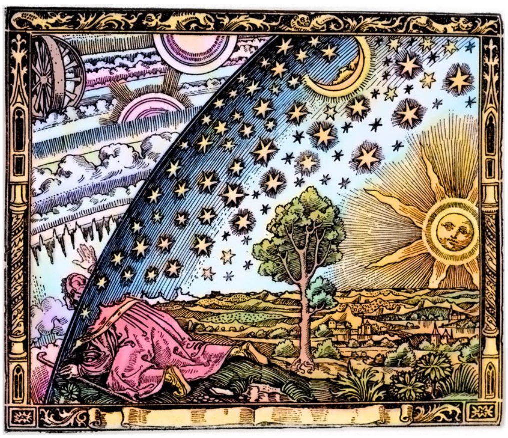


Figure 1.1: Image from Tøndering. Known as "The Woodcut", this engraving first appeared in Flammarion (1888) and depicts someone from the middle ages peering out to the Heavens at the edge of a flat Earth.

still people who thought the Earth was flat, the majority accepted this was not the case. It was therefore apparent from an early point in time, any observations across large enough distances that the curvature of the Earth would need to be taken into account would need a good understanding of the sphere. Since much of the world was still being discovered, maps were continuously being updated, so accuracy was crucial for setting sail on the open seas. Not only maps of the Earth, but maps of the skies above were needed for navigation.

Spherical trigonometry mainly came about as a tool to help ancient astronomers. Astronomy played a major part in everyday life, from time keeping to navigation, long before there were theories and equations to describe it. Since the earliest civilizations, when farming and trade were expanding, connections were made between the phases of the moon and vegetation. This caused people to monitor the moon by creating lunar calendars, but also increased interest in the bodies around us. By simply looking up at the sky, day or night, we see that the celestial bodies, such as the moon and stars or the sun, move in circular arcs across it. Initially, after realizing that the rising sun was the same each day, and that certain constellations appeared on a nightly basis, it was thought that the celestial sphere rotated around the Earth. Although we know this now not to be true, it was still clear that a method of calculating the positions of each object, as well as their trajectories, would involve extensive knowledge of the sphere.

The first major breakthrough for spherical trigonometry came in the first century AD, by Greek mathematician and astronomer Menelaus. Properties of the sphere had been known for thousands of years prior to this, but his groundbreaking results lead to great exploration of the Heavens and paved the way for future mathematicians to unearth further results. Astronomers relied on Menelaus' results for almost 1000 years. Medieval Middle Eastern scholars were able to simplify these results, and what's more discovered important formulas, including the spherical Law of Sines. These findings were not just vital to the development of spherical mathematics, religious followers depended on their applications too.

Considering we are looking into the history of spherical trigonometry, the question of why and when the trigonometric functions themselves came about is one expected to arise. The non-existence of calculators meant trigonometric tables needed to be produced, and this was done to various degrees of accuracy. Such tables were originally constructed to give the length of chords.

Hipparchus of Nicaea, who lived around 150 BC, invented the chord function whilst trying to model the Sun's orbit around the Earth. Following ancient belief that the universe rotated around the Earth, he had placed the

Earth at the centre of the orbit, however came to realize that this meant the number of days between the spring and autumn equinoxes, and vice versa, should be equal, but he knew this not to be true. Knowing how far to move the Earth from the centre of the orbit in order to reproduce the observed inter-equinoctial times required a new function, and so trigonometry was born. The sine would later replace the chord in India around the year 500, and soon after the rest of the world (Pingree, 1976).

The calculation of the table of sines was the main requirement in calculating a trigonometric table, since all other functions can be found from this. This is an arduous task when done by hand, although it is one that can be reduced by acknowledging the periodicity of the sine function and that it is an even function between 0 and $180\,^{\circ}$.

The years leading up to the 20th century saw an explosion in mathematical developments. The invention of the logarithm as well as the derivation of the spherical law of cosines left users of spherical trigonometry with a complete set of tools. The subject remained popular in schools and universities until the beginning of the 1900s. World War I and II arose, where the applications of spherical trigonometry to navigation for the military ensured that it remained a very relevant and important topic up until the 1950s. Since then it has rarely been explored, being replaced by its not so distant younger sister: trigonometry in the plane. As we will find in the forthcoming chapters, many of the planar results we know today are actually special cases of their spherical counterparts. The advances of computers and software leave little reason for people to apply spherical trigonometry to solve real life problems, like finding the rising times of the sun, for example: the answers are just a click away. It is therefore surprising, but also insightful, to notice that computer graphics, animation and GPS all use results from spherical trigonometry in their programs today. As we will see, the spherical world yields some interesting and amazing results.

Note on historical sources: Statements regarding the history of mathematics and spherical trigonometry are not individually referenced. The following sources were use for this material:

- Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry, Van Brummelen (2013);
- A Concise History of Mathematics, Struik (1967);
- The MacTutor website, O'Connor and Robertson.

Chapter 2

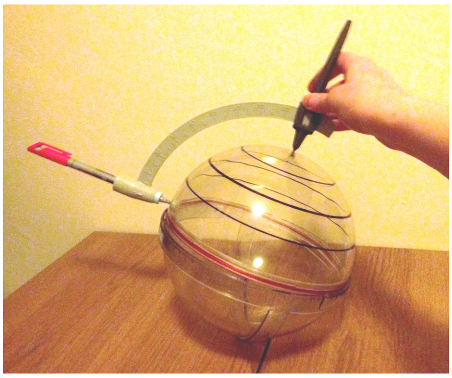
Geometry on the Sphere Part I

The method of writing numbers as we do today, where *place* is an indication of a number's value, was brought into fruition c. 2100 BC in Mesopotamia. This dates back to the Sumerian period and replaced a system that mirrors Roman numerals, where a string of symbols must be added to give the value. Interestingly, a sexagesimal system was imposed, i.e. each position represents 1, 60, 3600, and 60^{-1}, 60^{-2}, which we still use in part today for time. Furthermore, we use this system for measuring angles, with a whole circle representing $360\,^{\circ}$, each degree 60 minutes, and each minute 60 seconds. This is transformed into coordinates on the surface of the Earth. The reason for the use of base 60 may be unclear, but there is no doubt that its relationship with the sphere and spherical applications is something of beauty. The proofs in this chapter follow the methodology of Van Brummelen (2013).

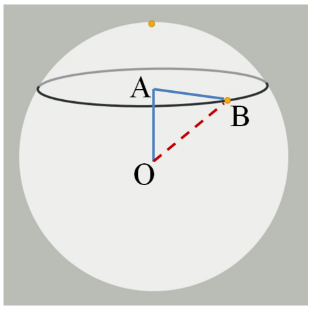
2.1 Cross-sections and Measurements

Before we begin working on the surface of the sphere, we need to know the properties lines and shapes have when drawn upon it. This raises some important questions: How can we draw a straight line on the sphere? What is the simplest shape that can be drawn onto the surface of the sphere? Both of these questions can be answered with an example. It is recommended to have a spherical object to hand in the forthcoming chapters to help visualise the spherical geometry. A useful object to have is a Lénart sphere, in use in Figure 2.1a. It comes with a spherical compass and spherical protractor, both pictured, to draw triangles and shapes on the sphere.

If we take a plane and intersect the sphere at any point, we see that the



(a) A Lénart sphere.



(b) Proof of Theorem 1.

Figure 2.1: Cross-sections of the sphere.

cross-section is always a circle, Figure 2.1a. This can easily be proved with an application of Pythagoras' Theorem using Figure 2.1b.

Proof. Take any cross-section of the sphere, centre O, by a plane. Drop a perpendicular from a point A on the plane to O and extend lines from both points to a point B on the edge of the cross-section. This implies that the angle OAB is a right-angle, so applying Pythagoras' Theorem we can write

$$AO^2 + AB^2 = OB^2.$$

We also know that as its endpoints are fixed, AO is constant. Since OB is the radius of the sphere, this too is constant, which implies that AB is constant. As B is an arbitrary point on the cross-section, this is true for any point on the cross-section. We have therefore defined any cross-section to be a circle. □

Theorem 1. *Every cross-section of a sphere by a plane is a circle.*

This is also the simplest shape we can create on the sphere, as it consists of just one side. Depending on where the plane intersects the sphere, we get circles of different radii. Figure 2.1a shows circles with a radius less than that of our sphere as small circles (black), but the most important circle for us is a great circle (red).

Great circles are cross-sections of the sphere whose centres coincide with the centre of the sphere. They have the greatest circumference and therefore will take the role of straight lines upon the sphere. Clearly, each sphere has an infinite number of great circles and small circles that can be drawn upon it. Given that great circles are straight lines on the sphere, great circle arcs

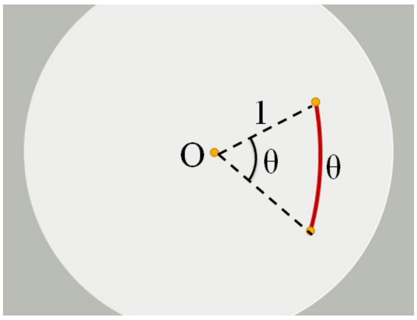


Figure 2.2: A great circle arc of length θ.

are straight line segments. We can therefore say that since the shortest distance between two points in the plane is a straight line, the shortest distance between two points on the sphere is a great circle arc.

Theorem 2. *Great circles represent straight lines on the surface of the sphere. Great circle arcs are line segments.*

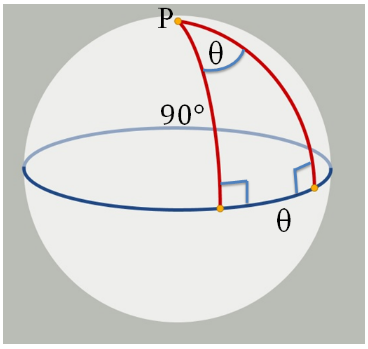
Now that we have the tools to construct shapes on the surface of the sphere, we need to make a note about their measurements. Not only the angles of a shape, which in our case will be triangles, but the lengths of the sides are measured in degrees.

Take a sphere of radius 1, Figure 2.2 (for simplicity, all spheres we consider will have unit radius). If we have an angle θ at the centre, O, we can project this onto the surface of the sphere to give an arc length of θ. In a circle, arc length $= r\theta$. A great circle arc is part of the cross-section of the sphere that is a great circle, so in Figure 2.2 the arc length is θ. This observation allows us to define a rule for converting arc lengths into angles.

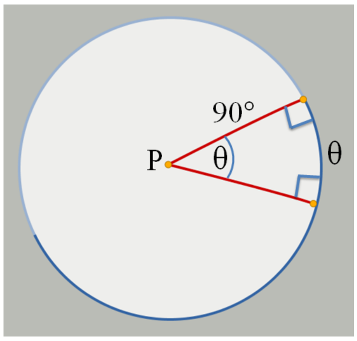
We start with a great circle and its pole, P, Figure 2.3a. A pole is the perpendicular axis of a great circle, so if we drop two line segments from P onto the great circle, they will each be of length $90\,^\circ$ and meet the great circle at right-angles. A bird's-eye view in Figure 2.3b, with P at the centre, gives a similar picture to Figure 2.2. If the angle between the two line segments at P is θ, the distance between these two lines on the corresponding great circle to P is θ. So we have our result.

