Generalised ζ-functions

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Abstract

We briefly look at the Riemann zeta function and the more general Hurwitz zeta function. Using complex plane integration techniques we analytically continue the zeta function for the Bessel operator to the negative real axis, and then apply this method to calculate the zeta function determinant of the Laplacian on the 3-dimensional ball with Dirichlet boundary conditions.

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1 Introduction

Zeta function regularisation is a method that can be applied to assign finite values to otherwise divergent series or products. It was first introduced by Ray and Singer[1] and was later used by Hawking[2]. In particular it can be used to calculate the determinant of an otherwise divergent differential operator, L. If the operator has eigenvalues λ_i, then the determinant of L is given by

$$\det L = \prod_i \lambda_i. \tag{1.1}$$

In general this product is divergent, but to make sense of it we can form the generalised zeta function for the operator, defined as

$$\zeta(s) = \sum_i \lambda_i^{-s}. \tag{1.2}$$

This is equivalent to defining

$$\ln \det L = -\zeta'(0). \tag{1.3}$$

We can see this is the case by differentiating the generalised zeta function:

$$\begin{aligned} \zeta'(s) &= -\sum_i \lambda_i^{-s} \ln \lambda_i \\ \implies \zeta'(0) &= -\sum_i \ln \lambda_i \\ &= -\ln \det L. \end{aligned}$$

The generalised zeta function must then be analytically continued in the exponent s to the point $s = 0$, in order to evaluate the determinant. We will see how this method can be used to calculate the functional determinant of the Laplace operator on the 3-dimensional ball with Dirichlet boundary conditions.

We will also look at the zeta function for the Bessel operator and the analytic continuation to the negative real axis, and use this to calculate the residues of the zeta function at negative odd integers and its finite values at negative even integers.

1.1 Analytic Continuation

Analytic continuation is a technique to extend the domain over which a function is defined. Consider two functions, f_1 and f_2, with f_1 defined over some domain

Ω_1 and f_2 defined over some domain Ω_2. If $f_1 = f_2$ on the intersection of these domains then we say f_2 is the analytic continuation of f_1 over Ω_2 and vice versa. Furthermore if the analytic continuation of a function exists, then it is unique. To understand the process we can look at a few simple examples.

Consider the function

$$f(s) = \sum_{n=0}^{\infty} s^n. \tag{1.4}$$

This converges when $|s| < 1$ to $1/(1-s)$. However the function

$$g(s) = \frac{1}{1-s} \tag{1.5}$$

is analytic everywhere excluding $s = 1$. We know that $f(s) = g(s)$ on the disk $|s| < 1$, therefore $g(s)$ can be viewed as the analytic continuation of $f(s)$ to all s, excluding $s = 1$.

The Gamma function is defined as

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} \, \mathrm{d}t, \quad \mathrm{Re}(s) > 0. \tag{1.6}$$

This integral definition converges for $\mathrm{Re}(s) > 0$ but it can be analytically continued to all s. We can show this by finding the functional equation $\Gamma(s)$ satisfies. Consider $\Gamma(s+1)$:

$$\begin{aligned}
\Gamma(s+1) &= \int_0^{\infty} t^s e^{-t} \, \mathrm{d}t \\
&= \left[-t^s e^{-t}\right]_0^{\infty} + \int_0^{\infty} s t^{s-1} e^{-t} \, \mathrm{d}t \\
&= s\Gamma(s) \\
\implies \Gamma(s+1) &= s\Gamma(s). \qquad (1.7)
\end{aligned}$$

We can then use equation (1.7) to find the values of the Gamma function for $-1 < \mathrm{Re}(s) \leq 0$, with a simple pole at $s = 0$. By repeating this step infinitely many times we find the analytic continuation for the Gamma function on the complex plane, with simple poles at negative integers.

2 Preliminaries

2.1 Hurwitz ζ-Function

The Hurwitz ζ-function is defined by

$$\zeta_{\mathcal{H}}(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}, \quad \Re(s) > 1,\ \Re(a) > 0. \tag{2.1}$$

It is possible to obtain an integral representation for $\zeta(z,a)$ using the gamma function. Starting from equation (1.6), we make the change of variable $t = (n+a)x$, resulting in

$$\Gamma(s) = (n+a)^s \int_0^{\infty} e^{-(n+a)x} x^{s-1}\, dx. \tag{2.2}$$

Then by rearranging this and summing over n we get

$$\sum_{n=0}^{\infty}(n+a)^{-s}\Gamma(s) = \sum_{n=0}^{\infty}\int_0^{\infty} e^{-(n+a)x} x^{s-1}\, dx. \tag{2.3}$$

The left-hand side of this equation is simply the Hurwitz ζ-function multiplied by the gamma function, and by interchanging the summation and the integral on the right-hand side and rearranging we obtain

$$\begin{aligned}\zeta_{\mathcal{H}}(s,a) &= \frac{1}{\Gamma(s)}\int_0^{\infty} e^{-ax} x^{s-1}\sum_{n=0}^{\infty} e^{-nx}\, dx \\ &= \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{e^{-ax}x^{s-1}}{1-e^{-x}}\, dx. \end{aligned} \tag{2.4}$$

It is possible to evaluate $\zeta(s,a)$ for certain values of s. For example when s is a negative integer we have[3]

$$\zeta_{\mathcal{H}}(-n,a) = -\frac{B_{n+1}(a)}{n+1}, \quad n = 0,1,2,\ldots, \tag{2.5}$$

where $B_{n+1}(a)$ is a Bernoulli polynomial.

We also require the derivative of $\zeta_{\mathcal{H}}(s,a)$ with respect to the second argument

a. This is a shift in the first argument:

$$\begin{aligned}\frac{\partial}{\partial a}\zeta_{\mathcal{H}}(s,a) &= \frac{\partial}{\partial a}\sum_{l=0}^{\infty}(l+a)^{-s} \\ &= -s\sum_{l=0}^{\infty}(l+a)^{-s-1} \\ &= -s\zeta_{\mathcal{H}}(s+1,a). \end{aligned} \tag{2.6}$$

2.1.1 Representing $\zeta_{\mathcal{H}}(s,a)$ as a Contour Integral

To represent equation (2.4) as a contour integral, we consider it as a function of a complex variable t, which is analytic at all points except $t=0$, where there is a branch point. The contour γ must therefore enclose this point. We will take the branch cut along the negative real axis, shown in figure 1.

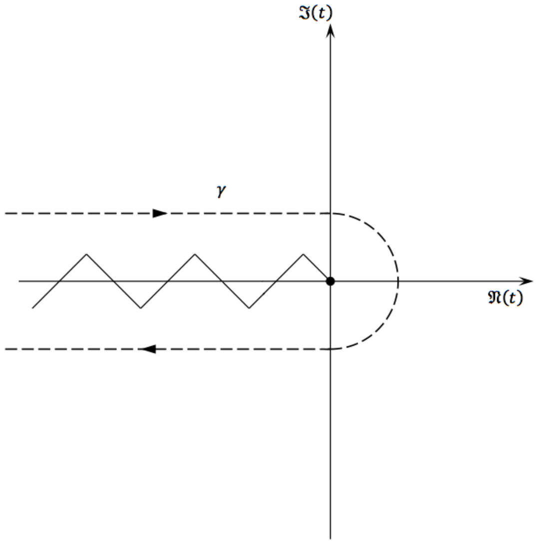

Figure 1: Figure showing the branch cut along the negative real axis, and the contour γ.

There will be a phase change of 2π as we circle the origin, therefore above the branch cut we have $t = e^{\mathrm{i}\pi}\tau$ where $\tau \in [\infty, 0]$, and below the branch cut we have $t = e^{-\mathrm{i}\pi}\tau$ where $\tau \in [0, \infty]$. Then splitting up the integral into two parts, one above the branch cut and one below, we have

$$\int_\gamma \frac{t^{s-1}e^{-at}}{1-e^{-t}}\,dt = \int_\infty^0 \frac{e^{a\tau}(e^{\mathrm{i}\pi}\tau)^{s-1}}{1-e^{\tau}}\,d\tau + \int_0^\infty \frac{e^{a\tau}(e^{-\mathrm{i}\pi}\tau)^{s-1}}{1-e^{\tau}}\,d\tau$$
$$= \int_0^\infty \frac{e^{-\mathrm{i}\pi s}\tau^{s-1}e^{a\tau}}{1-e^{\tau}}\,d\tau - \int_0^\infty \frac{e^{\mathrm{i}\pi s}\tau^{s-1}e^{a\tau}}{1-e^{\tau}}\,d\tau. \tag{2.7}$$

Then replacing a with $1-a$ and using the fact that $2\mathrm{i}\sin x = e^{\mathrm{i}x} - e^{-\mathrm{i}x}$ we can write this as

$$\int_\gamma \frac{t^{s-1}e^{(a-1)t}}{1-e^{-t}}\,dt = -2\mathrm{i}\sin\pi s \int_0^\infty \frac{\tau^{s-1}e^{(1-a)\tau}}{1-e^{\tau}}\,d\tau$$
$$= 2\mathrm{i}\sin\pi s \int_0^\infty \frac{\tau^{s-1}e^{-a\tau}}{1-e^{-\tau}}\,d\tau. \tag{2.8}$$

Therefore using equation (2.4) we have

$$\zeta_{\mathcal{H}}(s,a) = \frac{1}{2\mathrm{i}\sin\pi s\,\Gamma(s)} \int_\gamma \frac{t^{s-1}e^{(a-1)t}}{1-e^{-t}}\,dt. \tag{2.9}$$

2.2 Riemann ζ-function

2.2.1 Definition

The Riemann ζ-function is a special case of the Hurwitz ζ-function, obtained by setting $a = 1$. It is defined by

$$\zeta(s) = \zeta_{\mathcal{H}}(s,1) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1, \tag{2.10}$$

and its analytic continuation to all s. Although this function was introduced by Leonhard Euler as a function of a real argument, Bernhard Riemann was the first to study it as a function of a complex argument in his 1859 article *"On the Number of Primes Less Than a Given Magnitude"*.

The analytic continuation of the ζ-function can be found via its functional equation, as with the gamma function. In his paper Riemann showed that $\zeta(s)$ satisfied the following[4]:

$$\Lambda(s) = \Lambda(1-s), \tag{2.11}$$

where

$$\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{2.12}$$

2.2.2 Key Properties

The values of $\zeta(s)$ at negative integers can be expressed using equation (2.5). Setting $a = 1$ we have

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}, \tag{2.13}$$

where B_{n+1} is a Bernoulli number. The first few Bernoulli numbers are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42}. \tag{2.14}$$

For all odd integers n excluding $n = 1$, $B_n = 0$. Then equation (2.13) tells us that $\zeta(s)$ vanishes at even negative integers. These are known as the trivial zeros of the Riemann zeta function.

If we consider the divergent series $1 + 2 + 3 + 4 + \ldots$, we see that this is the series obtained by setting $s = -1$ in equation (2.10). However we can now assign a finite result to this series using (2.13), and we obtain

$$\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}. \tag{2.15}$$

2.2.3 The Riemann Hypothesis

The Riemann hypothesis concerns the non-trivial zeros of $\zeta(s)$. Proposed by Riemann in 1859[4], it states that the real part of every non-trivial zero of the Riemann zeta function is equal to 1/2. We can show that every non-trivial zero s satisfies $0 < \Re(s) < 1$, called the critical line.