Theorem 3. *An angle between two lines on the surface of the sphere can be converted into a length by moving* $90\,^\circ$ *away along both lines and joining the endpoints.*

A practical example of the properties of the sphere we have seen so far is the Earth. The equator and lines of longitude are great circles, and the lines of latitude are small circles. The north and south poles are the two poles to the equator, they are at a latitude of $\pm 90\,^\circ$, respectively. Therefore, the



(a) Line segments from the pole, P, to the corresponding great circle.



(b) A bird's-eye view of the pole.

Figure 2.3: Converting between angles and lengths.

distance between two lines of longitude on the equator is equal to the angle between them at the north (or south) pole.

2.2 Triangles on the Sphere

When first imagining a triangle, one might think of a shape with three straight sides, be it equilateral, isosceles or scalene, with the sum of the internal angles equal to $180\,°$. When we move onto the surface of the sphere, where the sides of a spherical triangle are great circle arcs, things get interesting. Can you imagine a triangle with an internal angle greater than $180\,°$? Or a side as long as the circumference of a great circle? We need some constraints to our spherical triangles; let us start with how big we can build them.

2.2.1 What is the Largest Possible Spherical Triangle?

Since the sides of a triangle can be as small as we like, we need only consider how large the perimeter of a spherical triangle can be.

Picture a triangle on the surface of the sphere, Figure 2.4, left. Gradually increase the internal angles at each vertex, which will in turn increase the side lengths so they remain great circle arcs; the triangle will begin to straighten out until we have a great circle, Figure 2.4, right. Increasing the angles beyond this point would begin to make the perimeter of the triangle smaller again. We have therefore reached an upper limit for the perimeter of a spherical triangle: $360\,°$. We will now look at the reasons behind this limit

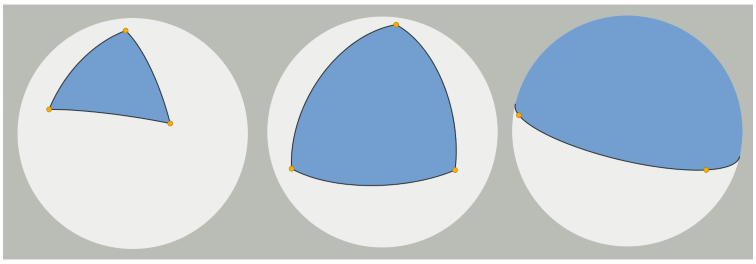
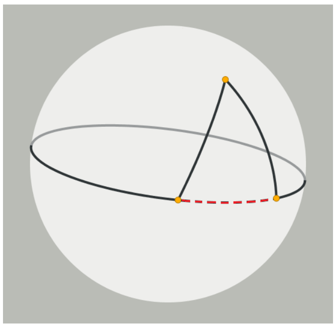
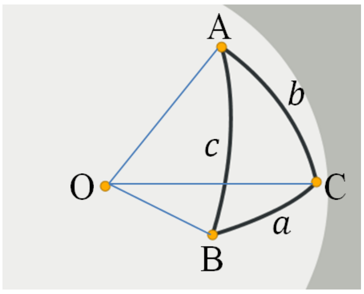


Figure 2.4: Increasing the edges of a spherical triangle.



(a) A spherical triangle with one side $> 180\,^\circ$ and its compliment (red, dashed).



(b) A spherical triangle with vertices connected to the centre of the sphere.

Figure 2.5: Spherical triangles.

in more detail.

On the surface of a sphere, the longest straight line we can draw is a great circle: length $360\,^\circ$. Intuitively, it is impossible to have three sides this length, so we have to go slightly shorter. In Figure 2.5a, we see a triangle with two short sides and one side greater than $180\,^\circ$. This is where we establish our first constraint.

Where the side of a spherical triangle is greater than $180\,^\circ$*, we replace it with its supplementary side.*

As this can be done with every possible triangle, it is unnecessary to allow triangles with individual sides longer than $180\,^\circ$. This also keeps the geometry on the surface of the sphere relatively simple. Our train of thought then tells us that if each side is less than $180\,^\circ$, the upper bound of the perimeter is $3 \times 180\,^\circ = 540\,^\circ$. Again this is not possible, if two sides are of length $180\,^\circ$, the third would have to be zero, or very small.

We construct a triangle in Figure 2.5b where the angles at the centre of the sphere, O, are equal to the side lengths. If we collapse the side OA onto the plane below, the sum of the angles at O is $AOB + AOC = BOC$.

Bringing the side OA back up to its original position we are increasing the angles AOC and AOB, so we have $AOB + AOC > BOC$. This gives us our second constraint.

The sum of any two sides in a spherical triangle must be greater than the third.

This is equivalent to proposition 20 in Euclid's *Elements* in planar maths, that the third side of a planar triangle cannot exceed the sum of the other two. The proof of this is an example of one that works for spherical geometry as well as geometry in the plane.

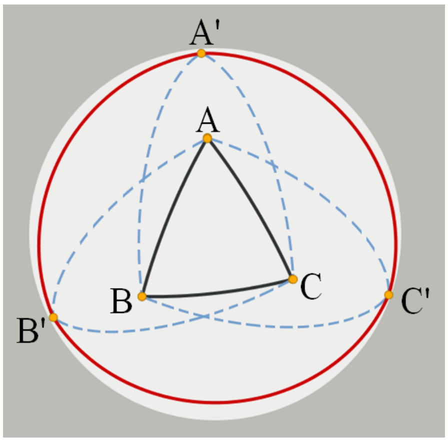
Proof. We continue with the tetrahedron formed by joining the vertices of the spherical triangle in Figure 2.5b to the centre of the sphere. Since the three sides connected to the centre, O, are triangles, their angles must sum to $3 \times 180\,^\circ = 540\,^\circ$. It can be shown, using the same argument for the angles at one of the vertices, say A, as for the justification that two sides of a spherical triangle must be greater than the third, that the sum of the two angles OAB and OAC is greater than the angle BAC (similarly for B and C). Therefore,

$$\begin{aligned}\text{Perimeter of } ABC &= \text{angles at } O \\ &= 540\,^\circ - (\text{angles at } A + \text{angles at } B + \text{angles at } C) \\ &< 540\,^\circ - (A + B + C) \\ &= 540\,^\circ - 180\,^\circ \\ &= 360\,^\circ. \quad \square\end{aligned}$$

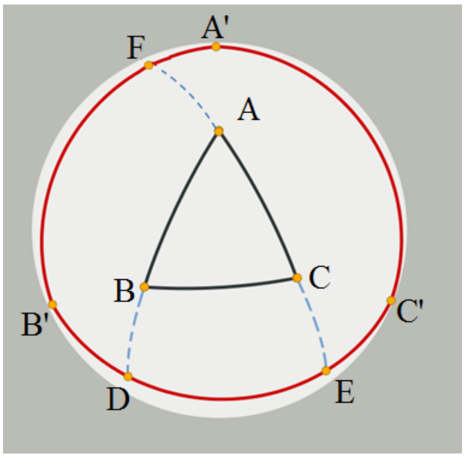
2.2.2 The Sum of Spherical Angles

Can an equilateral triangle consist of three right-angles? On the plane this is impossible, but the curved surface of the sphere allows for strange geometry. We turn our attention to the smallest and largest possible sums of angles in spherical triangles.

As the length of a great circle arc tends to zero, the arc tends to a straight line. Therefore the smallest possible triangle we can construct on the sphere will resemble a planar triangle, whose sum of internal angles is $180\,^\circ$. This is the lower bound of the sum of angles in a spherical triangle, as increasing the side lengths will increase the size of the angles. We found that the largest perimeter of a spherical triangle is $360\,^\circ$, whose sum of angles is $3 \times 180\,^\circ = 540\,^\circ$. This can be taken as the maximum sum of angles by imposing the same constraint as used for the perimeter: where an angle is greater than $180\,^\circ$, we replace it with its complimentary angle. This argument suffices



(a) Constructing the polar triangle (red) of triangle ABC (black).



(b) The Polar Duality theorem.

Figure 2.6: The polar triangle.

for the upper bound, but we will look closer at the lower bound in order to introduce the polar triangle.

To do this we go back to the 10th century in medieval Persia. At this time, there was a profound development in mathematics in the Muslim world, with many theorems and ideas being re-invented. It is here that Abū Naṣr Manṣūr ibn ʻAlī ibn ʻIrāq constructed the first polar triangle.

Take an arbitrary triangle, Figure 2.6a. Mark the pole of each side by moving 90° perpendicularly away from it (blue dashed lines), choosing the pole on the same side as the original triangle. The point C' is the pole of AB and we define A' and B' in the same manner. Joining A', B' and C' (red lines) forms the *polar triangle* of triangle ABC. Here we introduce the first of two polar triangle results.

Theorem 4. *The polar triangle of a polar triangle is the original triangle.*

Proof. In Figure 2.6a, since C' is the pole of AB and A' is the pole of BC, B is 90° from both A' and C', so B is the pole of $A'C'$. This also holds for the other two sides in relation to A and B. The result follows. $\square$

An object is said to have a *dual* relationship with the original when taking a transformation of the original and then applying the same transformation to the result brings us back to our starting point. We can often learn a lot from the dual of an object that we would not have known from studying the original. This brings us to the **Polar Duality Theorem**.

Theorem 5. *The sides of a polar triangle are equal to the supplements of the angles of the original. The angles of a polar triangle are equal to the supplements of the sides of the original.*

Proof. In Figure 2.6b, extending the arcs from A to points D and E on $B'C'$ means we can say the angle at A is equal to DE, as A is the pole of $B'C'$. Since C' is the pole of AD, $C'D$ is also of length $90\,^\circ$, similarly for $B'E$ since B' is the pole of AE. Therefore

$$B'C' = B'E + C'D - DE = 180\,^\circ - DE = 180\,^\circ - A. \tag{2.1}$$

Likewise for the other two sides of the polar triangle and the corresponding angles of the original triangle opposite them.

The second part of Theorem 5 follows from noting that the angle $B' = EF$. We know $AE = CF = 90\,^\circ$, so

$$B' = AE + CF - AC = 180\,^\circ - AC.$$

The result follows for A' and C' too. □

We can now complete the proof for the lower bound of the sum of angles in a spherical triangle.

Proof. Since the perimeter of a spherical triangle is less than $360\,^\circ$, this holds for a polar triangle, so we write

$$A'B' + B'C' + A'C' < 360\,^\circ.$$

Thus, using Theorem 5 and Eq. (2.1),

$$(180\,^\circ - C) + (180\,^\circ - A) + (180\,^\circ - B) < 360\,^\circ \quad \Rightarrow \quad A + B + C > 180\,^\circ. \quad \square$$

Now that we understand some of the geometry on the sphere, we can begin exploring theorems for spherical triangles upon it.

Chapter 3

Menelaus of Alexandria

When introducing this topic in Chapter 1 we mentioned Hipparchus of Nicaea, the founding father of trigonometry. As Hipparchus lived over 2000 years ago, there is not much that we can say about him. He was born in Nicaea, now Iznik in modern day Turkey, in 190 BC and is believed to have died on the island of Rhodes around 120 BC. His work as an astronomer and mathematician is mainly known of from references to him in Claudius Ptolemy's *Almagest*. Another, slightly less elusive, character also praised by Ptolemy lived around the first century AD, and is whom this chapter is about.

It is speculated that Menelaus was born around 70 AD in Alexandria, Egypt, due to the scholars Pappus and Proclus referring to him as Menelaus of Alexandria. He later made some astronomical observations in Rome according to Ptolemy's *Almagest* in 98 AD, and died around the year 130. Menelaus was an astronomer and mathematician who produced many books on geometry and trigonometry, however only one, *Sphaerica*, survives today in its Latin and Arabic translations. Luckily, this is the most important for us, as it concerns spherical triangles and their application to astronomy.

Although *Sphaerica* was not the first book on spherical trigonometry, it was the most innovative. Up until this point, no astronomer or mathematician had the crucial tool of planar trigonometry as Menelaus did. This left their works in vague territory where one arc was longer than another, but no one could calculate by how much.

As the original Greek text of *Sphaerica* is lost, we are unsure of how precise the translated copies are. It is likely that the translators altered some of the text and historical content to suit their audiences, evident from the use of the sine rather than the chord in the summary text. The main message however is still there, which leads us to our first trigonometric theorem. The proof of Menelaus' Theorem A, Thābit's proof of Theorem B, and information on the celestial sphere are inspired by Van Brummelen (2013).

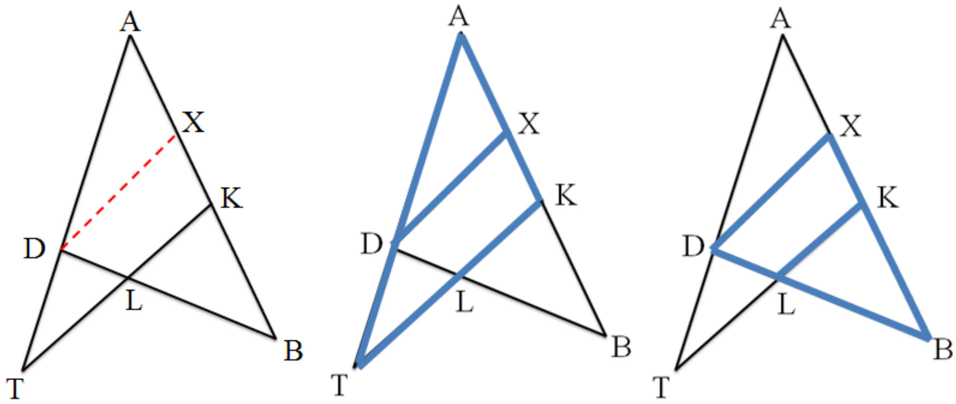


Figure 3.1: The Menelaus configuration.

3.1 Menelaus' Theorem

We begin with the Menelaus configuration, Figure 3.1. This construction yields **Menelaus' planar theorem**, which is stated in book II of *Sphaerica*, but not proved. This suggests that the planar configuration and theorem was already known. We will prove it for completeness and to use as an intermediate theorem in projecting the Menelaus configuration onto the sphere.

Proof. In Figure 3.1, by extending a line from the point D, parallel to that of KL, to a point X on the line AK, we highlight two sets of similar triangles, illustrated.

Rewriting the ratio of the sides AK and KB and eliminating terms involving X, we obtain the required result.

$$\frac{AK}{KB} = \frac{AK}{XK} \cdot \frac{XK}{BK} = \frac{AT}{TD} \cdot \frac{DL}{LB}. \tag{3.1}$$

□

To move Figure 3.1 onto the surface of the sphere, we place the planar version behind its spherical counterpart, effectively cutting the sphere by a plane in the shape of the Menelaus configuration, Figure 3.2. Here, the planar configuration meets the surface of the sphere at points A, B and D. Therefore, the lines and points falling in between these boundaries are within the sphere, and those outside, point T, are outside of the sphere. Menelaus' planar theorem involves three straight lines and their ratios, so to complete the transformation our aim is to 'pop' the straight lines onto their corresponding arcs. That is, $AKB \to AZB$, $ADT \to ADG$, and $DLB \to DEB$. The first and third have line segments that fall within

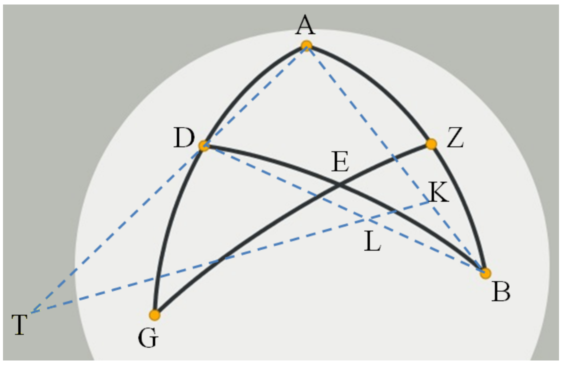
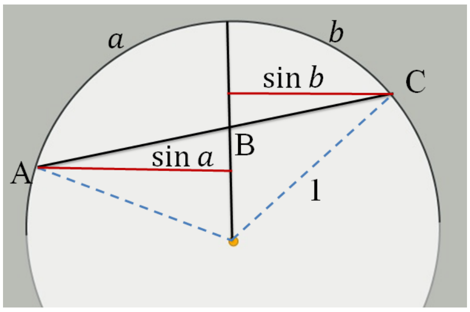
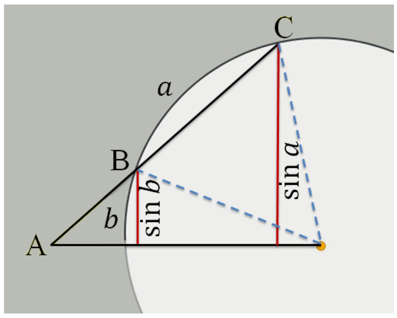


Figure 3.2: The Menelaus configuration on the surface of the sphere.



(a) Proof for the internal lines.



(b) Proof for the partially external lines.

Figure 3.3: Popping the line segments onto the sphere.

the sphere, but the second line segment to be projected onto the sphere is partially outside of it. For this we will need two separate methods; let us generalise these so as to use them for any arbitrary line segment. We will first deal with the internal lines.

Proof. Figure 3.3a shows an arc of length $a+b$ with the planar line underneath it. This is part of a circle of radius 1, so the angles at the centre are a and b. Using planar trigonometry, the two red lines in Figure 3.3a are of length $\sin a$ and $\sin b$, respectively, since we are dealing with two right-angled triangles.

As the two triangles about AC are similar, we can take the ratio of these sides to get an immediate relationship between the straight line segment and the arc:

$$\frac{AB}{BC} = \frac{\sin a}{\sin b}. \qquad \square$$

A similar proof follows for the partially external line segment using Figure 3.3b.

Proof. Here we have the length a going from the horizontal line at A to C

and b going from the same horizontal line to B. Again the angles at the centre are equal to the corresponding lengths, and by similar triangles we yield the result

$$\frac{AC}{AB} = \frac{\sin a}{\sin b}. \qquad \square$$

Both of these results tell us that to pop the line segments onto the arcs on the sphere, we need to take the sines of the arc lengths. Doing this to the lines in Menelaus' planar theorem, Eq. (3.1), we obtain **Menelaus' Theorem A**:

$$\frac{\sin AZ}{\sin ZB} = \frac{\sin AG}{\sin DG} \cdot \frac{\sin DE}{\sin EB}. \tag{3.2}$$

This is the first proposition in the third and final installment of *Sphaerica*, but with Theorem A comes **Menelaus' Theorem B**.

$$\frac{\sin AB}{\sin AZ} = \frac{\sin BD}{\sin DE} \cdot \frac{\sin GE}{\sin GZ}. \tag{3.3}$$

Menelaus' Theorem B was used by Ptolemy in the *Almagest*, but he did not prove it. There are several ways of getting to the result, and since Menelaus himself did not prove it, we are free to choose any of them. One way would be to go back to the Menelaus configuration, Figure 3.4a.

Proof. This time, we extend a line from K to a point Y, parallel to DL in Figure 3.4a, rather than parallel to KL from D. We can then manipulate the new set of similar triangles to give

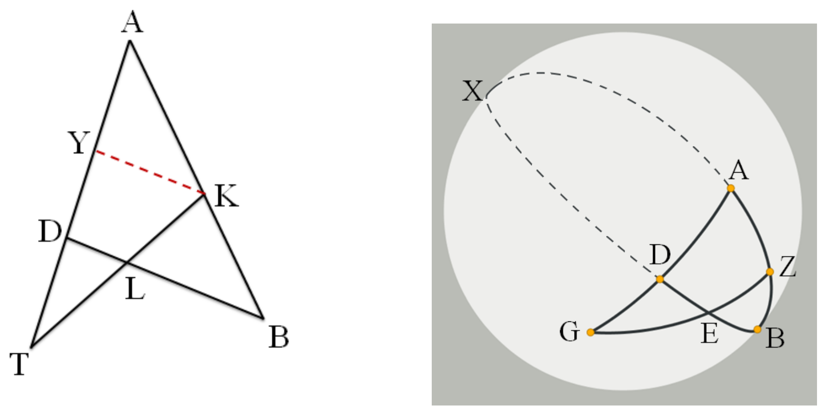
$$\frac{AB}{AK} = \frac{AB}{AK} \cdot \frac{TK}{TL} \cdot \frac{LT}{TK} = \frac{BD}{YK} \cdot \frac{YK}{DL} \cdot \frac{LT}{TK} = \frac{BD}{DL} \cdot \frac{LT}{TK}.$$

Replacing these line segments with the sines of the corresponding arc lengths as before, we have Menelaus' Theorem B. $\square$

We also could have gotten to this point by using Theorem A, Eq. (3.2). This method was favoured by Thābit ibn Qurra, who used this proof long after Menelaus' time, in the 9th century. It goes like this.

Proof. Begin with the Menelaus configuration on the surface of the sphere, Figure 3.4b. Extend arcs BA and BD until the two arcs meet at point X, directly opposite B on the sphere. We now have the Menelaus configuration $XAZEGD$ to which we can apply Theorem A.

$$\frac{\sin AZ}{\sin AX} = \frac{\sin GZ}{\sin GE} \cdot \frac{\sin DE}{\sin DX}.$$



(a) Planar Menelaus' Theorem B.

(b) Thābit's proof of Theorem B.

Figure 3.4: Proving Menelaus' Theorem B.

As BX is a semicircle (for both arcs), $AX = 180° - AB$ and therefore $\sin AX = \sin AB$. Similarly, $\sin DX = \sin BD$. Making these substitutions and rearranging eliminates our X term, and we are left with Theorem B, Eq. (3.3). □

These two equations were known by many as "regula sex quantitatem" during medieval times, translated as "the rule of six quantities". But how was this strange configuration useful to astronomy?