The Riemann zeta function can also be expressed as the following infinite product[5]:

$$\begin{aligned} \zeta(s) &= \prod_{p,\ prime} \left(\frac{1}{1-p^{-s}} \right) \qquad (2.16) \\ &= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \ldots \cdot \frac{1}{1-p^{-s}} \cdot \ldots, \end{aligned}$$

and as this is a convergent infinite product of non-zero factors we can conclude that $\zeta(s)$ has no zeros for $\Re(s) > 1$. It can also be shown that $\zeta(s)$ satisfies the following functional equation[5]:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{2.17}$$

From this we can identify the trivial zeros of $\zeta(s)$ at the even negative integers. We can also see that for any non-trivial zero s then $1 - s$ is also a solution to

$\zeta(s) = 0$. However we already know that there are no zeros with real part greater than 1, therefore all non-trivial zeros lie in the critical strip $0 < \Re(s) < 1$.

The Riemann hypothesis remains one of the most important unsolved problems in mathematics, and is one of the Clay Mathematics Institute's Millennium Prize Problems. It implies results about the distribution of prime numbers; the prime number theorem approximates how many primes there are less than a given number, and the Riemann hypothesis tells us how far away from the true value this is.

3 Zeta Function of the Bessel Operator on the Negative Real Axis

In this section we will look at the generalised ζ-function formed from the zeros of the Bessel function. By finding the analytic continuation of the ζ-function on the negative real axis we can calculate its values at negative even integers, and although the calculation of its values at odd negative integers is complicated, the residue at these values can be found quite simply.

3.1 Introduction to Bessel Functions

Bessel functions are the solutions to Bessel's differential equation, given by[6]

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0, \tag{3.1}$$

where ν is called the order of the Bessel function. One solution to this equation can be found using the method of Frobenius, and is called the Bessel function of the first kind, of order ν:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{3.2}$$

This series converges for all $x \geq 0$. Bessel functions of the second kind, denoted Y_ν, diverge as $x \to 0$.

3.2 Definition of the Generalised ζ-Function

In this section we will look at the ζ-function defined by

$$\zeta_\nu(s) = \sum_{n=1}^{\infty} j_{\nu n}^{-s}, \quad \Re(s) > 1, \tag{3.3}$$

where $j_{\nu n}$ is the n^{th} zero of the Bessel function J_ν for arbitrary real ν.

It can be shown that the values of $\zeta_\nu(s)$ at positive even integers satisfies the following recursive law, called the linear law[7]:

$$\frac{1}{4n!(n+\nu+1)!} = \sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{(n-k)!(n-k+\nu)!} \zeta_\nu(2k+2), \quad n \geq 0. \tag{3.4}$$

We will use this to calculate the values of $\zeta_\nu(s)$ for $s = 2, 4$ and 6. By setting $n = 0$ we obtain an expression for $\zeta_\nu(2)$:

$$\zeta_\nu(2) = \frac{\nu!}{4(\nu+1)!} = \frac{1}{4(\nu+1)}. \tag{3.5}$$

Now setting $n = 1$ gives us an expression for $\zeta_\nu(4)$ in terms of $\zeta_\nu(2)$:

$$\frac{1}{4(\nu+2)!} = \frac{1}{(1+\nu)!}\zeta_\nu(2) - \frac{4}{\nu!}\zeta_\nu(4). \tag{3.6}$$

Substituting in (3.5) and rearranging we find

$$\zeta_\nu(4) = \frac{1}{2^4(\nu+1)^2(\nu+2)}. \tag{3.7}$$

Finally by setting $n = 2$ and substituting in our expressions for $\zeta_\nu(2)$ and $\zeta_\nu(4)$ we find that

$$\zeta_\nu(6) = \frac{1}{2^5(3+\nu)(2+\nu)(1+\nu)^3}. \tag{3.8}$$

3.3 Analytic Continuation of the Zeta Function

However we are interested in the behaviour of $\zeta_\nu(s)$ on the negative part of the real axis, and in this section we will find the analytic continuation of this function in s to the left. We would first like to express $\zeta_\nu(s)$ as a contour integral, and this is done by applying the residue theorem.

Consider a function $F(z)$, with simple zeros at $z = z_n$, $n = 0, 1, 2, \ldots$. Then by expanding $F(z)$ about each $z = z_n$ we can write

$$\frac{F'(z)}{F(z)} \approx \frac{1}{z - z_n}. \tag{3.9}$$

Then, assuming the contour, γ, contains all values z_n but no poles of a second function $G(z)$, we can use the residue theorem to state the following:

$$\int_\gamma G(z)\frac{F'(z)}{F(z)}\,dz = 2\pi\mathrm{i}\sum_n G(z_n). \tag{3.10}$$

Using this method we have the following contour integral representation for $\zeta_\nu(s)$:

$$\zeta_\nu(s) = -\frac{1}{2\pi\mathrm{i}}\int_{C_\varepsilon}\frac{J_\nu'(t)}{J_\nu(t)}t^{-s}\,dt, \tag{3.11}$$

where the negative sign comes from having reversed the contour[7]. However we can simplify this by noticing that

$$\frac{J_\nu'(t)}{J_\nu(t)} = \frac{d}{dt} \ln J_\nu(t). \tag{3.12}$$

This allows us to write equation (3.11) as

$$\begin{aligned}
\zeta_\nu(s) &= \frac{\mathrm{i}}{2\pi} \int_{C_\varepsilon} t^{-s} \frac{d}{dt} \left[\ln J_\nu(t)\right] \, dt \\
&= \frac{\mathrm{i}}{2\pi} \int_{C_\varepsilon} \left[\frac{d}{dt} \left(t^{-s} \ln J_\nu(t)\right) + st^{-s-1} \ln J_\nu(t)\right] \, dt \\
&= \frac{\mathrm{i}s}{2\pi} \int_{C_\varepsilon} t^{-s-1} \ln J_\nu(t) \, dt. \qquad (3.13)
\end{aligned}$$

Figure 2 shows the contour C_ε, where $0 < \varepsilon < j_{\nu 1}$ and $R \to \infty$. Call the part of the contour above the real axis $C_{+(\varepsilon)}$ and the part below $C_{-(\varepsilon)}$.

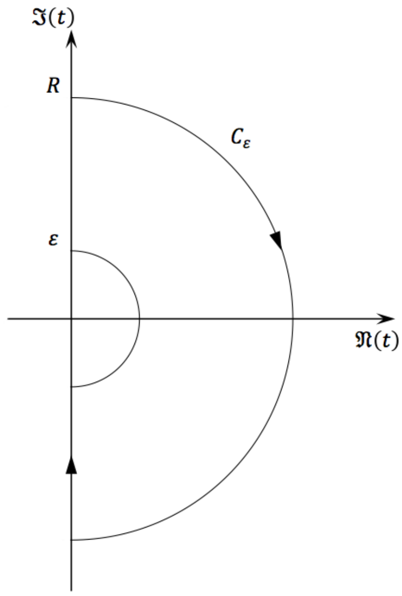

Figure 2: Figure showing the contour that is used: two arcs of circumference centred at the origin, one of radius R and one of radius ε, and two radial lines connecting them.

The behaviour of the Bessel function J_ν along $C_{\pm(\varepsilon)}$ is given by[7]

$$J_\nu(t) = \frac{1}{\sqrt{2\pi t}} e^{\mp \mathrm{i}t} e^{\pm \mathrm{i}(\nu+1/2)\pi/2} \left[1 + O\left(\frac{1}{t}\right)\right]. \tag{3.14}$$

Therefore

$$\sqrt{2\pi t}\; e^{\pm[t-(\nu+1/2)\pi/2]} J_\nu(t) \sim 1 + O\left(\frac{1}{t}\right), \tag{3.15}$$

and by rewriting equation (3.13) as

$$\begin{aligned}\zeta^{(\pm)}_{\nu\varepsilon}(s) = \frac{\mathrm{i}s}{2\pi}\Bigg\{ &\int_{C_{\pm(\varepsilon)}} t^{-s-1} \ln\left[\sqrt{2\pi t}\; e^{\pm \mathrm{i}[t-(\nu+1/2)\pi/2]} J_\nu(t)\right] dt \\ &- \int_{C_{\pm(\varepsilon)}} t^{-s-1} \ln\left[\sqrt{2\pi t}\; e^{\pm \mathrm{i}[t-(\nu+1/2)\pi/2]}\right] dt \Bigg\}\end{aligned} \tag{3.16}$$

we have removed the divergent behaviour of the integrand as $|t| \to \infty$, when $\Re(s) > -1$ in the first term. However the second term is still only analytical for $\Re(s) > 1$[7].

First we will consider the second term, which is comprised of two integrals along $C_{+(\varepsilon)}$ and $C_{-(\varepsilon)}$. It is clear that the integrand vanishes for $|t| \to \infty$, therefore only the section of the contour running along the imaginary axis will contribute to the integrals. Then adding together the contributions from the two contours we have[7]

$$\begin{aligned}\zeta_\nu(s) &= \zeta^{(+)}_{\nu\varepsilon}(s) + \zeta^{(-)}_{\nu\varepsilon}(s) \\ &= \frac{\mathrm{i}s}{2\pi} \sum_{(\pm)} \int_{C_{\pm(\varepsilon)}} t^{-s-1} \ln\left[\sqrt{2\pi t}\; e^{\pm \mathrm{i}[t-(\nu+1/2)\pi/2]} J_\nu(t)\right] dt \\ &\qquad + \frac{\mathrm{i}s}{2\pi}\left[\frac{\mathrm{i}\varepsilon^{1-s}}{1-s} + \mathrm{i}\left(\nu + \frac{1}{2}\right)\frac{\pi}{2s}\varepsilon^{-s}\right].\end{aligned} \tag{3.17}$$

We can see from this that $\zeta_\nu(s)$ has a simple pole at $s = 1$, with residue $1/\pi$. Also at $s = 0$, there is only one term which does not vanish, and so we have $\zeta_\nu(0) = -1/2(\nu + 1/2)$.

In the limit $\varepsilon \to 0$, the final term vanishes and we are left with the sum over the two integrals. We will restrict s to the strip $-1 < \Re(s) < 0$. There is a branch point located at $t = 0$, and we will take the branch cut along the negative real axis, as shown in figure 3. There will be a phase change of π moving from $\Im(t) < 0$ to $\Im(t) > 0$, therefore we will make the following change of variables:

$$\begin{aligned} t &= e^{\frac{\mathrm{i}\pi}{2}}\rho \quad \text{on} \quad C_+, \\ t &= e^{-\frac{\mathrm{i}\pi}{2}}\rho \quad \text{on} \quad C_-. \end{aligned} \tag{3.18}$$

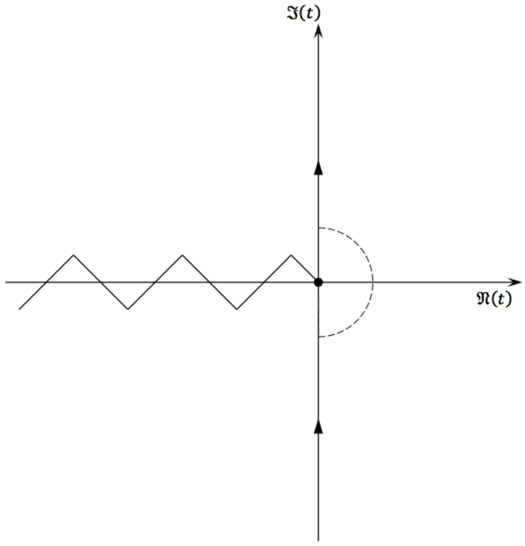

Figure 3: Figure showing the branch cut along the negative real axis.