3.2 Astronomy and the Menelaus Configuration: The Celestial Sphere

The most important use of Menelaus' Theorems for ancient and medieval astronomers was converting between different coordinate systems on the celestial sphere. The different lines of reference mean that there are several different ways of locating stars and planets. The celestial sphere embodies the Sun, the Moon, and all of the stars and planets visible from Earth. When observing these bodies, we can quickly draw three conclusions: all move in circles; their speeds are constant; and the Earth is at the centre of these motions.

We can account these properties back to Aristotle through Ptolemy, as he describes them in the *Almagest*. The axis of these rotations is not dissimilar to our own; the North Pole is the north star, Polaris, which can be seen as an extension of our own North Pole. We can also think of the celestial equator

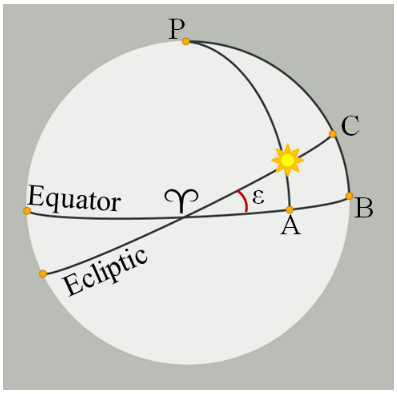


Figure 3.5: Application of Menelaus' Theorem.

as a projection of our own equator. Due to the infinitely large nature of the universe, the Earth is modelled as a pin prick at the centre of the celestial sphere. The horizon of the celestial sphere is dependent on where on Earth you are standing. The dome of universe surrounding you is unique to where you are; the point where this dome meets the surface of the Earth is the horizon. When standing in the northern hemisphere, the altitude of the North Star from the horizon is equal to the latitude of the observer.

If we were to record the position of the Sun at the same time everyday for a year, we would find that it changes slightly each day; it follows the path of the *ecliptic*, shown in Figure 3.5 along with the North Pole, P, and the equator. The Sun moves approximately 1° across the ecliptic each day. As the ancient Babylonians used the sexagesimal system, and 360 is a multiple of 60, dividing the sphere into 360 parts makes sense and coincides with the number of days in a year. The spring equinox, where the ecliptic intersects the equator, occurs at the point Υ, the symbol for the zodiac sign Aries. The two lines make an angle of ε between them, which is the obliquity of the ecliptic. This is equal to the tilt of the Earth's axis, now around 23.44°.

Looking at figure 3.5, we can see the Menelaus configuration $PCBA\Upsilon S$, where S is the Sun. We can apply Menelaus' Theorem to this to find the equitorial coordinates of the Sun, given by the distances ΥA and AS.

Chapter 4

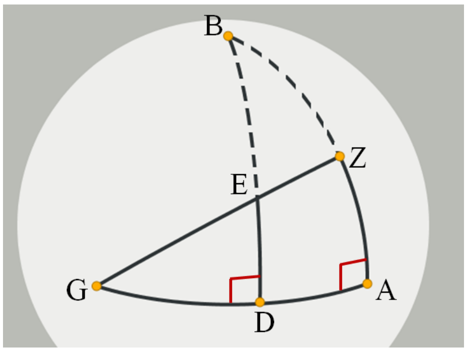
Medieval Islamic Mathematics

We fast forward to the 10th century. Menelaus' Theorems stood the test of time for the rest of the first millennium, but something exciting was happening in the Muslim world. It is difficult to determine who came up with our next result first; different proofs and publications of the theorem appeared almost simultaneously.

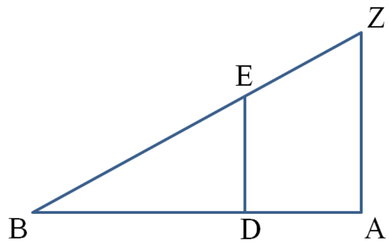
There were many influential Arab scholars in the medieval Middle East, one of whom we met in Chapter 2: Abū Naṣr Manṣūr. He was born in 970 in Persia, and died around 66 years later in Ghazna, now Ghazni in Afghanistan. Abū Naṣr was a talented mathematician, the teacher of al-Bīrūnī, and also a prince. His family ruled over Khwarazm, the region where Abū Naṣr studied under Abū 'l-Wafā'. When civil war broke out in Khwarazm, it is likely that Abū Naṣr fled the area, along with al-Bīrūnī. He was later said to be employed at the court of Ali ibn Ma'mun (where al-Bīrūnī was also employed) whose state was soon to be taken over by the Ghaznavids, whose capital was Ghazna. As well as discovering the polar triangle (Chapter 2), Abū Naṣr was responsible for preserving and simplifying many of Menelaus' findings. He is most famous for his work alongside al-Bīrūnī. The proofs in this chapter follow the style of Van Brummelen (2013).

4.1 The Rule of Four Quantities

Whilst Menelaus' Theorem gives us a "rule of six quantities" and was used by many generations of astronomers, it was not the easiest to remember. Without it, however, we would not have Abū Naṣr's **Rule of Four Quantities**. Abū Naṣr's original work on the topic appeared in the *Book of the Azimuth*, but the only surviving reference we have to this is through al-Bīrūnī's *Keys to Astronomy*.



(a) Proof on the sphere.



(b) In the plane.

Figure 4.1: The Rule of Four Quantities.

Proof. Figure 4.1a shows a Menelaus configuration where B is the pole to AG, giving us two embedded right-angled triangles. Applying Menelaus' Theorem B, Eq. (3.3), to the diagram, we get

$$\frac{\sin AB}{\sin AZ} = \frac{\sin BD}{\sin DE} \cdot \frac{\sin GE}{\sin GZ}.$$

As BD and AB are of length $90\,°$, the sines of these sides are equal to 1, so

$$\frac{\sin DE}{\sin AZ} = \frac{\sin EG}{\sin GZ}. \tag{4.1}$$

□

Aside It is here we have our first example of the *principle of locality*, which is where we we can make a direct comparison of a spherical result with its planar case.

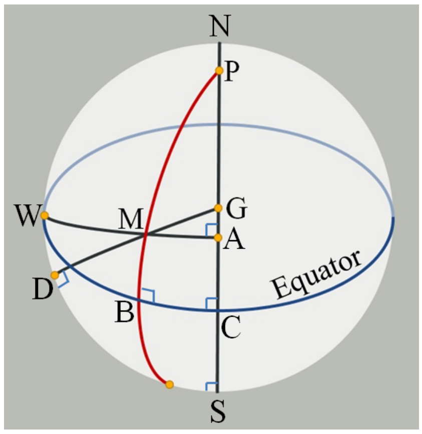
On the graph of $\sin x$, as $x \to 0$, $\sin x \to x$. (The exact value of $\sin x$ actually only tends to x when using radians, but seeing as we are not using numbers here we need not worry about it too much.) Applying this to the arrangement in the Rule of Four Quantities, we get

$$\frac{\sin DE}{\sin AZ} = \frac{\sin EG}{\sin GZ} \to \frac{DE}{AZ} = \frac{EG}{GZ},$$

which is equivalent to the well known relation between taking the ratios of the sides of similar triangles in the plane, shown in Figure 4.1b.

Clearly the Rule of Four Quantities is much easier to apply than the Menelaus Theorems, which allowed for huge advances in astronomy. The celestial sphere is not the only sphere that directly concerned mathematicians though, the surface of the Earth too is spherical.

(a) At a glance. Image adapted from Google Maps (map, 2014).



(b) Al-Bīrūnī's determination using the Rule of Four Quantities.

Figure 4.2: Finding the qibla.

4.2 Finding the Qibla

This particular application of the Rule of Four Quantities is in cartography, but it came about for a religious purpose. People of the Islamic faith pray facing the direction of Mecca five times each day. In doing this, they are actually facing the Ka'ba, the religious building that all Muslims are required to visit at least once in their lives, and the direction their bodies must face when buried after death, amongst other acts that must be performed in this direction (King and Lorch, 1992). This is known as the qibla.

The problem that arose for Muslims was how to calculate the qibla. At face value, it seems as though we can draw a right-angled triangle on the map; we know the location of both Mecca and the worshipper, so can calculate the differences in latitude and longitude of the two. Unfortunately, this is not the case. Turning our attention to Figure 4.2a, we have the qibla from Ghazna highlighted on the map of the Middle East, with the right-angled triangle constructed from the differences in coordinates of the two locations. The issue with this is that the bottom side of the triangle is not a great circle arc, it is an arc of a circle of latitude. Therefore, the image we are looking at is not even a triangle.

Various approximate solutions to this problem were proposed, some even taking the triangle in Figure 4.2a to be planar. Around the 10th century, solutions using Menelaus' Theorems began to surface. After the Rule of Four quantities was derived, methods of calculating the qibla were simplified. The familiar figure of al-Bīrūnī was able to solve this problem in four different ways. He was no stranger to applying his mathematical knowledge to geographical problems. In his work *Determination of the Coordinates of Cities* he explains how to find the circumference of the Earth using only planar trigonometry. It is in this book that he also calculates the qibla. His methodology was then applied to over 3000 places across the Muslim world by Shams al-Dīn al-Khalīlī. The qibla for each of these places was recorded in a table spanning sixteen pages, for any struggling worshipper to look up.

We will follow in the steps of al-Bīrūnī and calculate the direction of Mecca from Ghazna, using Figure 4.2b and the Rule of Four Quantities, Eq. (4.1). Not only was Ghazna the location of al-Bīrūnī at the time, it was also the capital of the Ghaznavid Empire and so a place of great importance.

Consider a bird's-eye view above Ghazna, G, from the celestial sphere. The outer circle of the sphere in Figure 4.2b is the horizon of Ghazna, and the line that connects the north point on the horizon to the south is the meridian of Ghazna, passing through the North Pole, P. The point M is where Mecca would be if we were directly above it, with its associated meridian the arc going through points PMB. The line going through W, M and A connects the west point on the horizon to the meridian of Ghazna. GD is the great circle arc connecting Ghazna to Mecca.

We know the latitudes of Ghazna, $\varphi_G = 33.58\,°$, and Mecca, $\varphi_M = 21.67\,°$, and from their longitudes we can calculate their difference in longitude as $\Delta\lambda = 27.37\,°$. From these values we can determine all necessary distances that we need for the qibla.

We begin with the $CAPMB$ configuration. We know the difference in longitude of the two locations, which is equal to the angle between the two meridians at P. Therefore, $BC = \Delta\lambda$. We also know $PM = 90\,° - MB$, so

$$\frac{\sin PM}{\sin MA} = \frac{\sin PB}{\sin BC} \Rightarrow \frac{\sin(90\,° - \varphi_M)}{\sin MA} = \frac{1}{\sin \Delta\lambda},$$

which gives us $\sin MA = \cos\varphi_M \sin\Delta\lambda$ and we get $MA = 25.29\,°$.

Using the same method for the $WMACB$ configuration, we find $\sin AC = \sin\varphi_M / \cos MA$, which gives us $AC = 24.11\,°$. From this we can deduce that $GA = GC - AC = 9.47\,°$.

It is at this point that it is useful to think about what it is we want to find. The qibla is a direction, not a length, so how does the Rule of Four Quantities enable us to find this? It comes from looking out to the horizon.

Using configuration $WMASD$, we find $\sin MD = \cos MA \cos GA$, so $MD = 63.10\,°$. We can now take the final step in our calculation, by finding the distance DS from the configuration $GMDSA$. Doing so, we get $\sin DS = \sin MA / \cos MD$, which tells us $DS = 70.79\,°$.

This solves our problem; when standing in Ghazna, one must turn $70.79\,^\circ$ west from the south in order to face Mecca.

4.3 The Spherical Law of Sines

As well as its extensive practical applications, the Rule of Four Quantities was used to make further advances in spherical trigonometry. There was plenty of controversy over who discovered the spherical Law of Sines, mainly between Abū Naṣr and his teacher, Abū 'l-Wafā'. Al-Bīrūnī scorned Abū 'l-Wafā' in his *Keys to Astronomy* for taking credit for the theorem, but this opinion may of course be biased, given al-Bīrūnī's relationship with Abū Naṣr.

Abū 'l-Wafā' al-Būzjānī was born in 940 in Būzjān, Iran, and died in 998 in Baghdad, Iraq. He first moved to Baghdad to work in the court of the ruler of Iran and most of Iraq at the time, Adud ad-Dawlah, alongside other respected scientists. Amongst Abū 'l-Wafā's works was his very own *Almagest*, the same title as that by Ptolemy. His version, however, was much more comprehensive and covered exciting new ideas, such as applying the tangent function and the inverse trigonometric functions; the secant, cosecant and cotangent, to astronomy. Along with these innovative applications, Abū 'l-Wafā proved the **spherical Law of Sines**.

Proof. We begin with an arbitrary triangle on the surface of the sphere in Figure 4.3a, with angles A, B and C, and sides of length a, b and c. If we choose a vertex, say C, to drop a perpendicular from, we split our triangle into two right-angled triangles, Figure 4.3b.

We now choose A and B to be poles, and draw their corresponding great circle arcs by extending their adjoining sides to be $90\,^\circ$ long, Figure 4.3c. It is now possible to apply the Rule of Four Quantities, Eq. (4.1), twice, by using the line CD to create two sets of embedded right-angled triangles. Doing so, we get

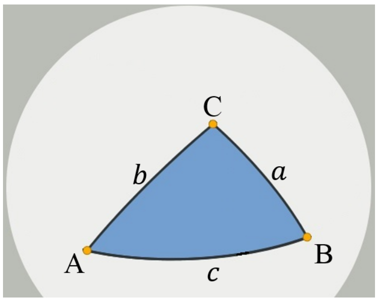
$$\frac{\sin CD}{\sin EZ} = \frac{\sin AC}{\sin AZ} \Rightarrow \sin CD = \sin A \sin b,$$

and

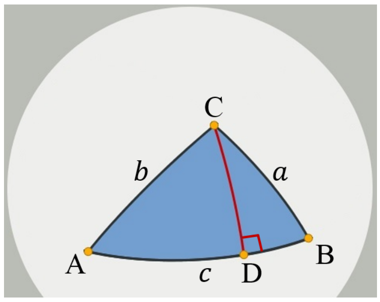
$$\frac{\sin CD}{\sin TH} = \frac{\sin BC}{\sin BT} \Rightarrow \sin CD = \sin B \sin a.$$

Combining these two equations and rearranging gives us

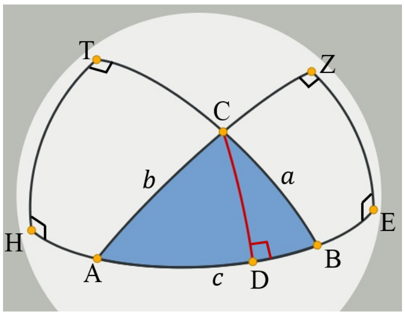
$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$



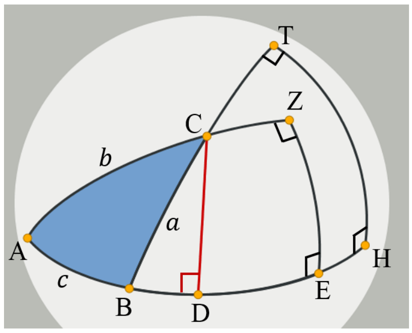
(a) An arbitrary triangle with acute angles.



(b) Dropping a perpendicular from C.



(c) Making A and B poles.



(d) Obtuse-angled triangle.

Figure 4.3: Proof of the spherical Law of Sines.

Of course, we could have dropped the perpendicular from any of the vertices in Figure 4.3a, which gives us the complete result:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{4.2}$$

□

But what if the perpendicular we drop does not fall within the triangle? If we have an obtuse angle in the triangle, the line dropped from the opposite vertex does not fall within the boundaries of the triangle itself, Figure 4.3d. We therefore need to take a slightly different route in proving the spherical Law of Sines.

Proof. In Figure 4.3d, drop a perpendicular from the point C onto an extension of the line c. From here, define the pole and great circle arc for A as before, but take the great circle arc of B to the opposite side, instead extending c to the right. We again apply the Rule of Four Quantities, and the result follows. □

What is special about this result is that the spherical Law of Sines was the first equation of its kind; up until now we have not known of an equation that gives us a relationship between sides *and* angles of a spherical triangle. It would not be surprising if the spherical Law of Sines went on to transform the work of scientists from this point onwards, being able to apply it to any triangle, as opposed to purely right-angled triangles, must have been eye-opening. Unfortunately, this was not the case. The Rule of Four Quantities was still the main tool in any astronomers pocket, as the configuration was somewhat more useful for the quantities they were trying to find. It was not until the Renaissance in Europe that the spherical Law of Sines took off.

Principle of Locality Aside from the lacklustre set of applications of the spherical Law of Sines, we can find its planar brother. Using the same argument as in Section 4.1 for the Rule of Four Quantities, reducing the sides of a spherical triangle reduces the spherical Law of Sines to the **planar Law of Sines**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Chapter 5

Right-Angled Triangles

It is not often that in every day life a triangle has an angle of exactly $90\,^\circ$. John Napier, however, did not see working purely with right-angled triangles as a limitation, realising that any oblique triangle can be split into two right-angled triangles by dropping a perpendicular from one of the vertices—just as we do in planar trigonometry today. Napier was a Scottish Protestant who lived from 1550 to 1617. He was greatly interested in theology, the study of the nature of God and religious belief, and attended St. Andrew's University at the age of just 13.

Despite his discoveries, Napier was not primarily a mathematician; he was an inventor. The mathematical ideas he came up with were mainly tools to help him find something else quicker. The logarithm is a prime example of one of these inventions. Napier published *Mirifici logarithmorum canonis descriptio* in 1614, a book dedicated to easing the difficulty of arduous multiplication and division of trigonometric functions, a task that dominated all work with spherical trigonometric formulae, by adding them instead. His idea was this: to create a system of two sets of numbers such that when $x = x_1 + x_2$, $y = y_1 y_2$. He eventually managed this with the help of Henry Briggs of Gresham College, London, finding $y = 10^x$ gave the correct result. This innovation made the work of future astronomers and mathematicians, including that of Johannes Kepler, much simpler. We follow the proofs in Van Brummelen (2013) to see some interesting results in this chapter.

5.1 Pythagoras' Theorem on the Sphere

At the beginning of the 12th century lived Jābir ibn Aflaḥ of Seville, Spain. Jābir was not known for excelling in his scientific findings, however he did rewrite Ptolemy's *Almagest* in his most famous work, *Correction of the Al-*

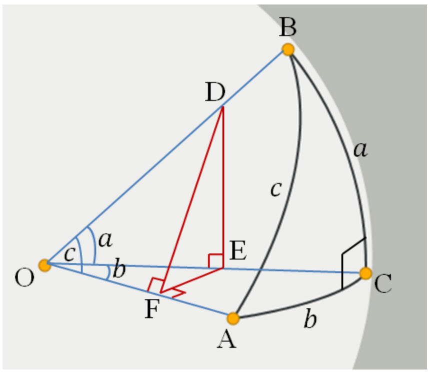


Figure 5.1: Proving Geber's Theorem.

magest. Fortunately for Jābir, this was found by astronomers in late medieval Europe, bringing his ten fundamental identities of spherical right-angled triangles to light, known as Geber's Theorem.

To find these spherical identities, we need to use planar trigonometric results. To do this, we must express the spherical angles A and B in Figure 5.1 as planar angles between two line segments. Create a planar right-angled triangle in the configuration in Figure 5.1 by dropping a perpendicular from a point D on the line OB, then join D and E to a point F on OA. This constructs angle A at point F and subsequently produces three more planar right-angled triangles on the faces attached to the centre of the sphere, O.

From here, applying trigonometry at each vertex of the four right-angled planar triangles, we obtain four results relating the sides and angles of right-angled triangles on the sphere:

$$\begin{aligned} \sin a &= \sin A \sin c, \\ \sin b &= \tan a \cot A, \\ \cos A &= \tan b \cot c, \\ \cos c &= \cos a \cos b. \end{aligned} \tag{5.1}$$

The remaining 6 identities can be derived in a similar fashion. The last of the above identities is particularly important.

Principle of Locality Using the Maclaurin series expansion of the cosine,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,$$

tells us that for small x we can approximate $\cos x$ by $1 - x^2/2$. Substituting this expression into each term of $\cos c = \cos a \cos b$ and rearranging, we get

$$c^2 = a^2 + b^2 + \frac{a^2 b^2}{2}.$$

This simplifies to $c^2 = a^2 + b^2$ because as the sides of the triangle diminish, the last term becomes infinitesimally small. This is the planar Pythagorean Theorem, so we can deduce that

$$\cos c = \cos a \cos b \tag{5.2}$$

is the **spherical Pythagorean Theorem**.

Pythagoras' Theorem in planar triangles has actually been known and in use since the ancient Babylonians. Specific cases of using a 3,4,5 triangle in building ancient settlements date back much further, but lacked a formula to describe the relation.

Clearly the application of Napier's logarithm to the multiple products of trigonometric functions in Geber's Theorem, Eq. (5.1), was indispensable, minimising hours of work. Pierre Siméon de Laplace was quoted praising the work of Napier long after his death saying "by shortening the labours, [logarithms] doubled the life of the astronomer".

5.2 The Spherical Law of Cosines

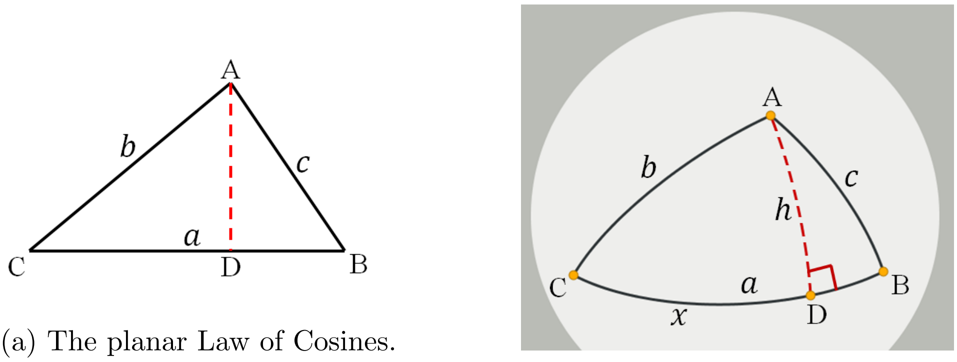
We have seen the relevance of Euclid's *Elements*, Section 2.2.1, when discussing properties of both spherical and planar triangles. To look again at Euclid's work for inspiration on the Law of Cosines seems counter-intuitive, seeing as the *Elements* was written in the 3rd century BC, before trigonometry had been invented. Book II of the *Elements*, however, can be seen as a statement about algebraic geometry, Proposition 13 about acute-angled triangles is of particular interest. The statement translates as this, in reference to Figure 5.2a:

> In triangle ABC with an acute angle at C and a perpendicular dropped from A onto BC (defining D), $c^2 = a^2 + b^2 - 2CD \cdot BC$.