Then in the limit as $\varepsilon \to 0$ equation (3.17) becomes

$$\begin{aligned}
\zeta_\nu(s) = \frac{\mathrm{i}s}{2\pi} \Bigg\{ & \int_0^\infty \left(e^{\mathrm{i}\pi/2}\rho\right)^{-s-1} \ln\left[\sqrt{2\pi\mathrm{i}\rho}\, e^{\mathrm{i}[\mathrm{i}\rho-(\nu+1/2)\pi/2]} J_\nu(\mathrm{i}\rho)\right] e^{\mathrm{i}\pi/2}\, d\rho \\
& - \int_0^\infty \left(e^{-\mathrm{i}\pi/2}\rho\right)^{-s-1} \ln\left[\sqrt{-2\pi\mathrm{i}\rho}\, e^{-\mathrm{i}[-\mathrm{i}\rho-(\nu+1/2)\pi/2]} J_\nu(-\mathrm{i}\rho)\right] e^{-\mathrm{i}\pi/2}\, d\rho \Bigg\} \\
= \frac{\mathrm{i}s}{2\pi} \Bigg\{ & \int_0^\infty e^{-\mathrm{i}\pi s/2} \rho^{-s-1} \ln\left[\sqrt{2\pi\rho}\, e^{-t}\, \mathrm{i}^{-\nu} J_\nu(\mathrm{i}\rho)\right]\, d\rho \qquad (3.19) \\
& - \int_0^\infty e^{\mathrm{i}\pi s/2} \rho^{-s-1} \ln\left[\sqrt{2\pi\rho}\, e^{-t}\, \mathrm{i}^{\nu} J_\nu(-\mathrm{i}\rho)\right]\, d\rho \Bigg\}.
\end{aligned}$$

Defining the modified Bessel function, $I_\nu(k)$, as

$$\mathrm{i}^\nu J_\nu(-\mathrm{i}k) = i^{-\nu} J_\nu(\mathrm{i}k) = I_\nu(k), \qquad (3.20)$$

and recalling that

$$e^{\mathrm{i}\pi s} - e^{-\mathrm{i}\pi s} = 2\mathrm{i}\sin(\pi s) \qquad (3.21)$$

means that we can express $\zeta_\nu(s)$ as

$$\zeta_\nu(s) = \frac{s}{\pi} \sin\left(\frac{\pi s}{2}\right) \int_0^\infty \rho^{-s-1} \ln\left[\sqrt{2\pi\rho}\, e^{-\rho} I_\nu(\rho)\right]\, d\rho, \quad -1 < \Re(s) < 0. \quad (3.22)$$

We would now like to continue this in z to the left of the real axis. We will define the function $G(\rho)$ to be

$$G(\rho) = \ln\left[\sqrt{2\pi \mathrm{i}}\, e^{-\rho} I_\nu(\rho)\right]. \tag{3.23}$$

Then

$$\rho^2 G'' + \rho^2 (G')^2 + 2\rho^2 G' + \frac{1}{4} - \nu^2 = 0. \tag{3.24}$$

Proof: We need to find the first and second derivatives of G with respect to ρ. First we have

$$\begin{aligned} G' &= \frac{\sqrt{2\pi}\rho^{-1/2} I_\nu(\rho) - 2\sqrt{2\pi\rho} I_\nu(\rho) + 2\sqrt{2\pi\rho} I'_\nu(\rho)}{2\sqrt{2\pi\rho} I_\nu(\rho)} \\ &= \frac{I_\nu(\rho) - 2\rho I_\nu(\rho) + 2\rho I'_\nu(\rho)}{2\rho I_\nu(\rho)} \end{aligned} \tag{3.25}$$

$$\implies (G')^2 = \frac{I_\nu^2(\rho)(1 - 4\rho + 4\rho^2) + I_\nu(\rho) I'_\nu(\rho)(4\rho - 8\rho^2) + 4\rho^2 (I'_\nu(\rho))^2}{4\rho^2 I_\nu^2(\rho)}. \tag{3.26}$$

Differentiating once more we obtain

$$G'' = \frac{-2I_\nu^2(\rho) + 4\rho^2 I_\nu(\rho) I''_\nu(\rho) - 4\rho^2 (I'_\nu(\rho))^2}{4\rho^2 I_\nu^2(\rho)}. \tag{3.27}$$

Therefore we have

$$\begin{aligned} \rho^2 G'' + \rho^2 (G')^2 + 2\rho^2 G' + \frac{1}{4} - \nu^2 &= \frac{-2I_\nu^2(\rho) + 4\rho^2 I_\nu(\rho) I''_\nu(\rho) - 4\rho^2 (I'_\nu(\rho))^2}{4I_\nu^2(\rho)} \\ &+ \frac{I_\nu^2(\rho)(1 - 4\rho + 4\rho^2) + I_\nu(\rho) I'_\nu(\rho)(4\rho - 8\rho^2) + 4\rho^2 (I'_\nu(\rho))^2}{4I_\nu^2(\rho)} \\ &+ \frac{\rho I_\nu(\rho) - 2\rho^2 I_\nu(\rho) + 2\rho^2 I'_\nu(\rho)}{I_\nu(\rho)} + \frac{1}{4} - \nu^2 \\ &= 4I_\nu(\rho) \frac{\rho^2 I''_\nu(\rho) + \rho I'_\nu(\rho) - (\rho^2 + \nu^2) I_\nu(\rho)}{4I_\nu^2(\rho)}. \end{aligned} \tag{3.28}$$

However, the modified Bessel function, $I_\nu(\rho)$, satisfies the following differential equation[6]:

$$\rho^2 I''_\nu(\rho) + \rho I'_\nu(\rho) - (\rho^2 + \nu^2) I_\nu(\rho) = 0. \tag{3.29}$$

Therefore
$$\rho^2 G'' + \rho^2 (G')^2 + 2\rho^2 G' + \frac{1}{4} - \nu^2 = 0,$$
as required.

As $\rho \to \infty$, the function $G(\rho)$ has an asymptotic expansion of the form[7]
$$G(\rho) \sim \sum_{n=1}^{\infty} a_n \rho^{-n}. \tag{3.30}$$

Substituting this into equation (3.24) will enable us to find expressions for the coefficients a_n. Differentiating we obtain
$$G' = -\sum_{n=1}^{\infty} n a_n \rho^{-n-1}, \tag{3.31}$$

$$G'' = \sum_{n=1}^{\infty} n(n+1) a_n \rho^{-n-2}. \tag{3.32}$$

For $(G')^2$ we then have
$$(G')^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nm a_n a_m \rho^{-n-m-2}. \tag{3.33}$$

To write this as a single summation let $k = n + m$, for $k = 2, 3, 4, \ldots$, then we have $m = k - n$, for $1 \leq n \leq k - 1$. We can then write $(G')^2$ as
$$\begin{aligned} (G')^2 &= \sum_{k=2}^{\infty} \sum_{n=1}^{k-1} n(k-n) a_n a_{k-n} \rho^{-k-2} \\ &= \sum_{k=2}^{\infty} b_k \rho^{-k-2}, \quad \text{where} \quad b_k = \sum_{n=1}^{k-1} n(k-n) a_n a_{k-n}. \end{aligned} \tag{3.34}$$

Substituting these into equation (3.24) we obtain
$$\sum_{n=1}^{\infty} n(n+1) a_n \rho^{-n} + \sum_{k=2}^{\infty} b_k \rho^{-k} - 2 \sum_{n=1}^{\infty} n a_n \rho^{-n+1} + \frac{1}{4} - \nu^2 = 0. \tag{3.35}$$

Relabelling indices to allow us to collect powers of ρ together, we have
$$\sum_{n=1}^{\infty} n(n+1) a_n \rho^{-n} + \sum_{n=2}^{\infty} b_n \rho^{-n} - 2 \sum_{n=0}^{\infty} a_{n+1}(n+1) \rho^{-n} + \frac{1}{4} - \nu^2 = 0. \tag{3.36}$$

By evaluating the $n = 0$ term we can obtain a_1, and we find

$$
\begin{aligned}
-2a_1 + \frac{1}{4} - \nu^2 &= 0 \\
\implies a_1 &= \frac{1}{2}\left(\frac{1}{4} - \nu^2\right).
\end{aligned}
\tag{3.37}
$$

The $n = 1$ term then gives us an expression for a_2 in terms of a_1, which we can substitute in:

$$
\begin{aligned}
2a_1 - 4a_2 &= 0 \\
\implies a_2 &= \frac{1}{4}\left(\frac{1}{4} - \nu^2\right).
\end{aligned}
\tag{3.38}
$$

For all other terms we have

$$
\sum_{n=2}^{\infty}[n(n+1)a_n + b_n - 2a_{n+1}(n+1)]\rho^{-n} = 0. \tag{3.39}
$$

This gives us the following recursion relation:

$$
2(n+1)a_{n+1} = \sum_{k=1}^{n-1} k(n-k)a_k a_{n-k} + n(n+1)a_n, \quad n \geq 2. \tag{3.40}
$$

Deleting and adding a term $\varPhi(\rho - 1)\sum_{k=1}^{p} a_k \rho^{-k}$ to the $G(\rho)$ in the integrand in equation (3.22) gives us

$$
\begin{aligned}
\zeta_\nu(s) = \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Bigg\{\int_0^\infty \rho^{-s-1}\left[G(\rho) - \varPhi(\rho-1)\sum_{k=1}^{p} a_k\rho^{-k}\right] d\rho \\
+ \int_0^\infty \rho^{-s-1}\,\varPhi(\rho-1)\sum_{k=1}^{p} a_k\rho^{-k}\, d\rho\Bigg\}.
\end{aligned}
\tag{3.41}
$$

where $\varPhi(\rho - 1)$ denotes the unit step function switching on at $\rho = 1$. Doing so removes the poles in the integrand, therefore the integral is now convergent.