Since $BC = a$ and $CD = b\cos C$, Euclid actually had the planar Law of Cosines right under his nose, he just did not have the trigonometric functions to express it as we do today.

Proof. An application of Pythagoras' Theorem on the RHS right-angled triangle of Figure 5.2a quickly gives us the required result. □

We use the same principle for the spherical version.



(a) The planar Law of Cosines.

(b) The spherical Law of Cosines.

Figure 5.2: Proof of the Law of Cosines.

Proof. Copying the planar configuration, including the perpendicular dropped from A, onto the sphere, Figure 5.2b, we apply the spherical Pythagorean Theorem, Eq. (5.2), to both right-angled triangles, giving

$$\cos c = \cos h \cos(a - x) \qquad \text{and} \qquad \cos b = \cos h \cos x.$$

Rearranging for $\cos h$ and applying the cosine subtraction law to $cos(a - x)$, we can equate these two statements

$$\cos c \cos x = \cos b(\cos a \cos b + \sin a \sin b).$$

We want to eliminate terms involving x in order to generalise our formula to any ABC triangle. Dividing through by $\cos x$, we obtain a $\tan x$ term in our equation. To get rid of this final term involving x, we turn to the third of the fundamental trigonometric identities, Eq. (5.1); $\cos A = \tan b \cot c$. Putting this in terms of the right-angled triangle on the LHS of Figure 5.2b and substituting for $\tan x$, we obtain **the spherical Law of Cosines**

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{5.3}$$

□

As with every major theorem on the sphere so far, the planar version of the spherical Law of Cosines is not far away.

Principle of Locality Section 5.1 uses the Maclaurin series to relate the cosine function to an expression for small x; $\cos x \approx 1 - x^2/2$. We also know from Section 4.1 that for small x, $\sin x$ reduces to x, approximately.

Applying these results to the spherical Law of Cosines, we have

$$1 - \frac{c^2}{2} = \left(1 - \frac{a^2}{2}\right)\left(1 - \frac{b^2}{2}\right) + ab\cos C.$$

Thus, we have the planar equivalent

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

This particular relation between the three sides of a triangle and one of the angles lends itself well to problems on the surface of the Earth and on the celestial sphere. Most textbooks on the subject require students to calculate distances on the Earth, such as popular sea routes, like that of the *Titanic.* The trick is to draw the great circle arc between the start and end point of the journey, the arc CB in figure 5.2b, then connect both at the North Pole, A. Doing so gives us two of the side lengths of the triangle and the angle between them, which can then be put into the spherical Law of Cosines to obtain the distance required.

When the situation arises that we know two angles in a triangle and the side between them in the plane, the planar Law of Cosines does not help. Instead we find the third angle and apply the planar Law of Sines. We cannot do this on the sphere as the sum of the internal angles of a spherical triangle is not one size fits all. We can however use the Polar Duality Theorem (Theorem 5, Section 2.2.2).

This allows us to convert between sides and angles on the sphere, which is easily done to the spherical Law of Cosines to give us a formula relating the three angles and one side of a spherical triangle:

$$\begin{aligned}\cos(180^\circ - C) = {} & \cos(180^\circ - A)\cos(180^\circ - B) \\ & + \sin(180^\circ - A)\sin(180^\circ - B)\cos(180^\circ - c).\end{aligned}$$

This gives us the **Law of Cosines for Angles**

$$\cos C = -\cos A\cos B + \sin A\sin B\cos c.$$

This eased the worries of mathematicians wanting to find the remaining angle when a side and two angles were known. We now have a complete set of spherical trigonometric tools for working with sides and angles of spherical triangles.

Chapter 6

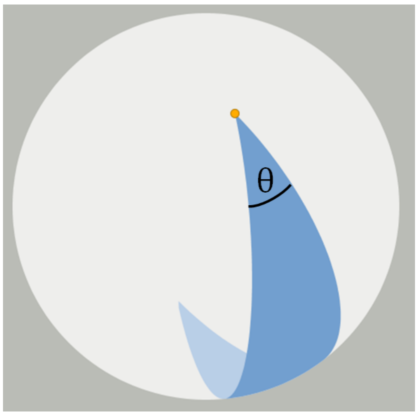
Geometry on the Sphere Part II

The application of spherical trigonometry to the Earth and Heavens consisted mainly of two things: calculating the length of sides of spherical triangles to find distances between points of interest; and finding the angle between two sides to calculate the direction we must face to find them. Something of little interest to astronomers and cartographers was the area of the triangles they were working with. As mathematicians, a thorough look at any topic is the only way to approach it, as interesting results may be lying just beneath the surface. We will therefore fill the final gap in our knowledge by finding the area of spherical triangles and polygons, Part II of our geometrical quest.

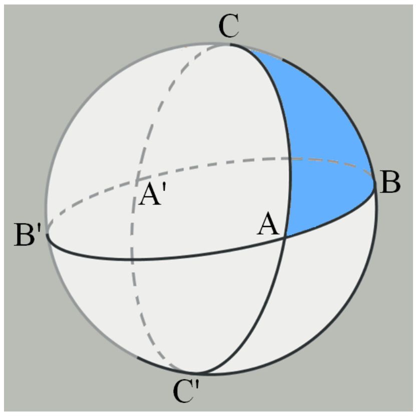
6.1 Areas of Spherical Polygons

The simplest shape we can draw on the surface of the sphere, as discussed in Section 2.1, is a circle as it consists of just one side. In particular, a great circle has been of most interest to us, as it takes the role of a straight line on the sphere. Although this is not a polygon as such, it *is* a triangle with three angles of $180\,^\circ$, which is a polygon. A great circle splits the sphere into two hemispheres, both with area half of that of the surface area of the sphere. In our case of the unit sphere, this is $1/2 \cdot 4\pi r^2 = 2\pi$.

Almost as obscure as a one sided spherical polygon is a spherical polygon with just two sides. This occurs by joining two great circle arcs of length $180\,^\circ$ at their *antipodal* points—two points directly opposite one another on the sphere, Figure 6.1a. The lune formed when doing this has an area depending on the size of the angle, θ, between the two semicircles, which is relatively easy to find. From Todhunter and Leathem (1949), the ratio of θ over $360\,^\circ$



(a) The area of a lune.



(b) The area of a spherical triangle.

Figure 6.1: Calculating the area of a spherical polygon.

is equal to the ratio of the area of the lune over the area of the sphere, so

$$\textbf{area of lune} = \frac{4\pi \cdot \theta}{360\,^\circ} = \frac{\pi\theta}{90}. \tag{6.1}$$

The area of a spherical triangle was not discovered until the 17th century by Albert Girard. Girard was a French born mathematician who was forced to move to the Netherlands because of his Protestant faith, something that saddened him for much of his life. He made large contributions to the areas of algebra, arithmetic and trigonometry, including being the first to use the "sin", "sec" and "tan" abbreviations in his *Trigonométrie* in 1629. This is where we also find his work on spherical triangle areas. His idea was this, in reference to Figure 6.1b (Van Brummelen, 2013):

Proof. Extend the three sides of triangle ABC into their respective great circles. A', B' and C' are the antipodal points of A, B and C, so the triangle $A'B'C'$ is the same as the original triangle ABC. The great circle of BC is the outer circle of the sphere. We now have several triangles and lunes on the surface of the sphere. Each lune comes from extending the adjoining lines of the vertices of the original triangle to their antipodal point, so we have 12 in total—4 at each point. Focusing on the lune from each point that includes the original triangle, we add the triangles involved to get

$$3 \cdot ABC + A'BC + AB'C + ABC'.$$

Another pair of equal triangles in this configuration is $A'BC$ and ABC', so we can swap the former for the latter in the above equation. Doing so

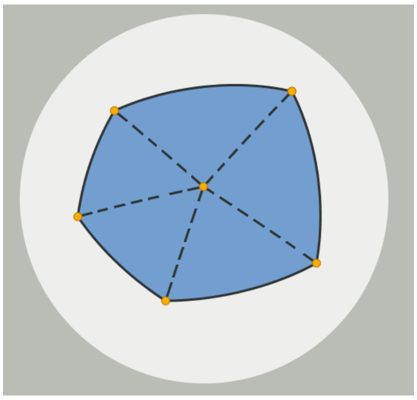


Figure 6.2: Finding the area of a spherical polygon.

gives us the 4 triangles that make up the front hemisphere of Figure 6.1b, so $3 \cdot ABC + A'BC + AB'C + ABC' = 2 \cdot ABC + 2\pi$. Since this is still the sum of three lunes, we can set it equal to the sum of their areas, using Eq. (6.1),

$$2 \cdot ABC + 2\pi = \frac{\pi}{90}(A + B + C),$$

which gives us the area we require

$$\textbf{area of triangle} = \frac{\pi}{180}(A + B + C - 180\,°). \qquad \square$$

So the area of a spherical triangle is proportional to the amount it's angles exceed 180 °—the sum of the angles in a planar triangle.

In the plane, to find the area of a polygon we can divide it up into triangles. The same applies to polygons on the sphere. Here we will extend the formula for the area of a triangle.

Take the arbitrary polygon in Figure 6.2 and divide it into triangles by picking a random point inside it. If the polygon has n sides, its area is n times the area of a triangle (Van Brummelen, 2013)

$$\begin{aligned}\textbf{area of polygon} &= \frac{\pi}{180}(\text{sum of triangles' angles} - n \cdot 180\,°),\\ &= \frac{\pi}{180}(\text{sum of polygon's angles} + 360\,° - n \cdot 180\,°),\\ &= \frac{\pi}{180}[\text{sum of polygon's angles} - (n-2) \cdot 180\,°]. \end{aligned} \tag{6.2}$$

Similar to the spherical triangle, the area of a spherical polygon is proportional to the amount it's angles exceed $(n-2) \cdot 180\,°$. This is unsurprising as the sum of the angles of a planar polygon is equal to $(n-2) \cdot 180\,°$.

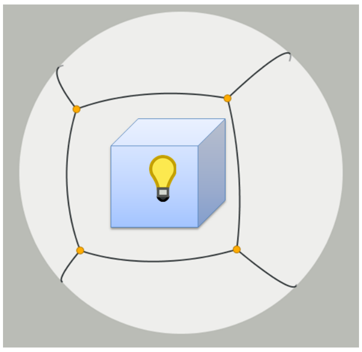


Figure 6.3: Legendre's proof of Euler's Polyhedral Formula.

6.2 Euler's Polyhedral Formula

Leonhard Euler was a Swiss born mathematician that lived between 1707 and 1783. He is probably the most productive mathematician of all time, not just the 18th century, making developments in almost all areas of mathematics. His father was a Protestant minister, who introduced Euler to mathematics before sending him to university in Basel, Switzerland, to get some further education before doing a degree in theology. It was here that Euler discovered his love and talent for mathematics, owing his father's change of mind that he could instead do a mathematical degree to Johann Bernoulli, who had previously studied with Euler's father and worked at the University of Basel. Euler wrote an average of three pages of mathematics per day throughout his life, including the time after he became blind. Amazingly, it is during this time that Euler developed his Polyhedral Formula.

A convex polyhedron is a 3-dimensional shape whose sides are made up of polygons, with none denting inwards. There are five (and no more) regular polyhedra, each of whose sides are made up purely of identical regular polygons, but there are many more irregular polyhedra, where the sides can be made up of a mixture of different regular or irregular polygons.

Theorem 6. *For any polyhedron, with V vertices, E edges and F faces,*

$$V - E + F = 2.$$

This is Euler's Polyhedral Formula. Euler's proof of this theorem for all polyhedra was not considered rigorous enough when he initially wrote it, but there are now a total of 19 proofs. The first came from a completely unexpected area of mathematics, in Book 7 of *Éléments de Géométrie* entitled *The Sphere* by Adrien-Marie Legendre. We will use Legendres proof as shown in Van Brummelen (2013).

Proof. Legendre began his proof by projecting a polyhedron onto the sphere, as demonstrated in Figure 6.3 with the projection of the cube. The polyhedron is at the centre of the sphere, with a light source at its centre casting shadows of its edges and vertices onto the surface of the sphere. The edges are great circle arcs, and form a *spherical polyhedron.*

Using the area of a spherical polygon, Eq. (6.2), we can sum all of the faces of the spherical polyhedron to give the surface area of the unit sphere:

$$\sum \frac{\pi}{180}[\text{sum of polygon's angles} - (n-2)\cdot 180\,^\circ] = 4\pi,$$

which we can expand to give

$$\text{sum of all angles} - \sum[n \cdot 180\,^\circ - 2\cdot 180\,^\circ] = 720\,^\circ.$$

Since the number of polygons is equal to the number of faces of the spherical polyhedron and each edge of the polyhedron is counted twice, we have $\sum[n \cdot 180\,^\circ - 2\cdot 180\,^\circ] = 2E - 2F\cdot 180\,^\circ$. All of the angles are at vertices, so the sum of the angles is $V \cdot 360\,^\circ$. Thus,

$$V\cdot 360\,^\circ - 2E + 2F\cdot 180\,^\circ = 720\,^\circ \quad \Rightarrow \quad V - E + F = 2. \qquad \square$$

Clearly as this holds for spherical polyhedra it holds for planar polyhedra too, since the former is a generalisation of the latter.

Chapter 7

Quaternions

For many years, algebra was thought of as an extension of normal arithmetic, obeying the laws of commutativity, associativity and closure (a property that says numbers stay in the same set they are originally in under operation, e.g. if a and b are real numbers, so is $a+b$) under algebraic operations. The 16th century saw the introduction of complex numbers by Bombelli, an Italian algebraist, with the algebra of complex numbers not being cemented into mathematics until the work of Gauss in the 19th century. We can interpret the complex number $z = a + \mathbf{i}b$, where a and b are real numbers, as the vector (a, b) in the plane. The magnitude of z is the length of the vector from the origin and the angle between the vector and the x-axis is the angle of z. Multiplication of complex numbers can be thought of as a rotation in the plane.

William Rowan Hamilton was particularly interested in these rotations and wanted to define a similar relationship for 3-dimensional space. In 1843, he opened up the world of algebra with his introduction of the quaternion, a *hyper-complex* number of rank 4 that violated the law of commutativity under multiplication. Whilst walking one morning along the Royal Canal in Dublin with his wife, Hamilton had a stroke of inspiration; something that he was so excited by, he carved it into the bridge they were passing by:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{7.1}$$

This was the invention of the quaternion, a complex number consisting of three complex components and four real numbers.

7.1 Some Useful Properties

The tree of quaternions has many branches; we will only look at the relevant properties needed in order to apply quaternions to the sphere. A good explanation of quaternions in all their glory, and the source for the information in this chapter, is Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality, Kuipers (1999), as well as Quaternions and Rotation Sequences, Kuipers (2000).

We denote a quaternion with a lower case letter, say q, and write it as an element of R^4: $q = (q_0, q_1, q_2, q_3)$, where the *components* $q_0, \ldots, q_3$ are real numbers. The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote the orthonormal basis of R^3, so the quaternion q is the sum of a *scalar part*, q_0, and a *vector part*, $\mathbf{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$. The scalar part of a *pure quaternion* is zero.

7.1.1 The Quaternion Product

Addition of quaternions follows the usual algebraic rules of adding each of the corresponding components, as with multiplying a quaternion by a scalar—each element is multiplied by the scalar. The difference occurs when taking the product of two quaternions. Along with Hamilton's Eq. (7.1), there are 3 more equations implied by this to make note of:

$$\begin{aligned}\mathbf{ij} &= \mathbf{k} = -\mathbf{ji},\\ \mathbf{jk} &= \mathbf{i} = -\mathbf{kj},\\ \mathbf{ki} &= \mathbf{j} = -\mathbf{ik}.\end{aligned}$$

Keeping these in mind, the product of two quaternions, say p and q, with some algebra, can be written as

$$\begin{aligned}pq &= (p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3)(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)\\ &= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + p_0(\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) + q_0(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3)\\ &\quad + \mathbf{i}(p_2q_3 - p_3q_2) + \mathbf{j}(p_3q_1 - p_1q_3) + \mathbf{k}(p_1q_2 - p_2q_1).\end{aligned}$$

This can be written more concisely by using the notation of the cross and dot products of two vectors, $\mathbf{p}$ and $\mathbf{q}$, given that $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$;

$$pq = p_0q_0 - \mathbf{p}\cdot\mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}. \tag{7.2}$$

This is known as the quaternion product.

7.1.2 The Complex Conjugate and Inverse Quaternions

Similar to the complex conjugate of any complex number, the quaternion has its own complex conjugate. For the quaternion $q = q_0 + \mathbf{q}$, we define its complex conjugate as

$$q^* = q_0 - \mathbf{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3.$$

After a lot of algebra, it can also be shown that the complex conjugate of a product of quaternions is the product of the complex conjugates of the two quaternions in reverse order. That is $(pq)^* = q^* p^*$.

The norm of a quaternion q is the scalar denoted by $N(q)$ or $|q|$. This represents the length of the quaternion, defined as

$$N(q) = \sqrt{q^* q}, \qquad \text{or} \qquad N^2(q) = q^* q.$$

As with ordinary complex numbers, if a quaternion has a norm of 1 it is said to be a unit quaternion or normalized.

Combining the ideas of the complex conjugate and the norm of quaternions, we can define the inverse of a quaternion. By the definition of an inverse, we have

$$q^{-1} q = q q^{-1} = 1.$$

Multiplying this by the complex conjugate,

$$q^{-1} q q^* = q^* q q^{-1} = q^* \qquad \Rightarrow \qquad q^{-1} = \frac{q^*}{N^2(q)}.$$

Therefore, if q is normalized its inverse is equal to its complex conjugate: $q^{-1} = q^*$. To take advantage of this property, all quaternions discussed from here on will be unit quaternions. We ensure this is the case by writing

$$q = q_0 + \mathbf{q} = \cos\theta + \mathbf{u}\sin\theta, \tag{7.3}$$

where $\mathbf{u} = \mathbf{q}/|\mathbf{q}|$, since $\sin^2\theta + \cos^2\theta \equiv 1$ for all θ, so $N(q) = 1$.

7.2 Rotation Sequences

Now that we have defined the basis of what a quaternion is and the necessary properties to apply them to spherical trigonometry, we can begin to relate them to rotations in the R^3 space. Since the product of complex numbers, which consist of the sum of a scalar and *one* imaginary part, results in a rotation of a vector in the plane, it makes sense that a rotation in 3-dimensional space will require a scalar part and *three* imaginary components: the product of quaternions.

By introducing the unit quaternion, Eq. (7.3), we introduced the cosine and sine of θ in order to ensure a length of unity, which gives us the perfect opportunity to relate θ to the angle of rotation. Kuipers (1999, p. 127–132) does just this, stating that the unit quaternion acting on a vector represents a rotation of the vector by an angle of 2θ about $\mathbf{q}$, the vector part of q. From this, we can say

$$q_{z,\theta} = \cos\frac{\theta}{2} + \mathbf{k}\sin\frac{\theta}{2}$$

represents a rotation of θ about the z-axis. Similarly for the y and x axes, we have

$$\begin{aligned} q_{y,\theta} &= \cos\frac{\theta}{2} + \mathbf{j}\sin\frac{\theta}{2}, \\ q_{x,\theta} &= \cos\frac{\theta}{2} + \mathbf{i}\sin\frac{\theta}{2}. \end{aligned} \tag{7.4}$$

Note the presence of θ in the subscript of q and $\theta/2$ in the equation. Here we have introduced a new notation, where the subscript of q tells us the axis of rotation as well as the angle. This will come in handy soon.

To denote a rotation sequence we will adopt another new notation, shown in Figure 7.1a. This represents a clockwise rotation about the z-axis by an

(a) A single rotation. (b) A rotation sequence.

Figure 7.1: Rotation notation.

angle of α, which we write using quaternions as $q_{z,\alpha}$. It takes us from the original coordinate frame to a new one. A sequence of rotations is then a string of these symbols, read from left to right, Figure 7.1b, with the corresponding sequence of quaternions also written in this order: $q_{z,\alpha}q_{y,\beta}$. This is an *open* rotation sequence, as the final coordinate frame is not the same as the initial frame.

A *closed* rotation sequence is one that, after transformation, takes us back to the original coordinate frame. Intuitively, an obvious way to do this is to do the inverse of the previous rotations, Figure 7.2. Rotating the coordinate

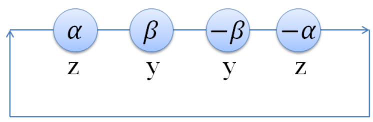


Figure 7.2: A closed rotation sequence.

frame by an angle of α about the z-axis, followed by an angle of β about the y-axis, to recover the original frame we must rotate by $-\beta$ about y then $-\alpha$

about z. Let us check this with quaternions. The inverse of a sequence of unit quaternions is equal to the complex conjugate. Taking the conjugate of a product reverses the order, so $(q_{z,\alpha}q_{y,\beta})^* = q^*_{y,\beta}q^*_{z,\alpha} = q_{y,-\beta}q_{z,-\alpha}$.