Performing the integration in the second term gives

$$\begin{aligned}
\frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\int_0^\infty \rho^{-s-1}\Phi(\rho-1)\sum_{k=1}^{p} a_k\rho^{-k}\ d\rho &= \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\int_1^\infty \rho^{-s-1}\sum_{k=1}^{p} a_k\rho^{-k}\ d\rho \\
&= \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\sum_{k=1}^{p} a_k\left[-\frac{\rho^{-s-k}}{s+k}\right]_1^\infty \\
&= \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\sum_{k=1}^{p}\frac{a_k}{s+k}. \qquad (3.42)
\end{aligned}$$

Therefore we now have our final expression for $\zeta_\nu(s)$:

$$\begin{aligned}
\zeta_\nu(s) = \frac{s}{\pi}\sin\left(\frac{\pi}{2}s\right)\Bigg\{\int_0^\infty \rho^{-s-1}\left[G(\rho)-\Phi(\rho-1)\sum_{k=1}^{p} a_k\rho^{-k}\right]\ d\rho \qquad (3.43) \\
+\sum_{k=1}^{p}\frac{a_k}{s+k}\Bigg\}.
\end{aligned}$$

3.4 The ζ-Function at the Negative Integers

Equation (3.43) is valid for $-1 < \Re(s) < 0$, but it gives an explicit analytic continuation to $-p-1 < \Re(s) < 0$. We will first consider the values at even negative integers. At $s = -2l$ where $l \in \mathbb{N}^*$, we have $\sin(\pi s/2) = 0$, therefore the first term vanishes. Letting $s \to s+\varepsilon$ and taking the limit as $\varepsilon \to 0$ for the second term we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left\{\frac{\varepsilon-2l}{\pi}\sin\left[\frac{\pi(\varepsilon-2l)}{2}\right]\sum_{k=1}^{p}\frac{a_k}{\varepsilon-2l+k}\right\} \\
&\quad= \lim_{\varepsilon\to 0}\left\{\frac{\varepsilon-2l}{\pi}\left[\sin(-l\pi)+\varepsilon\frac{\pi}{2}\cos\left(\frac{\pi(\varepsilon-2l)}{2}\right)+O(\varepsilon^2)\right]\sum_{k=1}^{p}\frac{a_k}{\varepsilon-2l+k}\right\} \\
&\quad= -\frac{2l}{\pi}\left[\frac{\pi}{2}\cos(-l\pi)a_{2l}\right] \\
&\quad= -(-1)^l l a_{2l}. \qquad (3.44)
\end{aligned}$$

Therefore

$$\zeta_\nu(-2l) = -(-1)^l l a_{2l}, \qquad l \in \mathbb{N}^*. \qquad (3.45)$$

At negative odd integers, $s = -(2l+1)$ where $l \in \mathbb{N}$, the integral in equation (3.43) is well defined but difficult to evaluate. It is easier to work out the residue at these values:

$$\begin{aligned}\mathrm{Res}[\zeta_\nu(s), s = -(2l+1)] &= -\frac{(2l+1)}{\pi}(-1)^{l-1} a_{2l+1} \\ &= \frac{(2l+1)}{\pi}(-1)^{l} a_{2l+1}. \end{aligned} \tag{3.46}$$

4 Zeta Function Determinant of the Laplace Operator on the 3-Dimensional Ball

As we have already seen, when the determinant of an operator is divergent we can still evaluate it if it is possible to analytically continue the generalised ζ-function of that operator. In this section we will look at the generalised ζ-function formed from the eigenvalues of the Laplace operator, on Dirichlet boundary conditions.

4.1 Solving Laplace's Equation in Spherical Coordinates

We will consider the operator $(-\nabla^2 + m^2)$ on the 3-dimensional ball. The eigenvalues λ_k are therefore determined from the following equation:

$$(-\nabla^2 + m^2)\phi_k(x) = \lambda_k \phi_k(x), \tag{4.1}$$

or

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) - \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) - \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} + (m^2 - \lambda)\phi = 0. \tag{4.2}$$

Consider a seperable solution of the form

$$\phi = f(r)g(\theta)h(\varphi). \tag{4.3}$$

Substituting this into equation (4.2) gives us the following, where $'$ denotes the derivative with respect to the relevant argument:

$$-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 ghf') - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta fhg') - \frac{1}{r^2\sin^2\theta}fgh'' + (m^2-\lambda)fgh = 0. \tag{4.4}$$

We can simplify this slightly by applying the product rule, resulting in the following equation:

$$-\frac{2}{r}ghf' - ghf'' - \frac{\cos\theta}{r^2\sin\theta}fhg' - \frac{1}{r^2}fhg'' - \frac{1}{r^2\sin^2\theta}fgh'' + (m^2-\lambda)fgh = 0. \tag{4.5}$$

Dividing through by ϕ/r^2 ($= fgh/r^2$) and rearranging to collect terms involving r on one side gives

$$\frac{1}{f}[r^2 f'' + 2rf' - r^2 f(m^2-\lambda)] = -\frac{1}{g}\left(\frac{\cos\theta}{\sin\theta}g' + g''\right) - \frac{1}{h\sin^2\theta}h''. \tag{4.6}$$

Now, as the LHS is dependent on r only and the RHS dependent on θ and φ only we can set both sides of this equation equal to a separation constant, $l(l+1)$. Concentrating first on the r dependence we are left with the equation

$$r^2 f'' + 2rf' - f[r^2m^2 - r^2\lambda + l(l+1)] = 0. \tag{4.7}$$

To solve this we look for solutions of the form

$$f = r^{-\frac{1}{2}}\psi(r). \tag{4.8}$$

After substituting this into equation (4.7) we are left with

$$r^2\psi'' + r\psi' + \psi\left[r^2(\lambda - m^2) - \left(l + \frac{1}{2}\right)^2\right] = 0. \tag{4.9}$$

This equation is actually Bessel's equation (3.1), and we can see this more clearly by making the change of variables

$$r' = r\omega,\ \omega = \sqrt{\lambda - m^2}. \tag{4.10}$$

Doing so reduces equation (4.9) to

$$(r')^2\psi'' + r'\psi' + \psi\left[(r')^2 - \left(l + \frac{1}{2}\right)^2\right] = 0, \tag{4.11}$$

with solutions

$$\psi_{l,n} = J_{l+1/2}(r'_{l,n}) = J_{l+1/2}(r\omega_{l,n}). \tag{4.12}$$

We ignore Bessel functions of the second kind as we assume the solution is finite at the origin. Substituting this into (4.8) gives us the radial component of our solution,

$$f_{l,n} = r^{-1/2}J_{l+1/2}(r\omega_{l,n}). \tag{4.13}$$

Now going back to equation (4.6), the angular components have the form

$$-\frac{1}{g}\left(\frac{\cos\theta}{\sin\theta}g' + g''\right) - \frac{1}{h\sin^2\theta}h'' = l(l+1), \tag{4.14}$$

and rearranging this to have terms depending on θ on one side of the equation and terms depending on φ on the other, we obtain

$$\frac{\sin^2\theta}{g}g'' + \frac{\sin\theta\cos\theta}{g}g' + l(l+1)\sin^2\theta = -\frac{1}{h}h''. \tag{4.15}$$

Once again the LHS is only dependent on θ and the RHS only dependent on φ, meaning that both sides must be equal to a constant which we will call k^2. The equation for $h(\varphi)$ becomes

$$h'' + k^2h = 0, \tag{4.16}$$

with solutions

$$h(\varphi) = e^{\pm ik\varphi},\ k = 0, 1, 2, \ldots \tag{4.17}$$

We are now left with the equation for $g(\theta)$. We have

$$\sin^2\theta g'' + \sin\theta\cos\theta g' + g[l(l+1)\sin^2\theta - k^2] = 0. \tag{4.18}$$

Introducing the following change of variables will simplify things by eliminating the trigonometric functions[8]:

$$t = \cos\theta,\ g(\theta) = P(\cos\theta) = P(t). \tag{4.19}$$

Since $0 \le \theta \le \pi$ we have $|t^2| \le 1$, and $\sin\theta = \sqrt{1-t^2}$. Then

$$\begin{aligned} \frac{dg}{d\theta} &= -\sin\theta\frac{dP}{dt} \\ &= -\sqrt{1-t^2}\frac{dP}{dt}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} \frac{d^2g}{d\theta^2} &= \sin^2\theta\frac{d^2P}{dt^2} - \cos\theta\frac{dP}{dt} \\ &= (1-t^2)\frac{d^2P}{dt^2} - t\frac{dP}{dt}. \end{aligned} \tag{4.21}$$

Equation (4.18) becomes

$$(1-t^2)^2\frac{d^2P}{dt^2} - 2t(1-t^2)\frac{dP}{dt} + [l(l+1)(1-t^2) - k^2]P = 0. \tag{4.22}$$

This equation is the Legendre differential equation of order k. l must be either 0 or a positive integer in order to have finite solutions on the interval $|t^2| \le 1$, and the integer k can take values $-l, -(l-1), \ldots, 0, \ldots, (l-1), l$[9]. The solutions are called associated Legendre functions and take the form

$$P_l^k(t) = (-1)^k\frac{(1-t^2)^{k/2}}{2^l l!}\frac{d^{l+k}}{dt^{l+k}}(t^2-1)^l. \tag{4.23}$$

Now we have all the components of our separable solution in (4.3), however the angular components are often combined to form the spherical harmonics*, Y_l^k:

$$Y_l^k(\theta,\varphi) = P_l^k(\cos\theta)e^{ik\varphi}. \tag{4.24}$$

Finally we combine equations (4.13) and (4.24) to produce the following separable solution to equation (4.2):

$$\phi_{k,l,n} = r^{-1/2}J_{l+1/2}(r\omega_{l,n})Y_l^k(\theta,\varphi), \quad \omega_{l,n}^2 = \lambda_{l,n} - m^2. \tag{4.25}$$

*the notation Y_l^m is standard, however we use k here to avoid confusion.

We want our solution to vanish on the boundary of the sphere, i.e. at $r = R$, and so we have

$$\phi_{k,l,n}(R,\theta,\varphi) = R^{-1/2} J_{l+1/2}(R\omega_{l,n}) Y_l^k(\theta,\varphi) = 0$$
$$\implies J_{l+1/2}(R\omega_{l,n}) = 0. \tag{4.26}$$

The generalised ζ-function is given by the following:

$$\zeta(s) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(2l+1)(\lambda_{l,n})^{-s}$$
$$= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(2l+1)(w_{l,n}^2 + m^2)^{-s}, \tag{4.27}$$

since for every value of l there are $(2l+1)$ values k can take, i.e. there are $(2l+1)$ independent spherical harmonics.

4.2 Analytic Continuation of $\zeta(s)$

The ζ-function is defined for $\Re(s) > 3/2$ as it is defined in (4.27)[10]. In order to calculate the determinant of the Laplace operator we must analytically continue this to the left. To do this we again begin by writing $\zeta(s)$ as a contour integral by applying the residue theorem. Using the same method described in section 3, we can write $\zeta(s)$ as the following contour integral:

$$\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(2l+1)(\omega_{l,n}+m^2)^{-s} = \sum_{l=0}^{\infty}(2l+1)\int_\gamma \frac{(k^2+m^2)^{-s}}{2\pi \mathrm{i}}\frac{J'_{l+1/2}(kR)}{J_{l+1/2}(kR)}\,dk$$
$$= \sum_{l=0}^{\infty}(2l+1)\int_\gamma \frac{(k^2+m^2)^{-s}}{2\pi \mathrm{i}}\frac{\partial}{\partial k}\ln\left[J_{l+1/2}(kR)\right]\,dk, \tag{4.28}$$

where the contour γ runs anticlockwise and encloses all solutions of equation (4.26) on the positive real axis. Before considering the summation over l we will first proceed with the integral alone. Defining

$$\zeta^\nu(s) = \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{(k^2+m^2)^{-s}}{2\pi \mathrm{i}}\frac{\partial}{\partial k}\ln\left[k^{-\nu}J_\nu(kR)\right]\,dk, \tag{4.29}$$

where $\nu = l+1/2$, will simplify things later. We can add the factor $k^{-\nu}$ into the logarithm without changing the result, as doing so will enclose no additional poles[11].

It is possible to shift the integration contour and place it along the imaginary axis. Equation (4.29) has branch points at $k = \pm\mathrm{i}m$, however only the branch point located at $k = \mathrm{i}m$ will concern us as our contour encloses the positive, real solutions of equation (4.26). Then placing the branch cut along the positive imaginary axis we form the contour shown in figure 4.