Closed rotation sequences can be started at any point in the sequence, as long as the order of rotations is maintained. It is because of this property that we can write a string of quaternions representing a closed rotation sequence equal to the identity. In the example of Figure 7.2, we write

$$q_{z,\alpha}q_{y,\beta}q^*_{y,\beta}q^*_{z,\alpha} = 1 \tag{7.5}$$

and, starting from the middle of the sequence,

$$q^*_{y,\beta}q^*_{z,\alpha}q_{z,\alpha}q_{y,\beta} = 1.$$

It is of great use to us to be able to split a rotation sequence up. This can be done very easily by multiplying a sequence on the right or left by the inverse of the last quaternion. Continuing with our closed loop example, we can multiply the LHS of Eq. (7.5) by $q^*_{z,\alpha}$, and the RHS by $q_{z,\alpha}$, giving

$$q_{y,\beta}q^*_{y,\beta} = q^*_{z,\alpha}q_{z,\alpha},$$

which we know is true as, for this example, both sides are equal to 1.

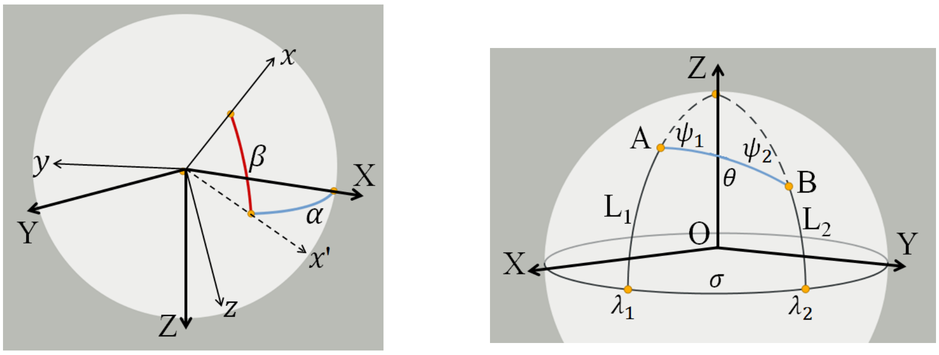
7.2.1 3-Dimensional Representation

Rotating a vector of constant length, with fixed origin, about a coordinate frame in 3-dimensional space maps out a circle, so when rotating normalized quaternions we are effectively drawing great circles on the surface of a unit sphere. The rotation sequence in Figure 7.1b looks like Figure 7.3a in 3-dimensional space. It is a transformation of the coordinate frame XYZ to xyz. Rotating about the Z-axis by angle α we arrive at the frame $x'yZ$, then a rotation of β about the y-axis takes us to the final coordinate frame of xyz. Here we have drawn two great circle arcs of length α and β.

It can be shown that the sequence in Figure 7.4 maps out an arbitrary great circle arc, shown in Figure 7.3b (Kuipers, 1999, p. 216–218). Here λ and L can be thought of as the longitude and latitude of the start and end points of the great circle arc: in this example, the arc goes from A at (λ_1, L_1) to B at (λ_2, L_2).

7.3 Deriving the Spherical Law of Sines

We are now ready to show how the use of quaternions and rotation sequences can give us spherical trigonometry results. We begin by mapping out a



(a) A rotation of α about the Z-axis followed by a rotation of β about the y-axis.

(b) Great circle path from A to B.

Figure 7.3: 3-dimensional rotation sequences.

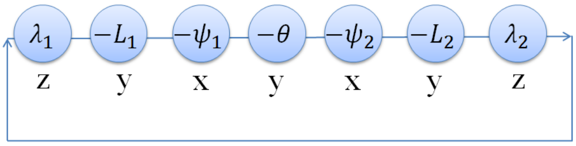


Figure 7.4: A great circle rotation sequence.

spherical triangle, Figure 7.5, shown on the surface of the unit sphere in Figure 7.6 (full details are shown in Kuipers (1999, p. 237–241)). This

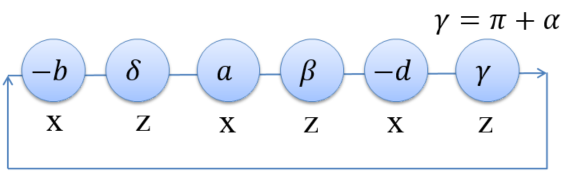


Figure 7.5: The rotation sequence for a spherical triangle.

produces the quaternion closed loop sequence of

$$q^*_{x,b} q_{z,\delta} q_{x,a} q_{z,\beta} q^*_{x,d} q_{z,\gamma} = 1,$$

which we can appropriately pre and post-multiply to give us

$$p = q_{z,\beta} q^*_{x,d} q_{z,\gamma} = q^*_{x,a} q^*_{z,\delta} q_{x,b} = r.$$

As we know how to write each quaternion given its axis and angle of rotation, Eq. (7.4), we can work out the components of p and r by expanding the

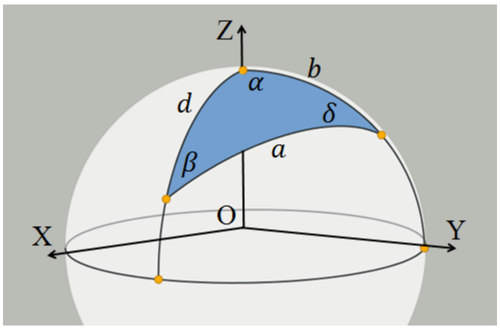


Figure 7.6: The spherical triangle of the rotation sequence in Figure 7.5.

following, using the quaternion product, Eq. (7.2):

$$\begin{aligned} p = p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3 &= q_{z,\beta} q^*_{x,d} q_{z,\gamma} \\ &= \left(\cos\frac{\beta}{2} + \mathbf{k}\sin\frac{\beta}{2}\right)\left(\cos\frac{d}{2} - \mathbf{i}\sin\frac{d}{2}\right)\left(\cos\frac{\gamma}{2} + \mathbf{k}\sin\frac{\gamma}{2}\right) \end{aligned}$$

and

$$\begin{aligned} r = r_0 + \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3 &= q^*_{x,a} q^*_{z,\delta} q_{x,b} \\ &= \left(\cos\frac{a}{2} - \mathbf{i}\sin\frac{a}{2}\right)\left(\cos\frac{\delta}{2} - \mathbf{k}\sin\frac{\delta}{2}\right)\left(\cos\frac{b}{2} + \mathbf{i}\sin\frac{b}{2}\right). \end{aligned}$$

We are now just a few steps away from obtaining our result. As p and r are equal, each of their components must be equal, so we can write $p_0 = r_0$, etc. With some algebra of the above equations for p and r, we get

$$p_0 = -\cos\frac{d}{2}\sin\frac{\alpha+\beta}{2} = \cos\frac{\delta}{2}\cos\frac{a-b}{2} = r_0, \tag{7.6}$$

$$p_1 = \sin\frac{d}{2}\sin\frac{\alpha-\beta}{2} = -\cos\frac{\delta}{2}\sin\frac{a-b}{2} = r_1, \tag{7.7}$$

$$p_2 = \sin\frac{d}{2}\cos\frac{\alpha-\beta}{2} = -\sin\frac{\delta}{2}\sin\frac{a+b}{2} = r_2, \tag{7.8}$$

$$p_3 = \cos\frac{d}{2}\cos\frac{\alpha+\beta}{2} = -\sin\frac{\delta}{2}\cos\frac{a+b}{2} = r_3. \tag{7.9}$$

Multiplying Eq. (7.7) by Eq. (7.9) and simplifying, we obtain

$$\sin d \sin\alpha + \sin d \sin\beta = \sin\delta \sin a + \sin\delta \sin b. \tag{7.10}$$

Similarly, multiplying Eq. (7.8) by Eq. (7.9), we get

$$\sin d \sin\alpha - \sin d \sin\beta = \sin\delta \sin a - \sin\delta \sin b. \tag{7.11}$$

Adding Eq. (7.10) and Eq. (7.11), then subtracting the two, we achieve our goal: the spherical Law of Sines

$$\frac{\sin d}{\sin\delta} = \frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta}.$$

Chapter 8

Conclusions

It is clear to see that the role of spherical trigonometry in the development of astronomy and Earth related measurements, as well as knowledge of the sphere, was crucial. As we have journeyed through ancient, medieval and modern times, we have touched upon some of the important applications of spherical trigonometry results, from calculating the position of the Sun to determining the direction of Mecca from any point on the surface of the Earth. Without the inquisitive minds of people who lived thousands of years ago, such as Hipparchus, trigonometric functions and knowledge and exploration of the heavens would not be as advanced as they are today.

We have seen how spherical trigonometry has grown and simplified over time, beginning in the first century with Menelaus' Theorem, a difficult theorem with six quantities to memorise and a less than obvious configuration to look for. Although it was revolutionary in gaining information of the positions of stars and planets, the invention of the Rule of Four Quantities from medieval 10th century mathematician Abū Naṣr simplified the work of astronomers greatly. This also lead to an efficient and accurate method of calculating the direction of Mecca, the qibla, from Ghazna by the work of al-Bīrūnī, which gave the rest of the Muslim world access to the qibla from their locations too. The Rule of Four Quantities is arguably the most important result of spherical trigonometry, as it also gave us the derivation of the spherical Law of Sines by Abū 'l-Wafā'—a result that finally related sides *and* angles of spherical triangles, a property results prior to this lacked. One thing all of these relations had in common was their extensive use of the sine function, requiring users of such relations to trawl through tables of sines to find their values, to then spend large proportions of time multiplying and dividing them—an arduous task without a calculator. This was the case up until the 16th century when Napier invented the logarithm in order to make calculations simpler. The first tables of logarithms in fact were not of pure numbers, but of logs of sines of angles. Finally we met the spherical Law of Cosines, a flexible result relating three sides and an angle of a spherical triangle, or conversely the spherical Law of Cosines for Angles, giving us the opposite. This result was ideal for navigation, enabling us to calculate distances on the surface of the Earth when our information is limited.

We then explored a completely different topic; Hamilton's quaternions. The ability to obtain spherical trigonometry results from modern areas of mathematics, as well as the publication of these, shows how important and versatile spherical trigonometry is.

It is equally important to note that the applications of spherical trigonometry did not die out when technology took over; it is still in use in sophisticated computer programming today.

In order to create a *sampling algorithm* in computer graphics, the Law of Cosines for Angles is applied. The problem arises from the angle subtended by a spherical polygon, and can only be solved by breaking the polygon up into spherical triangles. The use of stratified sampling, a measure of reducing variance, is applied to maintain a uniform distribution of random samples within the spherical triangles, giving the image a smoother appearance. (Arvo, 1995)

Of particular interest to us is animation, which combines the use of quaternions for their spherical applications to create 3-dimensional images. Spherical linear interpolation (SLERP) of quaternions is used to obtain the orientations given by a string of quaternions in order to rotate objects. This links directly to the rotation sequences we saw in Chapter 7. The use of quaternions and the SLERP approach, as opposed to linear interpolation, means that animations maintain a constant speed throughout their duration, as opposed to speeding up in the middle when using linear interpolation. (Barrera et al., 2004)

The application of spherical trigonometric results that were invented hundreds of years ago to things we see on a daily basis, such as computer graphics or animated films, shows how relevant the subject still is. The development of spherical trigonometry throughout the ages therefore plays a larger part in 21st century life than it is given credit for. Although it is now a fully described subject, with limited room for further development, the applications of spherical trigonometry are truly extensive.

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