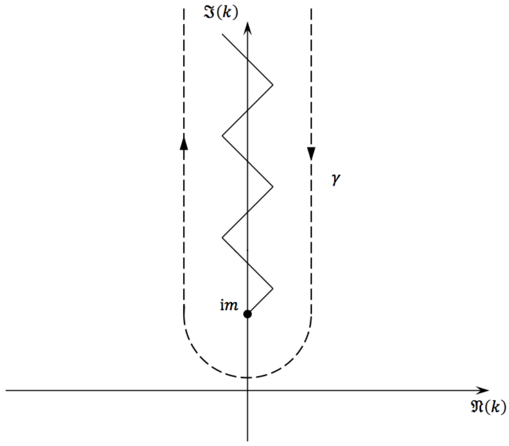

Figure 4: Figure showing the branch cut and the shifted contour along the imaginary axis.

We shift the contour by making the following change of variable:

$$k \to \mathrm{i}k, \quad \text{where } k \text{ varies between } m \text{ and } \infty. \tag{4.30}$$

$m^2 - k^2$ is negative, and as we circle the branch point its argument changes by 2π. Defining its argument to be $-\pi$ on the left-hand side of the branch cut, and π on the right-hand side of the branch cut, we have

$$k^2 + m^2 = e^{-\mathrm{i}\pi}(k^2 - m^2) \quad \text{on LHS,} \tag{4.31}$$

$$k^2 + m^2 = e^{\mathrm{i}\pi}(k^2 - m^2) \quad \text{on RHS.} \tag{4.32}$$

Then splitting the integral in equation (4.29) into two parts; one part to the right of the branch cut and one to the left, we have

$$
\begin{aligned}
\zeta^{\nu}(s) &= \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \frac{(k^2+m^2)^{-s}}{2\pi\mathrm{i}} \frac{\partial}{\partial k} \ln\left[k^{-\nu} J_{\nu}(kR)\right] dk \\
&= \frac{1}{2\pi\mathrm{i}} \int_{m}^{\infty} e^{\mathrm{i}\pi s}(k^2-m^2)^{-s} \frac{\partial}{\partial k} \ln\left[(\mathrm{i}k)^{-\nu} J_{\nu}(\mathrm{i}kR)\right] dk \\
&\quad - \frac{1}{2\pi\mathrm{i}} \int_{m}^{\infty} e^{-\mathrm{i}\pi s}(k^2-m^2)^{-s} \frac{\partial}{\partial k} \ln\left[(\mathrm{i}k)^{-\nu} J_{\nu}(\mathrm{i}kR)\right] dk.
\end{aligned} \tag{4.33}
$$

Recalling equations (3.20) and (3.21) allows us to write equation (4.33) as the following:

$$
\zeta^{\nu}(s) = \frac{\sin(\pi s)}{\pi} \int_{m}^{\infty} (k^2-m^2)^{-s} \frac{\partial}{\partial k} \ln\left[k^{-\nu} I_{\nu}(kR)\right] dk. \tag{4.34}
$$

This representation of $\zeta^{\nu}(s)$ is valid in the strip $1/2 < \Re(s) < 1$[11].

Our next step is to split this integral up into several pieces by adding and subtracting the first two terms of the asymptotic expansion for the modified Bessel function. For $\nu \to \infty$ at $z = k/\nu$ fixed, we have[12]

$$
I_{\nu}(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right], \tag{4.35}
$$

where

$$
t = \frac{1}{\sqrt{1+z^2}} \quad \text{and} \quad \eta = \sqrt{1+z^2} + \ln\left[\frac{z}{1+\sqrt{1+z^2}}\right]. \tag{4.36}
$$

The coefficients u_k are found from the recursion relation[12]

$$
u_{k+1}(t) = \frac{1}{2} t^2(1-t^2) u_k'(t) + \frac{1}{8} \int_0^t (1-5\tau^2) u_k(\tau)\, d\tau, \quad u_0(t) = 1. \tag{4.37}
$$

We will also define the coefficients D_n by

$$
\ln\left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right] \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n}. \tag{4.38}
$$

Now making the change of variable $k = \nu z/R$ and adding and subtracting the two leading terms of equation (4.35) for $\nu \to \infty$, equation (4.34) becomes

$$
\begin{aligned}
\zeta^\nu(s) &= \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \ln \left[\left(\frac{\nu z}{R} \right)^{-\nu} I_\nu(\nu z) \right] dz \\
&= \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left\{ \ln \left[\left(\frac{\nu z}{R} \right)^{-\nu} I_\nu(\nu z) \right] \right. \\
&\qquad \left. - \ln \left[\frac{z^{-\nu}}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \right] - \sum_{n=1}^{2} \frac{D_n(t)}{\nu^n} \right\} dz \\
&\quad + \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left\{ \ln \left[\frac{z^{-\nu}}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \right] \right. \\
&\qquad \left. + \sum_{n=1}^{2} \frac{D_n(t)}{\nu^n} \right\} dz.
\end{aligned}
\tag{4.39}
$$

This can be simplified slightly by noticing that certain terms within the logarithms have no z-dependence, and so will disappear when taking the partial derivative. There is also a cancellation of $z^{-\nu}$ within the first term. This leaves the following:

$$
\zeta^\nu(s) = Z^\nu(s) + \sum_{i=-1}^{2} A_i^\nu(s), \tag{4.40}
$$

with the definitions

$$
\begin{aligned}
Z^\nu(s) = \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} & \left\{ \ln \left[I_\nu(\nu z) \right] \right. \\
& \left. - \ln \left[\frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \right] - \sum_{n=1}^{2} \frac{D_n(t)}{\nu^n} \right\} dz,
\end{aligned}
\tag{4.41}
$$

$$
A_{-1}^\nu(s) = \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \ln \left(z^{-\nu} e^{\nu\eta} \right) dz, \tag{4.42}
$$

$$
A_0^\nu(s) = \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \ln (1+z^2)^{-1/4} dz, \tag{4.43}
$$

$$
A_i^\nu(s) = \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left(\frac{D_i(t)}{\nu^i} \right) dz, \quad \text{for } i = 1, 2. \tag{4.44}
$$

As was mentioned earlier, this representation of $\zeta^\nu(s)$ is well-defined on the strip $1/2 < \Re(s) < 1$. We are now in a position to show the analytic continuation in the parameter s to the whole of the complex plane can be performed, in terms of known functions. We will first focus on the $A_i^\nu(s)$ terms, for $i = -1, 0$. It can be shown that[11]

$$A_{-1}^\nu(s) = \frac{m^{-2s}}{2\sqrt{\pi}} Rm \frac{\Gamma(s-1/2)}{\Gamma(s)} {}_2F_1\left(-\frac{1}{2}, s-\frac{1}{2}; \frac{1}{2}; -\left(\frac{\nu}{mR}\right)^2\right) - \frac{\nu}{2} m^{-2s}, \quad (4.45)$$

$$A_0^\nu(s) = -\frac{1}{4} m^{-2s} {}_2F_1\left(1, s; 1; -\left(\frac{\nu}{mR}\right)^2\right) = -\frac{1}{4} m^{-2s} \left[\left(\frac{\nu}{mR}\right)^2 + 1\right]^{-s}, \quad (4.46)$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is a hypergeometric function, defined by[6]

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^\infty t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt. \quad (4.47)$$

For $A_0^\nu(s)$ we have also used that ${}_2F_1(\alpha, s; \alpha, x) = (1-x)^{-s}$[11].

Recalling that

$$\zeta(s) = \sum_{l=0}^{\infty} (2l+1) \left[Z^{l+1/2}(s) + \sum_{i=-1}^{2} A_i^{l+1/2}(s)\right], \quad (4.48)$$

we now need to consider the summation over l. Looking first at the $A_{-1}^\nu(s)$ term, this is best done using the following Mellin-Barnes type integral representation of the hypergeometric functions:

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)} (-z)^t\, dt, \quad (4.49)$$

where the poles of $\Gamma(a+t)\Gamma(b+t)/\Gamma(c+t)$ lie to the left of the contour $\mathcal{C}$, and the poles of $\Gamma(-t)$ lie to the right[6]. Then

$$\begin{aligned} A_{-1}(s) = -\sum_{l=0}^{\infty} (2l+1) \frac{m^{-2s}}{8\pi^{3/2}\mathrm{i}\Gamma(s)} Rm \int_{\mathcal{C}} \Bigg\{ & \frac{\Gamma(t-1/2)\Gamma(s+t-1/2)\Gamma(-t)}{(mR)^{2t}\Gamma(t+1/2)} \\ & \times \left(l+\frac{1}{2}\right)^{2t} \Bigg\}\, dt - \sum_{l=0}^{\infty} \left(l+\frac{1}{2}\right)^2 m^{-2s}. \end{aligned} \quad (4.50)$$

We would like to write this as a Hurwitz ζ-function, equation (2.1), however care must be taken when interchanging the summation over l and the integration. In order that the integral does not diverge we require $\Re(\mathcal{C}) < -1$[11]. If this was not the case then the factor of $(l+1/2)^{2t+1}$ we would have within the integrand would cause the integral to diverge. The argument $\Gamma(t-1/2)\Gamma(s+t-1/2)/\Gamma(t+1/2)$ has a pole at $t = 1/2$, therefore we require the contour to cross the real axis to the right of $t = 1/2$, and then again between $t = 0$ and $t = 1/2$ to ensure that the pole $t = 0$ of $\Gamma(-t)$ lies to the right. We therefore have a contour that encloses the pole at $t = 1/2$, cancelling the potentially divergent second term in $A_{-1}(s)$.

Interchanging the summation and integration we have

$$A_{-1}(s) = -\frac{m^{-2s}Rm}{4\pi^{3/2}\mathrm{i}\Gamma(s)}\int_{\mathcal{C}}\frac{\Gamma(t-1/2)\Gamma(s+t-1/2)\Gamma(-t)}{(mR)^{2t}\Gamma(t+1/2)}\zeta_{\mathcal{H}}\left(-2t-1,\frac{1}{2}\right)\,dt \\ - m^{-2s}\zeta_{\mathcal{H}}\left(-2,\frac{1}{2}\right). \tag{4.51}$$

We can perform the integration using the residue theorem. Considering first the pole at $t = 1/2$, the residue is

$$-\frac{m^{-2s}Rm}{4\pi^{3/2}\mathrm{i}\Gamma(s)}\frac{\Gamma(s)\Gamma(-1/2)}{mR\Gamma(1)}\zeta_{\mathcal{H}}\left(-2,\frac{1}{2}\right) \\ = \frac{m^{-2s}}{2\pi\mathrm{i}}\zeta_{\mathcal{H}}\left(-2,\frac{1}{2}\right). \tag{4.52}$$

Then multiplying this by $2\pi\mathrm{i}$ the we see that it will cancel with the last term in $A_{-1}(s)$. The other poles in the integrand occur at $t = 1/2 - s - j$, $j = 0, 1, 2\ldots$, and so using the fact that the residue of $\Gamma(-j)$ is $(-1)^j/j!$ we have

$$A_{-1}(s) = -2\pi\mathrm{i}\times\frac{m^{-2s}Rm}{4\pi^{3/2}\mathrm{i}\Gamma(s)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{\Gamma(-s-j)\Gamma(s+j-1/2)}{(mR)^{1-2s-2j}\Gamma(1-s-j)} \\ \times\zeta_{\mathcal{H}}\left(2j+2s-2,\frac{1}{2}\right). \tag{4.53}$$

Using the functional equation of the gamma function (1.7) and simplifying we can write this as

$$A_{-1}(s) = \frac{R^{2s}}{2\sqrt{\pi}\Gamma(s)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{\Gamma(s+j-1/2)}{s+j}\zeta_{\mathcal{H}}\left(2j+2s-2,\frac{1}{2}\right). \tag{4.54}$$

It is simpler to find A_0, as all we require is the binomial expansion. The binomial expansion for complex arguments is given by

$$(x+y)^{-s} = \sum_{j=0}^{\infty} \binom{-s}{j} x^{-s-j} y^j, \tag{4.55}$$

valid for $|x| > |y|$. The binomial coefficient for complex arguments can be written in terms of the Gamma function:

$$\binom{-s}{j} = \frac{\Gamma(1-s)}{\Gamma(j+1)\Gamma(1-s-j)} = \frac{\Gamma(1-s)}{j!\Gamma(1-s-j)}. \tag{4.56}$$

Therefore we have

$$\begin{aligned}
A_0(s) &= \sum_{l=0}^{\infty}(2l+1)\left[-\frac{1}{4}m^{-2s}\sum_{j=0}^{\infty}\frac{\Gamma(1-s)}{j!\Gamma(1-s-j)}\left(\frac{\nu}{mR}\right)^{-2s-2j}\right] \\
&= -\frac{1}{2}R^{2s}\sum_{j=0}^{\infty}(mR)^{2j}\frac{\Gamma(1-s)}{j!\Gamma(1-s-j)}\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)^{1-2s-2j},
\end{aligned} \tag{4.57}$$

valid for $|mR| < l + 1/2$. The minimum value of l is zero, therefore it is convergent for $|mR| < 1/2$.

This can be written in a slightly different form using the following functional relation[6]:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{4.58}$$

Using (4.58) we can write

$$\begin{aligned}
\Gamma(1-s-j) &= \frac{\pi}{\Gamma(s+j)\sin[\pi(s+j)]} \\
&= \frac{\pi}{(-1)^j\Gamma(s+j)\sin(\pi s)} \\
\implies \frac{\Gamma(1-s)}{\Gamma(1-s-j)} &= \frac{(-1)^j\Gamma(s+j)}{\Gamma(s)}.
\end{aligned} \tag{4.59}$$

Substituting this into our equation for $A_0(s)$, (4.57), we are left with

$$A_0(s) = -\frac{R^{2s}}{s\Gamma(s)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(mR)^{2j}\Gamma(s+j)\zeta_{\mathcal{H}}\left(2j+2s-1, \frac{1}{2}\right), \tag{4.60}$$

where we have noticed the summation over l was simply a Hurwitz ζ-function.

We now have expressions for $A_{-1}(s)$ and $A_0(s)$, however we have not yet considered the $A_i(s)$ terms, for $i = 1, 2$. From equations (4.37) and (4.38) it is clear that the $D_i(t)$ are polynomials in t, and so we can write

$$D_i(t) = \sum_{a=0}^{i} x_{i,a} t^{i+2a}, \tag{4.61}$$

where the coefficients $x_{i,a}$ can be calculated using equations (4.37) and (4.38) directly, or via the recursion relation given in reference [11]. Then we have

$$\begin{aligned} A_i^\nu(s) &= \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left(\frac{\sum_{a=0}^{i} x_{i,a} t^{i+2a}}{\nu^i} \right) dz \\ &= \frac{\sin(\pi s)}{\pi} \sum_{a=0}^{i} \frac{x_{i,a}}{\nu^i} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left(t^{i+2a} \right) dz \\ &= -m^{-2s} \frac{\sin(\pi s)}{2\pi} \sum_{a=0}^{i} \frac{x_{i,a}}{\nu^i} \frac{(i+2a)}{(mR)^{i+2a}} \frac{\Gamma(s+a+\frac{i}{2})\Gamma(1-s)}{\Gamma(1+a+\frac{i}{2})} \nu^{i+2a} \\ &\qquad \times \left[1 + \left(\frac{\nu}{mR} \right)^2 \right]^{-s-a-i/2}, \end{aligned} \tag{4.62}$$

where for the last step we have used the following identity[11]:

$$\begin{aligned} \int_{mR/\nu}^{\infty} \left[\left(\frac{\nu z}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} t^n \, dz = -m^{-2s} \frac{n}{2(mR)^n} \frac{\Gamma(s+n/2)\Gamma(1-s)}{\Gamma(1+n/2)} \nu^n \\ \times \left[1 + \left(\frac{\nu}{mR} \right)^2 \right]^{-s-n/2}. \end{aligned} \tag{4.63}$$

We can simplify our expression for $A_i^\nu(s)$ slightly further using equation (4.58), resulting in

$$A_i^\nu(s) = -\frac{m^{-2s}}{2(mR)^i \Gamma(s)} \sum_{a=0}^{i} x_{i,a} \frac{(i+2a)}{(mR)^{2a}} \frac{\Gamma(s+a+i/2)}{\Gamma(1+a+i/2)} \nu^{2a} \left[1 + \left(\frac{\nu}{mR} \right)^2 \right]^{-s-a-i/2}. \tag{4.64}$$

Our next step is to consider the summation over l. We start with

$$A_i(s) = -\frac{m^{-2s}}{(mR)^i \Gamma(s)} \sum_{a=0}^{i} \frac{x_{i,a}(i+2a)}{(mR)^{2a}} \frac{\Gamma(s+a+i/2)}{\Gamma(1+a+i/2)} \sum_{l=0}^{\infty} \left(l+\frac{1}{2}\right)^{2a+1} \times \left[1+\left(\frac{l+1/2}{mR}\right)^2\right]^{-s-a-i/2}, \tag{4.65}$$

where we have combined the factors of $l+1/2$. Now we can use equations (4.55) and (4.56) as we did for A_0 to write

$$\left[\left(\frac{l+1/2}{mR}\right)+1\right]^{-s-a-i/2} = \sum_{j=0}^{\infty} \frac{\Gamma(1-s-a-i/2)}{j!\,\Gamma(1-s-a-j-i/2)} (mR)^{2s+2a+2j+i} \times \left(l+\frac{1}{2}\right)^{-2s-2a-2j-i} \tag{4.66}$$

After substituting this into equation (4.65) and simplifying the result we are left with

$$A_i(s) = -\frac{R^{2s}}{\Gamma(s)} \sum_{j=0}^{\infty} \frac{(mR)^{2j}}{j!} \zeta_{\mathcal{H}}\left(-1+i+2j+2s, \frac{1}{2}\right) \sum_{a=0}^{i} x_{i,a}(i+2a) \times \frac{\Gamma(s+a+i/2)}{\Gamma(1+a+i/2)} \frac{\Gamma(1-s-a-i/2)}{\Gamma(1-s-a-j-i/2)}. \tag{4.67}$$

Once again we can use equation (4.58) to rewrite the gamma functions, which will lead to some cancellation. First we have

$$\Gamma\left(1-s-a-j-\frac{i}{2}\right) = \frac{\pi}{\Gamma(s+a+j+i/2)\sin\left[\pi(s+a+j+i/2)\right]}, \tag{4.68}$$

and expanding the sine function gives us

$$\Gamma\left(1-s-a-j-\frac{i}{2}\right) = \frac{\pi}{(-1)^j \Gamma(s+a+j+i/2)\sin\left[\pi(s+a+i/2)\right]}. \tag{4.69}$$

By substituting this into equation (4.67) along with the following:

$$\Gamma\left(s+a+\frac{i}{2}\right) = \frac{\pi}{\Gamma(1-s-a-i/2)\sin\left[\pi(s+a+i/2)\right]}, \tag{4.70}$$

we see that the sine factors will cancel along with some of the gamma functions. Then we are left with our final expression for $A_i(s)$:

$$A_i(s) = -\frac{R^{2s}}{\Gamma(s)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (mR)^{2j} \zeta_{\mathcal{H}} \left(-1 + i + 2j + 2s, \frac{1}{2} \right) \tag{4.71}$$

$$\times \sum_{a=0}^{i} x_{i,a} \frac{(i+2a)\Gamma(s+a+j+i/2)}{\Gamma(1+a+i/2)}.$$

4.3 Calculation of the Zeta Function Determinant $\zeta'(0)$

Now putting everything we have so far together, the following representation of $\zeta(s)$ is valid for $-1/2 < \Re(s) < 1$[10]:

$$\zeta(s) = \sum_{l=0}^{\infty} \nu Z^{\nu}(s) + \sum_{i=-1}^{2} A_i(s), \tag{4.72}$$

where $Z^{\nu}(s)$ is defined in (4.41) and the $A_i(s)$ terms are defined in (4.54) for $i = -1$, (4.60) for $i = 0$, and (4.71) for $i = 1, 2$. We have successfully analytically continued $\zeta(s)$ past the point $s = 0$, and therefore equation (4.72) is now a suitable starting point to calculate $\zeta'(0)$.

For definiteness we will consider the case where $m = 0$. In this limit it is clear that only the $j = 0$ term will survive in the terms $A_i(s)$, for $i = -1, 0, 1, 2$. Starting with $A_{-1}(s)$, we then have

$$\begin{aligned} A_{-1}(s) &= \frac{R^{2s}}{2\sqrt{\pi}\Gamma(s)} \frac{\Gamma(s-1/2)}{s} \zeta_{\mathcal{H}} \left(2s-2, \frac{1}{2} \right) \\ &= \frac{R^{2s}}{2\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s+1)} \zeta_{\mathcal{H}} \left(2s-2, \frac{1}{2} \right), \end{aligned} \tag{4.73}$$

where we have used the functional equation of the gamma function, equation (1.7). Differentiating this using the product rule we obtain

$$\frac{d}{ds}A_{-1}(s) = \frac{R^{2s}\ln R}{\sqrt{\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s+1)}\zeta_{\mathcal{H}}\left(2s-2,\frac{1}{2}\right) + \frac{R^{2s}}{2\sqrt{\pi}}\frac{\Gamma'(s-1/2)}{\Gamma(s+1)}\zeta_{\mathcal{H}}\left(2s-2,\frac{1}{2}\right) - \frac{R^{2s}}{2\sqrt{\pi}}\frac{\Gamma(s-1/2)\Gamma'(s+1)}{[\Gamma(s+1)]^2}\zeta_{\mathcal{H}}\left(2s-2,\frac{1}{2}\right) + \frac{R^{2s}}{2\sqrt{\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s+1)}\frac{d}{ds}\zeta_{\mathcal{H}}\left(2s-2,\frac{1}{2}\right). \tag{4.74}$$

However, when we set $s=0$ the first three terms of this will vanish, as $\zeta_{\mathcal{H}}(-2,1/2)=0$. Therefore we only need to consider the final term. It can be shown that

$$\zeta_{\mathcal{H}}\left(s,\,\frac{1}{2}\right) = (2^s-1)\zeta(s). \tag{4.75}$$

Proof: Recall that the definition of the Hurwitz ζ-function is

$$\zeta_{\mathcal{H}}\left(s,\,\frac{1}{2}\right) = \sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)^{-s} = 2^s\sum_{l=0}^{\infty}(2l+1)^{-s} \tag{4.76}$$

This is summing over positive odd integers, therefore we can write this as the Riemann ζ-function minus the sum over positive even integers:

$$\zeta_{\mathcal{H}}\left(s,\,\frac{1}{2}\right) = 2^s\left[\zeta(s)-\sum_{l=1}^{\infty}(2l)^{-s}\right] = 2^s\left[\zeta(s)-2^{-s}\zeta(s)\right] = (2^s-1)\zeta(s). \tag{4.77}$$

Using this we can therefore write

$$\zeta_{\mathcal{H}}\left(2s-2,\frac{1}{2}\right) = (2^{2s-2}-1)\zeta(2s-2)$$

$$\implies \frac{d}{ds}\zeta_{\mathcal{H}}\left(2s-2,\frac{1}{2}\right) = (2^{2s-2}-1)\times 2\,\zeta'(2s-2) + 2^{2s-1}\ln 2\,\zeta(2s-2). \tag{4.78}$$

Substituting this into equation (4.74) and evaluating at $s = 0$, remembering that $\zeta(s)$ vanishes at even negative integers, we obtain

$$\begin{aligned}\left.\frac{d}{ds}A_{-1}(s)\right|_{s=0} &= -\frac{3}{4}\frac{\Gamma(-1/2)}{\sqrt{\pi}}\zeta'(-2) \\ &= \frac{3}{2}\zeta'(-2).\end{aligned} \tag{4.79}$$

Now we need to consider $A_0(s)$. As with $A_{-1}(s)$, we can differentiate using the product rule to obtain

$$\frac{d}{ds}A_0(s) = -R^{2s}\ln R \times \zeta_{\mathcal{H}}\left(2s-1,\frac{1}{2}\right) - \frac{1}{2}R^{2s}\frac{d}{ds}\zeta_{\mathcal{H}}\left(2s-1,\frac{1}{2}\right). \tag{4.80}$$

Differentiating the Hurwitz ζ-function using the same method that we used for $A_{-1}(s)$ we have

$$\begin{aligned}\zeta_{\mathcal{H}}\left(2s-1,\frac{1}{2}\right) &= (2^{2s-1}-1)\zeta(2s-1) \\ \Longrightarrow \quad \frac{d}{ds}\zeta_{\mathcal{H}}\left(2s-1,\frac{1}{2}\right) &= (2^{2s-1}-1)\times 2\,\zeta'(2s-1) + 2^{2s}\ln 2\,\zeta(2s-1).\end{aligned} \tag{4.81}$$

Then using this we obtain

$$\left.\frac{d}{ds}A_0(s)\right|_{s=0} = -\ln R \times \zeta_{\mathcal{H}}\left(-1,\frac{1}{2}\right) - \zeta'(-1) + \ln 2\,\zeta(-1). \tag{4.82}$$

Equation (2.5) can be used to calculate the value of $\zeta_{\mathcal{H}}(s,a)$ at negative integers, and so

$$\left.\frac{d}{ds}A_0(s)\right|_{s=0} = -\frac{1}{24}\ln R + \frac{1}{2}\zeta'(-1) + \frac{1}{24}\ln 2. \tag{4.83}$$

Now considering $A_1(s)$, we have

$$A_1(s) = -\frac{R^{2s}}{\Gamma(s)}\zeta_{\mathcal{H}}\left(2s,\frac{1}{2}\right)\left[x_{1,0}\frac{\Gamma(s+1/2)}{\Gamma(3/2)} + 3x_{1,1}\frac{\Gamma(s+3/2)}{\Gamma(5/2)}\right] \tag{4.84}$$

However, we can see that differentiating this will result in all terms having $\Gamma(s)$ or $\Gamma^2(s)$ on the denominator, which immediately causes most terms to disappear when considering $s \to 0$, and cancels the divergent behaviour of the remaining term. Therefore

$$\frac{d}{ds} A_1(s)\bigg|_{s=0} = 0. \tag{4.85}$$

Now we need to consider $A_2(s)$, however the $\zeta_{\mathcal{H}}(2s+1, 1/2)$ term makes this tricky, as it is not defined for $s = 0$. This can be resolved by considering the series expansions of $\zeta_{\mathcal{H}}(2s+1, 1/2)$ and $1/\Gamma(s)$ about $s = 0$. Then we can write

$$\frac{\zeta_{\mathcal{H}}(2s+1, 1/2)}{\Gamma(s)} \approx \frac{1}{2} - s\psi\left(\frac{1}{2}\right) + \frac{1}{2}s\gamma + \ldots, \tag{4.86}$$

where ψ is the digamma function and γ is Euler's constant[6]. Higher order terms will not matter as they will vanish when setting $s = 0$. substituting this into our expression for $A_2(s)$ gives us

$$A_2(s) = -R^{-2s}\left[\frac{1}{2} - s\psi\left(\frac{1}{2}\right) + \frac{1}{2}s\gamma + \ldots\right]\sum_{a=0}^{2} x_{2,a}(2+2a)\frac{\Gamma(s+a+1)}{\Gamma(2+a)}. \tag{4.87}$$

Now differentiating using the product rule and setting $s = 0$ we have

$$\begin{aligned}\frac{d}{ds} A_2(s)\bigg|_{s=0} &= -\ln R \times \sum_{a=0}^{2} x_{2,a}(2+2a)\frac{\Gamma(a+1)}{\Gamma(a+2)} \\ &\quad - \left[\frac{1}{2}\gamma - \psi\left(\frac{1}{2}\right)\right]\sum_{a=0}^{2} x_{2,a}(2+2a)\frac{\Gamma(a+1)}{\Gamma(a+2)} \\ &\quad - \frac{1}{2}\left[\sum_{a=0}^{2} x_{2,a}(2+2a)\frac{\Gamma'(s+a+1)}{\Gamma(a+2)}\right]_{s=0}\end{aligned} \tag{4.88}$$

Using the recursion relation given in reference [11], we can calculate the coefficients $x_{2,a}$. Then by applying the recursion relation for the gamma function it is simple to show that

$$\sum_{a=0}^{2} x_{2,a}(2+2a)\frac{\Gamma(a+1)}{\Gamma(a+2)} = 0. \tag{4.89}$$

This reduces equation (4.88) to the following:

$$\begin{aligned}\frac{d}{ds}A_2(s)\Big|_{s=0} &= -\frac{1}{2}\sum_{a=0}^{2} x_{2,a}(2+2a)\frac{\psi(a+1)\mathit{\Gamma}(a+1)}{\mathit{\Gamma}(a+2)} \\ &= -\frac{1}{2}\sum_{a=0}^{2} x_{2,a}(2+2a)\frac{\psi(a+1)}{a+1} \\ &= -\frac{1}{16}\psi(1)+\frac{3}{8}\psi(2)-\frac{5}{16}\psi(3), \end{aligned} \tag{4.90}$$

where we have used[6]

$$\mathit{\Gamma}'(s) = \psi(s)\mathit{\Gamma}(s). \tag{4.91}$$

Finally evaluating the values of the digamma function we find that

$$\frac{d}{ds}A_2(s)\Big|_{s=0} = -\frac{3}{32}. \tag{4.92}$$

Adding together the contributions from each of the A_i terms we have

$$\frac{d}{ds}\sum_{i=-1}^{2} A_i(s)\Big|_{s=0} = -\frac{3}{32}+\frac{\ln 2}{24}-\frac{\ln R}{24}+\frac{3}{2}\zeta'(-2)+\frac{1}{2}\zeta'(-1). \tag{4.93}$$

We are now left with $Z^\nu(s)$. Using integration by parts we can write $Z^\nu(s)$ as

$$\begin{aligned} Z^\nu(s) = \frac{\sin(\pi s)}{\pi}&\left\{\left[f(z)\left(\left(\frac{z\nu}{R}\right)^2-m^2\right)^{-s}\right]_{mR/\nu}^{\infty}\right. \\ &\left.+\int_{mR/\nu}^{\infty}\frac{2z\nu s}{R}\left[\left(\frac{z\nu}{R}\right)^2-m^2\right]^{-s-1} f(z)\,dz\right\}, \end{aligned} \tag{4.94}$$

where

$$f(z) = \ln\left[I_\nu(\nu z)\right] - \ln\left[\frac{1}{\sqrt{2\pi\nu}}\frac{e^{\nu\eta}}{(1+z^2)^{1/4}}\right] - \sum_{n=1}^{2}\frac{D_n(t)}{\nu^n}. \tag{4.95}$$

By considering the analyticity of this around $s = 0$ we find that

$$\frac{d}{ds}Z^\nu(s)\Big|_{s=0} = -f\left(\frac{mR}{\nu}\right). \tag{4.96}$$

The polynomials $D_n(t)$ are found to be

$$D_1(t) = \frac{1}{8}t - \frac{5}{24}t^3, \tag{4.97}$$

$$D_2(t) = \frac{1}{16}t^2 - \frac{3}{8}t^4 + \frac{5}{16}t^6, \tag{4.98}$$

and recalling the definitions of t and η we have

$$\begin{aligned} f\left(\frac{mR}{\nu}\right) &= \ln[I_\nu(mR)] + \ln\sqrt{2\pi\nu} - \nu\sqrt{1+\frac{m^2R^2}{\nu^2}} - \nu\ln\left[\frac{mR}{\nu+\sqrt{\nu^2+m^2R^2}}\right] \\ &+ \ln\left[1+\frac{m^2R^2}{\nu^2}\right]^{1/4} - \frac{1}{8}(\nu^2+m^2R^2)^{-1/2} + \frac{5}{24}\nu^2(\nu^2+m^2R^2)^{-3/2} \\ &- \frac{1}{16}(\nu^2+m^2R^2)^{-1} + \frac{3}{8}\nu^2(\nu^2+m^2R^2)^{-2} - \frac{5}{16}\nu^4(\nu^2+m^2R^2)^{-3}. \end{aligned} \tag{4.99}$$

The series representation of the modified Bessel function, I_ν, is given by[6]

$$\begin{aligned} I_\nu(x) &= \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{\nu+2k} \\ &= \left(\frac{x}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k}. \end{aligned} \tag{4.100}$$

Therefore taking the logarithm of this we obtain

$$\ln[I_\nu(mR)] = \nu\ln\left(\frac{mR}{2}\right) + \ln\left\{\frac{1}{\Gamma(\nu+1)} + \frac{1}{\Gamma(\nu+2)}\left(\frac{mR}{2}\right)^2 + \dots\right\}. \tag{4.101}$$

Then in the limit as $m \to 0$ we have

$$\left.\frac{d}{ds}Z^\nu(s)\right|_{s=0} = \ln\Gamma(\nu+1) - \frac{1}{2}\ln(2\pi\nu) - \nu\ln\nu + \nu - \frac{1}{12\nu}. \tag{4.102}$$

We can rewrite this using the following integral representation of $\ln\Gamma(\nu)$ [6]:

$$\ln\Gamma(\nu) = \left(\nu - \frac{1}{2}\right)\ln\nu - \nu + \frac{1}{2}\ln 2\pi + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-t\nu}}{t}\,dt. \tag{4.103}$$

Then by using the recursion relation of the gamma function we have

$$\begin{aligned} \left.\frac{d}{ds}Z^\nu(s)\right|_{s=0} &= \ln\nu + \ln\Gamma(\nu) - \frac{1}{2}\ln(2\pi\nu) - \nu\ln\nu + \nu - \frac{1}{12\nu} \\ &= -\frac{1}{12\nu} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-t\nu}}{t}\,dt. \end{aligned} \tag{4.104}$$

It is possible to combine these terms by rewriting $1/(12\nu)$ as

$$\frac{1}{12\nu} = \int_0^\infty \frac{e^{-t\nu}}{12}\,dt, \tag{4.105}$$

then we have

$$\left.\frac{d}{ds}Z^\nu(s)\right|_{s=0} = \int_0^\infty \left(-\frac{t}{12} + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-t\nu}}{t}\,dt. \tag{4.106}$$

Now considering the sum over ν gives

$$\begin{aligned}\left.\frac{d}{ds}Z(s)\right|_{s=0} &= \sum_{l=0}^{\infty}(2l+1)\int_0^\infty \left(-\frac{t}{12} + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-t(l+1/2)}}{t}\,dt \\ &= \int_0^\infty \left(-\frac{t}{12} + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{1}{t}\sum_{l=0}^{\infty}(2l+1)e^{-t(l+1/2)}\,dt. \end{aligned} \tag{4.107}$$

Let $u = t/2$, and consider

$$f(u) = \sum_{l=0}^{\infty} e^{-u(2l+1)}. \tag{4.108}$$

Differentiating this gives us

$$f'(u) = -\sum_{l=0}^{\infty}(2l+1)e^{-u(2l+1)}, \tag{4.109}$$

but performing the sum in (4.108) results in

$$\begin{aligned} f(u) &= e^{-u}\sum_{l=0}^{\infty} e^{-2lu} \\ &= \frac{1}{e^u - e^{-u}} \\ \implies f'(u) &= -\frac{e^u + e^{-u}}{(e^u - e^{-u})^2}. \end{aligned} \tag{4.110}$$

Now equating (4.109) and (4.110) and substituting back into the integral we are left with

$$\begin{aligned}\left.\frac{d}{ds}Z(s)\right|_{s=0} = 2\int_0^\infty \Bigg(&-\frac{1}{2t^2} + \frac{2}{t^3} - \frac{e^{-t}}{t(1-e^{-t})} - \frac{e^{-t}}{t^2(1-e^{-t})} \\ &-\frac{e^{-2t}}{t(1-e^{-t})^2}\Bigg)\frac{e^{-t/2}}{1-e^{-t}}\,dt. \end{aligned} \tag{4.111}$$

Introducing a regularisation parameter here will simplify things, and we define

$$Z'(0,z) = 2\int_0^\infty t^z \left(-\frac{1}{2t^2} + \frac{2}{t^3} - \frac{e^{-t}}{t(1-e^{-t})} - \frac{e^{-t}}{t^2(1-e^{-t})} \right. \\ \left. - \frac{e^{-2t}}{t(1-e^{-t})^2} \right) \frac{e^{-t/2}}{1-e^{-t}}\, dt, \tag{4.112}$$

with

$$Z'(0,0) = \frac{d}{ds} Z(s)\bigg|_{s=0} = Z'(0). \tag{4.113}$$

Doing so then enables us to calculate individual pieces of the integral in (4.112) by using[6]

$$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{1-e^{-\beta x}}\, dx = \frac{\Gamma(\nu)}{\beta^\nu} \zeta_{\mathcal{H}}\left(\nu, \frac{\mu}{\beta}\right). \tag{4.114}$$

We also require the first and second derivatives of this with respect to β. Differentiating both sides of this gives

$$\int_0^\infty \frac{x^{\nu}e^{-\mu x}e^{-\beta x}}{(1-e^{-\beta x})^2}\, dx = \frac{\nu\Gamma(\nu)}{\beta^{\nu+1}} \zeta_{\mathcal{H}}\left(\nu, \frac{\mu}{\beta}\right) - \frac{\Gamma(\nu)}{\beta^\nu}\frac{\partial}{\partial\beta} \zeta_{\mathcal{H}}\left(\nu, \frac{\mu}{\beta}\right). \tag{4.115}$$

Using the formula for the derivative of the Hurwitz ζ-function (2.6) and the functional relation of the gamma function (1.7) we can write this as

$$\int_0^\infty \frac{x^{\nu}e^{-(\mu+\beta)x}}{(1-e^{-\beta x})^2}\, dx = \frac{\Gamma(\nu+1)}{\beta^{\nu+1}} \zeta_{\mathcal{H}}\left(\nu, \frac{\mu}{\beta}\right) - \frac{\Gamma(\nu+1)\mu}{\beta^{\nu+2}} \zeta_{\mathcal{H}}\left(\nu+1, \frac{\mu}{\beta}\right). \tag{4.116}$$

Now differentiating both sides of this with respect to β we have

$$-\int_0^\infty \frac{x^{\nu+1}e^{-x(\mu+\beta)}}{(1-e^{-\beta x})^2}\, dx - 2\int_0^\infty \frac{x^{\nu+1}e^{-x(\mu+2\beta)}}{(1-e^{-\beta x})^3}\, dx = -\frac{(\nu+1)\Gamma(\nu+1)}{\beta^{\nu+2}} \zeta_{\mathcal{H}}\left(\nu, \frac{\mu}{\beta}\right) \\ + \frac{\Gamma(\nu+1)}{\beta^{\nu+1}}\frac{\partial}{\partial\beta} \zeta_{\mathcal{H}}\left(\nu, \frac{\mu}{\beta}\right) + \frac{(\nu+2)\Gamma(\nu+1)\mu}{\beta^{\nu+3}} \zeta_{\mathcal{H}}\left(\nu+1, \frac{\mu}{\beta}\right) \\ - \frac{\Gamma(\nu+1)\mu}{\beta^{\nu+2}}\frac{\partial}{\partial\beta} \zeta_{\mathcal{H}}\left(\nu+1, \frac{\mu}{\beta}\right). \tag{4.117}$$

We can then use (4.116) to replace the first integral, resulting in

$$2\int_0^\infty \frac{x^{\nu+1}e^{-x(\mu+2\beta)}}{(1-e^{-\beta x})^3}\,dx = -\frac{\Gamma(\nu+2)}{\beta^{\nu+2}}\zeta_{\mathcal{H}}\left(\nu+1,\frac{\mu}{\beta}\right) + \frac{\Gamma(\nu+2)\mu}{\beta^{\nu+3}}\zeta_{\mathcal{H}}\left(\nu+2,\frac{\mu}{\beta}\right)$$
$$+\frac{(\nu+1)\Gamma(\nu+1)}{\beta^{\nu+2}}\zeta_{\mathcal{H}}\left(\nu,\frac{\mu}{\beta}\right) - \frac{\Gamma(\nu+1)\mu\nu}{\beta^{\nu+3}}\zeta_{\mathcal{H}}\left(\nu+1,\frac{\mu}{\beta}\right)$$
$$-\frac{(\nu+2)\Gamma(\nu+1)\mu}{\beta^{\nu+3}}\zeta_{\mathcal{H}}\left(\nu+1,\frac{\mu}{\beta}\right) + \frac{(\nu+1)\Gamma(\nu+1)\mu^2}{\beta^{\nu+4}}\zeta_{\mathcal{H}}\left(\nu+2,\frac{\mu}{\beta}\right), \tag{4.118}$$

where we have again used the derivative of the Hurwitz ζ-function (2.6).

Equations (4.114), (4.116) and (4.118) can then be used to calculate the pieces of the integral in (4.112). We obtain

$$Z'(0,z) = \zeta_{\mathcal{H}}\left(z-2,\frac{1}{2}\right)[4\Gamma(z-2) - 2\Gamma(z-1) - (z-1)\Gamma(z-1)]$$
$$+\zeta_{\mathcal{H}}\left(z-1,\frac{1}{2}\right)\left[-\Gamma(z-1) - 2\Gamma(z) + \Gamma(z-1) + \Gamma(z) + \frac{1}{2}(z-2)\Gamma(z-1)\right.$$
$$\left.+\frac{1}{2}z\Gamma(z-1)\right] + \zeta_{\mathcal{H}}\left(z,\frac{1}{2}\right)\left[\Gamma(z) - \frac{1}{2}\Gamma(z) - \frac{1}{4}(z-1)\Gamma(z-1)\right]. \tag{4.119}$$

Finally, we can simplify this further by using the functional equation of the gamma function (1.7), leaving us with

$$Z'(0,z) = \zeta_{\mathcal{H}}\left(z-2,\frac{1}{2}\right)\Gamma(z-2)(6+z-z^2) + \frac{1}{4}\zeta_{\mathcal{H}}\left(z,\frac{1}{2}\right)\Gamma(z). \tag{4.120}$$

To find $Z'(0)$ this must be evaluated at $z=0$. Using equation (4.75) we can write this as

$$Z'(0,z) = (2^{z-2}-1)\zeta(z-2)\Gamma(z-2)(6+z-z^2) + \frac{1}{4}(2^z-1)\zeta(z)\Gamma(z). \tag{4.121}$$

Then by Taylor expanding we can find an expression for $Z'(0)$:

$$Z'(0) = \lim_{z\to 0} Z'(0,z) = \lim_{z\to 0}\left\{(s^{z-2}-1)[\zeta(-2) + z\zeta'(-2) + \ldots]\right.$$
$$\left.\times\left(\frac{1}{2z}+\ldots\right)(6+z-z^2) + \frac{1}{4}(z\ln 2+\ldots)[\zeta(0)+z\zeta'(0)+\ldots]\left(\frac{1}{z}+\ldots\right)\right\}, \tag{4.122}$$

where we have used $2^z = e^{z \ln 2}$. Higher order terms are not important as they will vanish when $z \to 0$. from section 2.2 we know that $\zeta(-2) = 0$, and $\zeta'(0) = -\ln(2\pi)/2$[3]. Therefore

$$Z'(0) = -\frac{9}{4}\zeta'(-2) - \frac{1}{8}\ln 2. \tag{4.123}$$

Now we can combine (4.93) and (4.123) to obtain our final expression for $\zeta'(0)$, leaving us with

$$\zeta'(0) = -\frac{3}{32} - \frac{1}{12}\ln 2 - \frac{3}{4}\zeta'(-2) + \frac{1}{2}\zeta'(-1) - \frac{1}{24}\ln R. \tag{4.124}$$

5 Conclusion

Using complex-plane integration techniques we have analytically continued the zeta function for the Bessel operator to the negative real axis. We have then used our result to calculate its value at even negative integers, and the residues at odd negative integers.

Using the zeta function regularisation prescription we have calculated the zeta function determinant of the Laplacian on the 3-dimensional ball for Dirichlet boundary conditions. In general the ability to calculate determinants of differential operators is useful within quantum field theory[10]. The method applied can be generalised to different boundary conditions and higher dimensions, and explicit results for dimensions $D \leq 6$ are given in reference [10], on Dirichlet, Neumann and the more general Robin boundary conditions.

It was helpful to look at some properties of the Hurwitz zeta function and the Riemann zeta function to simplify our calculations. We also looked at a brief overview of the Riemann hypothesis, which still remains unsolved over a century and a half later, although there has been some progress towards proving it. For example In 1914 Hardy showed that there are an infinite number of non-trivial zeros which have real part equal to 1/2[13].

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